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Quasi-periodic solutions of nonlinear elliptic partial differential equations

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Abstract. In a recent paper [9] the KAM theory has been extended to non-linear partial differential equations, to construct quasi-periodic solutions. In this article this theory is illustrated with three typical examples: an elliptic partial differential equation, an ordinary differential equation and a difference equation related to monotone twist mappings.

1. Three examples

We begin with three problems illustrating the circle of questions to be discussed. They involve nonlinear elliptic partial differential equations on the one hand and the stability question of mechanics on the other. As first example we consider the elliptic differential equation

$$\Delta u = f(x,u), \quad x \in \mathbf{R}^n, \quad \Delta = \sum_{\nu=1}^{n} \partial_{x_\nu}^2 \tag{1.1}$$

where $f \in C^\infty(\mathbf{T}^{n+1})$ is a periodic function of period 1 in its n+1 variables $x_1, x_2, \dots, x_n$ and $x_{n+1} = u$ with mean value zero:

$$\int_{\mathbf{T}^{n+1}} f(\overline{x})d\overline{x} = 0; \tag{1.2}$$

here we denote by $\overline{x}$ the vector $(x_1, x_2, \dots, x_{n+1})$, $x = (x_1, x_2, \dots, x_n)$ and by $\mathbf{T}^{n+1}$ the torus $\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$.

However, we do not impose the usual boundary conditions and do not look for periodic solutions but rather ask for quasi-periodic solutions in the folowing sense: We require that the desired solutions $u = u(x)$ can be represented in terms of a vector $\alpha \in \mathbf{R}^n$ and a function $U = U(\overline{x})$ with $U(\overline{x}) - x_{n+1} \in C^\infty(\mathbf{T}^{n+1})$, $\partial_{x_{n+1}} U > 0$ in the form

$$u(x) = U(x, \alpha \cdot x); \quad \alpha \cdot x = \sum_{\nu=1}^{n} \alpha_\nu x_\nu. \tag{1.3}$$

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To explain the term "quasi-periodic" we represent $U - x_{n+1}$ in terms of a Fourier series

$$U(\bar{x}) = x_{n+1} + \sum_{\bar{j}\in\mathbb{Z}^{n+1}} c_{\bar{j}} e^{2\pi i \bar{j}\cdot\bar{x}}$$

so that (1.3) takes the form

$$u(x) = \alpha\cdot x + \sum_{\bar{j}\in\mathbb{Z}^{n+1}} c_{\bar{j}} e^{2\pi i (j+\alpha j_{n+1})\cdot x}.$$

Thus $u(x)-\alpha\cdot x$ is quasi-periodic in the usual sense with frequencies $j_\nu+\alpha_\nu j_{n+1}$, $\nu = 1,2,\dots,n$, $j_1, j_2,\dots, j_{n+1} \in \mathbb{Z}$. The function $u(x)$ grows linearly and actually it is $e^{2\pi i u}$ which is quasi-periodic, but since u is to be viewed as angular variable it is justified to simply refer to u as being quasi-periodic.

If $\alpha \in \mathbb{Q}^n$ is a vector with rational components then u can be viewed as a periodic function on the torus: indeed, if $q \in \mathbb{Z}\backslash(0)$ is a common denominator of the α_ν, i.e. $q\alpha \in \mathbb{Z}^n$, then one has

$$u(x+qj)-u(x) \in \mathbb{Z} \text{ for all } x\in\mathbb{R}^n,\ j\in\mathbb{Z}^n,$$

and the graph $\{(x,u(x)), x \in \mathbb{R}^n\}$ can be viewed as an n-dimensional torus in $\mathbb{T}^{n+1}$.

However, if $\alpha \notin \mathbb{Q}^n$ the situation is quite different and the graph of u is not compact and, in fact, dense in $\mathbb{T}^{n+1}$. Indeed, from the periodicity of f and U we conclude that with u given by (1.3) also

$$u(x+j) - j_{n+1} = U(x, \alpha\cdot x + \lambda(\bar{j})),\quad \lambda(\bar{j}) = \alpha\cdot j - j_{n+1}$$

is a solution of (1.1). Since, as is well known, the set $\{\lambda(\bar{j}) = \alpha\cdot j - j_{n+1} | \bar{j} \in \mathbb{Z}^{n+1}\}$ is dense on $\mathbb{R}$ precisely if $\alpha\notin\mathbb{Q}^n$ we obtain in this case a one parameter family

$$u(x,\lambda) = U(x, \alpha\cdot x+\lambda) \tag{1.4}$$

of solutions, which depends monotonically on λ and satisfies $u(x,\lambda+1) = u(x,\lambda)+1$.

One can view this situation differently: The invertible mapping

$$\bar{x} = (x, x_{n+1}) \to (x, U(x, x_{n+1}))$$

of the torus onto itself takes the family of affine hyperplanes $x_{n+1} = \alpha\cdot x+\lambda$ into the solutions (1.4) of (1.1). This is an example of a foliation on a manifold, here the torus, whose leaves are required to satisfy the differential equation (1.1) and have a prescribed direction vector α. The problem then is to find such solutions u, or equivalently the corresponding function U of $n+1$ variables. The analytic

requirements for the function U are the following:

(1.5)
i) $U(\bar{x}) - x_{n+1} \in C^\infty(\mathbb{T}^{n+1})$, i.e. has period 1 in $x_1, x_2, \dots, x_{n+1}$
ii) $U(x, x_{n+1})$ is strictly increasing in x_{n+1}
iii) U satisfies the differential equation

$$\sum_{\nu=1}^{n} (\partial_{x_\nu} + \alpha_\nu \partial_{x_{n+1}})^2 U = f(x, U(\bar{x}))$$

Note that the left hand side correspond to Δu if we observe (1.3).

This is not anymore an elliptic differential equation since it involves differentiations only tangential to the hyperplanes $x_{n+1} - \alpha\cdot x = \text{const}$.; it is a degenerate differential equation and consequently the conditions for its solvability are rather unusual. The vector α can not be prescribed arbitrarily, but one of the main results in this note is to show that there always exist $\alpha \notin \mathbb{Q}^n$ for which (1.5) has a solution (see the Corollary to Theorem 2, Section 3).

The second example is the ordinary differential equation of second order

$$\frac{d^2x}{dt^2} = f(t,x),\quad f\in C^\infty(\mathbb{T}^2) \tag{1.6}$$

where f has period 1 in t, x and mean value zero. A special case is the equation for the nonlinear pendulum

$$\ddot{x} = g(t)\sin 2\pi x$$

where the "gravitational force" $g(t)$ is assumed to vary periodically in time with period 1. The angle of deviation from the vertical is denoted by $x \pmod 1$. We ask the question whether the velocity $\dot{x}$ of any solution of (1.6) is bounded for all t. Of course, if f is independent of t then $f = \partial_x Q$ where $Q = Q(x)$ has period 1 and the equation (1.6) has the energy integral

$$\frac{1}{2}\dot{x}^2 - Q(x) = E,\ E = \text{const},$$

from which it follows that $\dot{x}^2 \le 2(E+\max Q)$ is bounded. But if f depends on t it is conceivable that the pendulum gets pumped up so as to rotate faster and faster about its pivot. To some degree this is indeed the case, however, we will see that the velocity $|\dot{x}|$ is always bounded. The key to the proof is the construction of quasi-periodic solutions of the form

$$x(t) = U(t, \alpha t, \lambda) \tag{1.7}$$

where $U(t,\theta)-\theta \in C^\infty(\mathbb{T}^2)$, $\partial_\theta U > 0$, and α is an irrational number. Of course, the equation is the special case of (1.1) obtained by setting $n=1$ and replacing x, u by t, x.

We will show that solutions of the type (1.7) always exist for certain irrational

α which are sufficiently large. This illustrates the principle that for sufficiently large frequency ratios resonances become harmless and stability prevails. For the special case $f = p(t) - \sin x$ this problem has recently been studied by M. Levi [11].

The third example is a discrete version of (1.6): we consider a sequence of real numbers $x_n, n \in \mathbb{Z}$ satisfying the difference equations

$$x_{n+1} - 2x_n + x_{n-1} = f(x_n) \tag{1.8}$$

where $f \in C^\infty(S^1)$ has period 1 and mean value zero. This sequence is related to the area-preserving mapping of the torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\phi\colon (x, y) \to (x + y + f(x), y + f(x)). \tag{1.9}$$

We view ϕ as a discrete dynamical system and denote an orbit by $\phi^n(x, y) = (x_n, y_n)$. By eliminating y_n one sees that the x_n of an orbit satisfy (1.8) and, conversely, that any solution of (1.8) gives rise to an orbit (x_n, y_n) with $y_n = x_n - x_{n-1}$. One may be tempted to view (1.9) as an approximation to the Poincaré mapping

$$(x(0), \dot{x}(0)) \to (x(1), \dot{x}(1))$$

of the differential equation (1.6). For small f this may be justified but we will see that, generally, (1.6) and (1.8) have very different behavior. For example, in contrast to (1.6) there are sequances $\{x_n\}$ satisfying (1.8) for which the sequence $y_n = x_n - x_{n-1}$ is unbounded. This is the case even for

$$f = \lambda \sin 2\pi x \tag{1.10}$$

if λ is large enough. According to the work by Mather [4] it suffices to take $\lambda > 2/3\pi$. The mapping ϕ corresponding to the choice (1.10) is sometimes called the "standard mapping"; it has been studied extensively, using analytical as well as numerical approaches.

In this expository article we want to illustrate the results of a more general theory of minimal foliations with some typical examples. This theory (see [8]) refers to more general differential equations and, more importantly, contains also a study of weak (i.e. discontinuous) solutions.

In section 2 we describe the connection with mechanics which is standard except for the fact that we consider the singular limit $|\alpha| \to \infty$. The situation can be handled particularly easily with the present approach in configuration space; in the appendix we supply the connection with the theorem on invariant curves for a twist mapping. The main results are contained in Section 3. The discussion of the difference equation (1.8) in Section 4 is based on the work of Mather [4] and contains nothing new. Finally, in Section 5, we show the effect of symmetries on the existence of quasi-periodic solutions.

2. The stability of the Pendulum

Quasi-periodic solutions can be used to answer the question posed for the example 2, i.e. to show that for any solution $x = x(t)$ of (1.6) one has $\sup_t |\dot{x}(t)| < \infty$. More precisely, we will show that for any large positive number N there exists a number $M = M(N)$, such that any solution $x(t)$ of (1.6) with $|\dot{x}(0)| \le M$ satisfies $|\dot{x}(t)| \le N$ for all $t \in \mathbb{R}$. We will call, for short, the system *stable* in this case. In particular, the nonlinear pendulum is stable in this sense. To be sure, this stability statement does not refer to the equilibrium $x = 0$ but to a fictitious state with $\dot{x} = \pm\infty$.

To show this we write (1.6) as a system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= f(t, x) \end{aligned} \tag{2.1}$$

which we consider as a vector field on $\mathbf{T}^2 \times \mathbb{R}$. Let ϕ^s denote the corresponding flow taking $(t, x(t), y(t))$ into $(t + s, x(t + s), y(t + s))$. If $\Omega_\alpha = \mathbf{T}^2 \times [-\alpha, +\alpha]$ we have to show that

$$\bigcup_s \phi^s(\Omega_M) \subset \Omega_N \quad \text{for some} \quad N = N(M). \tag{2.2}$$

Assume now that we have a quasi-periodic solution of the form (1.7) with U satisfying

$$(\partial_t + \alpha\partial_\theta)^2 U = f(t, U) \tag{2.3}$$

and consider the mapping

$$(t, \theta) \to (t, x = U(t, \theta), y = (\partial_t + \alpha\partial_\theta)U)$$

as an embedding of a torus in $\mathbf{T}^2 \times \mathbb{R}$. We denote this torus by Λ_α. On account of (2.3) the vector field (2.1) is tangential to Λ_α, hence $\phi^s(\Lambda_\alpha) = \Lambda_\alpha$. Such an invariant torus can be used to obtain bounds for $|y|$. Indeed, if

$$m_1 = \min(\partial_t + \alpha\partial_\theta)U, \quad m_2 = \max(\partial_t + \alpha\partial_\theta)U$$

one has

$$\Lambda_\alpha \subset \mathbf{T}^2 \times [m_1, m_2],$$

and any solution $(t, x(t), y(t))$ of (2.1) with $y(t_0) \le m_1$ for some t_0 satisfies $y(t) \le m_2$ for all $t \in \mathbb{R}$. Indeed, otherwise the orbit would pass from one side of Λ_α to the other which is impossible since Λ_α is invariant under the flow.

Therefore, to establish the stability of (1.6) we have to prove the existence of such invariant tori Λ_α in $\mathbf{T}^2 \times [M, +\infty)$ and $\mathbf{T}^2 \times (-\infty, -M]$ for any $M > 0$. This corresponds to finding quasi-periodic solutions for arbitrarily large $|\alpha|$ and this is precisely what we will do next.

3. An existence Theorem

First we show that the condition (1.2) is necessary for the existence of a solution U of the problem (1.5). For this purpose we note that any periodic function $f \in C^\infty(\mathbf{T}^{n+1})$ can be written in the form

$$f(\bar{x}) = \partial_{x_{n+1}} Q(\bar{x}) + \Delta q(x) + \bar{f} \tag{3.1}$$

where $\bar{f}$ is the mean value of f and $Q \in C^\infty(\mathbf{T}^{n+1})$, $q \in C^\infty(\mathbf{T}^n)$ are both periodic functions of period 1 in its variables. For the proof we assume $\bar{f} = 0$; then the function

$$g(x) = \int_0^1 f(x, \theta) d\theta$$

has the mean value zero and therefore can be written in the form $g = \Delta q$ with $q \in C^\infty(\mathbf{T}^n)$. Integrating $f(x, x_{n+1}) - g(x)$ with respect to x_{n+1} we obtain a periodic function $Q(\bar{x})$, proving (3.1). Thus the differential equation (1.5) can be written in the form

$$\sum_{\nu=1}^n D_\nu^2 (U - q) - Q_u(x, U) - \bar{f} = 0,$$

$$D_\nu = \partial_{x_\nu} + \alpha_\nu \partial_{x_{n+1}}.$$

Multiplying this expression with $U_{x_{n+1}}$, which we abbreviate by U', we obtain

$$\sum_{\nu=1}^n D_\nu (U' D_\nu (U - q)) - \partial_{x_{n+1}} \left\{ \frac{1}{2} \sum_{\nu=1}^n (D_\nu (U - q))^2 + Q(x, U) \right\} - \bar{f} U' = 0.$$

Since $D_\nu U, \partial_{x_{n+1}} U = U' \in C^\infty(\mathbf{T}^{n+1})$ are periodic we obtain by the divergence theorem after integration over $\mathbf{T}^{n+1}$.

$$-\bar{f} \int_{\mathbf{T}^{n+1}} U' d\bar{x} = 0$$

By (1.5i) the integral is equal to 1 showing that the condition $\bar{f} = 0$ is indeed necessary for existence of quasi-periodic solution of the above type.

Moreover, the above decomposition shows that one can reduce our problem to the case

$$f = \partial_{x_{n+1}} Q(x, x_{n+1}), \quad Q \in C^\infty(\mathbf{T}^{n+1}) \tag{3.2}$$

by replacing U by $U + q$.

Before formulating the relevant theorem we consider the case of small nonlinearities f. If $f \equiv 0$ the solution of (1.5) is given by $U = x_{n+1}$ up to an additive constant. Therefore we will have to require that the linearized equation

$$\sum_{\nu=1}^n D_\nu^2 \phi = g, \quad D_\nu = \partial_{x_\nu} + \alpha_\nu \partial_{x_{n+1}}$$

is solvable for any smooth and periodic g of mean value $\bar{g} = 0$, i.e. for all

$$g \in C_0^\infty(\mathbf{T}^{n+1}) = \{\psi \in C^\infty(\mathbf{T}^{n+1}), \ \bar{\psi} = 0\}.$$

Moreover, we will assume that the solution $\phi \in C^\infty(\mathbf{T}^{n+1})$ is unique up to an additive constant, i.e. that the operator

$$L = \sum_{\nu=1}^n D_\nu^2 : C_0^\infty(\mathbf{T}^{n+1}) \to C_0^\infty(\mathbf{T}^{n+1}) \tag{3.3}$$

is bijective. It is remarkable that this condition on L is sufficient for the solvability of the perturbation problem of the nonlinear equation (1.5) (see Theorem 1, below). One has to keep in mind that L is not an elliptic differential operator on $C^\infty(\mathbf{T}^{n+1})$ since it involves only tangential derivatives on the foliation $x_{n+1} - \alpha \cdot x = \text{const}$. As a matter of fact, it is an elliptic operator only when restricted to these leaves.

Lemma. *The operator L given by* (3.3) *is bijective if and only if the $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfies a Diophantine condition*

$$\sum_{\nu=1}^n (\alpha_\nu j_{n+1} - j_\nu)^2 \geq c_0^{-1} (1 + j_{n+1}^2)^{-\tau} \tag{3.4}$$

for some positive constants c_0, τ and for all

$$\bar{j} = (j_1, j_2, \ldots, j_{n+1}) \in \mathbb{Z}^{n+1} \backslash (0).$$

Proof. The space $X = C_o^\infty(\mathbf{T}^{n+1})$ is a Frechet space whose topology can be described by the Sobolev norms

$$\|\phi\|_r = \left(\sum_{\bar{j} \in \mathbb{Z}^{n+1}} \left|\bar{j}\right|^{2r} \left|\hat{\phi}_{\bar{j}}\right|^2 \right)^{1/2}, \tag{3.5}$$

where

$$\phi(\bar{x}) = \sum_{\bar{j}} \hat{\phi}_{\bar{j}} e^{2\pi i \bar{j} \cdot \bar{x}}, \quad \hat{\phi}_0 = 0.$$

Since L is bijective we conclude from the open mapping theorem that L^{-1} is continuous; in particular, there exist positive constants τ, c_1 so that

$$\|\phi\|_0 \leq c_1 \|L\phi\|_{2\tau} = c_1 \sum_{\nu=1}^n \|D_\nu \phi\|_\tau^2 \tag{3.6}$$

which is equivalent to the condition

$$\left|\bar{j}\right|^{-2\tau} \leq c_1 4\pi^2 \sum_{\nu=1}^n (j_\nu - \alpha_\nu j_{n+1})^2. \tag{3.7}$$

These inequalities imply the seemingly stronger inequalities (3.4) with $c_0 = \max\{2, 4\pi^2 c_1 (2|\alpha|^2 + 1)^\tau\}$. Indeed, if $|j|^2 \geq 2|\alpha|^2 j_{n+1}^2 + 1$ then

$$\sum_{\nu=1}^n (j_\nu - \alpha_\nu j_{n+1})^2 \geq \frac{1}{2}|j|^2 - |\alpha|^2 j_{n+1}^2 \geq \frac{1}{2}$$

and (3.4) is trivially satisfied; and if $|j|^2 \le 2|\alpha|^2 j_{n+1}^2 + 1$ then

$$|\bar{j}|^2 = |j|^2 + j_{n+1}^2 \le (2|\alpha|^2+1)(j_{n+1}^2+1)$$

so that (3.7) implies (3.4) with $c_0 \ge 4\pi^2 c_1(2|\alpha|^2+1)^\tau$.

On the other hand, the condition (3.4) implies that the mapping $L: X \to X$ is a bijection as one sees readily using Fourier representation. This proves the lemma.

As we pointed out L is not an elliptic operator; but if it is bijective it behaves like a "subelliptic" operator with a certain derivative loss, in the sense of (3.6). The condition (3.4) also leads to an estimate for the quadratic form

$$(L\phi,\phi) = \sum_{\nu=1}^{n} \|D_\nu \phi\|_0^2 \ge c_0^{-1} 4\pi^2 \|\phi\|_{-\tau}^2 \tag{3.8}$$

where

$$(\phi,\psi) = \int_{\mathbf{T}^{n+1}} \phi(\bar{x})\psi(\bar{x})d\bar{x}$$

and $\|\phi\|_{-\tau}$ is the norm (3.5) for negative $r = -\tau$; in the elliptic case one would have such an estimate for $r = 1$.

It is surprising that the bijective character of L leads to a Diophantine condition. For $n = 1$ it means that α should not be a "Liouville number". This is an irrational number α for which there exists a sequence of distinct rationals p_s/q_s such that

$$|q_s\alpha - p_s| < q_s^{-\omega_s} \quad \text{where} \quad \omega_s \to \infty.$$

For $n > 1$ the condition (3.4) means that the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ should not admit *simultaneous* approximation by rationals with the same denominator, i.e. it should not be possible to find a sequence of rationals $p_{\nu s}/q_s$ such that

$$\max_\nu |q_s\alpha_\nu - p_{\nu s}| < q_s^{-\omega_s}, \quad \omega_s \to \infty$$

holds. In this case some α_ν may be rational and it suffices that just one component is not a Liouville number (but this condition is *not* necessary). Almost all vectors $\alpha \in \mathbb{R}$ satisfy (3.4) for some constant c_0 if $\tau > \frac{1}{n}$.

Now we consider the nonlinear operator

$$E(U) = LU - f(x,U) \tag{3.9}$$

and look for a solution of $E(U) = 0$. Here we consider $L = \sum_{\nu=1}^n D_\nu^2$ also defined for arbitrary C^∞-functions. The main result is that $E(U)$ admits such a solution if it has an approximate solution U^* for which $E(U^*)$ is sufficiently small and if L is a bijection on $C_0^\infty(\mathbf{T}^{n+1})$.

Theorem 1. *Let the operator L on $X = C_0^\infty(\mathbf{T}^{n+1})$ be a bijection and U^* a function satisfying* (1.5i) *and* (1.5ii) *and let $\varepsilon > 0$. Then there exists a neighborhood N of 0 in $C^\infty(\mathbf{T}^{n+1})$ such that, if $f \in C_0^\infty(\mathbf{T}^{n+1})$ has the property*

$$E(U^*) = LU^* - f(x,U^*) \in N \tag{3.10}$$

then there exists an exact solution U of $E(U) = 0$, satisfying

$$U - U^* \in C^\infty(\mathbf{T}^{n+1}), \ |U - U^*|_{C^1} < \varepsilon.$$

The smallness condition (3.10) depends on α, U^* and ε; it is to be considered a condition on f. We illustrate the result with the choice $U^* = x_{n+1}$ which is a solution of $LU^* = 0$. Hence (3.10) requires that

$$f \in N \cap C_0^\infty(\mathbf{T}^{n+1}),$$

and in this case the theorem guarantees the existence of a solution of the nonlinear equation. More generally, we conclude that the set of $f \in C_0^\infty(\mathbf{T}^{n+1})$ for which a solution of (1.5) exists is an open set.

For the following we need a sharper version of this theorem in which the smallness condition is independent of α. Assuming that c_0, τ are given positive numbers we consider the set (Diophantine condition)

$$DC(c_0,\tau) = \{\alpha \in \mathbb{R}^n \text{ satisfying } (3.4)\}.$$

For $\tau > \frac{1}{n}$ this set is nonempty if c_0 is large enough.

Theorem 2. *Let $c_0, \tau > 0$ be given and $\alpha \in DC(c_0,\tau)$. Then there exist integers $a = a(\tau,n)$, $b = b(\tau,n)$ with the following properties: Let ε, M be positive constants and U^* a function satisfying* (1.5i), (1.5ii) *and*

$$\partial_{x_{n+1}} U^* > M^{-1}, \ |U^* - x_{n+1}|_{C^a} < M.$$

Then there exists a positive $\delta = \delta(c_0,\tau,n,M,\varepsilon)$ such that the inequality

$$|E(U^*)|_{C^b} < \delta \tag{3.11}$$

implies the existence of a solution U of $E(U) = 0$ with $U - U^* \in C^\infty(\mathbf{T}^{n+1})$ and

$$|U - U^*|_{C^1} < \varepsilon.$$

This result is sharper than Theorem 1 in several respects: The quantity δ is independent of α. Moreover the smallness condition (3.11) depends only on a finite number of derivatives, where b is also independent of α. If we apply this theorem to $U^* = x_{n+1}$ again we can assure the existence of a solution if

$$|f|_{C^b} < \delta, \ f \in C_0^\infty(\mathbf{T}^{n+1}).$$

We apply this theorem for large frequency vectors α. For this purpose note that the condition (3.4) is invariant under the translations $\alpha \to \alpha + \mathbb{Z}^n$, i.e. if $\alpha \in DC(c_0,\tau)$ then also $\alpha + k \in DC(c_0,\tau)$ for all $k \in \mathbb{Z}^n$.

Corollary. *Let c_0, τ be given such that $DC(c_0,\tau) \neq \varnothing$, and let $f \in C_0^\infty(\mathbf{T}^{n+1})$. Then there exists a large constant $A = A(c_0, \tau, n)$ such that for*

$$\alpha \in DC(c_0,\tau),\ |\alpha| > A$$

the equation

$$LU = f(x,u)$$

possesses a solution $U = U(\bar{x},\alpha)$ satisfying (1.5) *and*

$$\text{(3.12)}\qquad \begin{aligned} &U - x_{n+1} = O(|\alpha|^{-2}) \\ &D_\nu U - \alpha_\nu = O(|\alpha|^{-1}),\ \partial_{x_{n+1}} U - 1 = O(|\alpha|^{-1}) \end{aligned}$$

The advantage of this corollary is that no smallness restriction is imposed on f; it assures the existence of quasi-periodic solutions for any equation (1.1) provided f satisfies (1.2). Of course, the smallness condition is hidden in the requirement $|\alpha| > A$. To show that this corollary follows from Theorem 2 we construct an approximate solution U^* of $E(U^*) \sim 0$ for large $|\alpha|$. Considering $\alpha_\nu \partial_{x_{n+1}}$ as the dominant term in D_ν we determine U^* as the solution of the ordinary differential equation

$$|\alpha|^2 \partial^2_{x_{n+1}} U^* = f(x, x_{n+1}) = \partial_{x_{n+1}} Q(\bar{x})$$

where we assume, without loss of generality, that f is in the reduced form (3.2). Let $Z(x,x_{n+1})$ be the periodic function satisfying $\partial^2_{x_{n+1}} Z = f(x, x_{x_{n+1}})$ then we set

$$U^* = x_{n+1} + |\alpha|^{-2} Z.$$

Then one finds

$$LU^* = \partial^2_{x_{n+1}} Z + O(|\alpha|^{-1}) = f(x,x_{n+1}) + O(|\alpha|^{-1})$$

hence

$$E(U^*) = f(x,x_{n+1}) - f(x, x_{n+1} + |\alpha|^{-2} Z) + O(|\alpha|^{-1}) = O(|\alpha|^{-1}).$$

also

$$\partial_{x_{n+1}} U^* = 1 + |\alpha|^{-2} \partial_{x_{n+1}} Z > \frac{1}{2}$$

$$|U^* - x_{n+1}|_{C^a} = O(|\alpha|^{-2})$$

and by making $|\alpha|$ large we can achieve $|E(U^*)|_{C^b} < \delta$ proving that the corollary follows from Theorem 2.

Note that for $n = 1$ this corollary provides the proof of the existence of a quasi-periodic solution $U = U(t,x)$ with $(\partial_t + \alpha\partial_x)U = \alpha + O(|\alpha|^{-1})$ and therefore the existence on invariant tori Λ_α for which $y = \alpha + O(|\alpha|^{-1})$ is large. This is what was required for the stability proof for $\ddot{x} = f(t,x)$, which is therefore completed.

The idea to obtain bounds for the solution near infinity, here near $y = \pm\infty$, by

constructing invariant tori near infinity can be used in many situations. Such an approach was employed by Dieckerhoff and Zehnder [2] in a more subtle situation to prove the boundedness of the solutions of a nonlinear Duffing equation, and for the present situation by M. Levi [11].

The proof of these theorems follows a standard pattern of rapidly convergent iterations and is rather technical. Therefore if can not be described here but the details can be found in [9]. Here we will just explain why it suffices to impose conditions of invertibility on L, the linearized operator for $f = 0$, and not on the linearized operator

$$E'(U)V = LV - f_u(x,U)V.$$

Here one can assume that $E(U)$ is small, and we will simply take U to be a solution of $E(U) = 0$. The answer is related to the observation that our differential equation is invariant under the translation $T_\lambda : U(\bar{x}) \to U(\bar{x} + \lambda e_{n+1})$, $\lambda \in \mathbf{R}$. Therefore if $E(U) = 0$ then also $E(T_\lambda U) = 0$, hence

$$0 = \frac{d}{d\lambda} E(T_\lambda U)\Big|_{\lambda=0} = LU' - f_u(x,U)U' = 0$$

where $U' = \partial_{x_{n+1}} U > 0$. Therefore eliminating f_u we obtain, with $W = (U')^{-1}V$

$$E'(U)V = LV - (LU')W = L(U'W) - (LU')W.$$

Multiplying both sides by U' it is easy to rewrite this differential operator in the form

$$U'(E'(U)U'W) = \sum_{\nu=1}^{n} D_\nu(U'^2 D_\nu W) = \tilde{L}W$$

where $\tilde{L}$ is defined by this equation. If $U' > M^{-1}$ one obtains for this operator $\tilde{L}$ similar L^2-estimates as for L: indeed,

$$-(\tilde{L}W, W) \geq M^{-2} \sum_{\nu=1}^{n} \|D_\nu W\|_0^2 = M^{-2}(LW,W) \geq M^{-2} c_0^{-1} 4\pi^2 \|W\|_{-\tau}$$

from (3.8). By this simple trick, which was noted by S. M. Kozlov, one reduces the estimates for $E'(U)$ to those of L. Similarly, one obtains estimates for higher derivatives as they are required for the convergence proof.

These results depend strongly on the translation invariance of E: $E \circ T_\lambda = T_\lambda \circ E$, and therefore one can not expect to prove this way that the mapping $U \to E(U)$ is locally invertible, as it is proposed in Hamilton's work [3]. Aside from this technical point his results can also be adapted to this situation.

Finally, we want to point out that (3.12) can be considered as the first terms of an asymptotic expansion for U in negative powers of α, which can easily be computed formally. However, such expansions diverge in general and it would be desirable to adapt this method to establish the asymptotic character ot these expansions.

4. A difference equation

We turn to a brief discussion of the third example (1.8). If α is not a Liouville number and $f \in C_0^\infty(\mathbf{T}^1)$ is small enough one can show that there exist quasi-periodic sequences of the form

$$x_n = u(n\alpha + \theta_0)$$

where $u(\theta) - \theta \in C^\infty(\mathbf{T}^1)$ and $\partial_\theta u > 0$ satisfying the equation (1.8). For this the function u has to satisfy the difference equation

$$u(\theta + \alpha) - 2u(\theta) + u(\theta - \alpha) = f(u(\theta)) \tag{4.1}$$

which is the analogue of (1.5iii). Any solution of (4.1) gives rise to an invariant curve of the form

$$x = u(\theta), \; y = u(\theta) - u(\theta - \alpha) \tag{4.2}$$

of the mapping ϕ given by (1.9). This statement about the existence of such u for small f is the consequence of the KAM theory (see [6]) but could also be proven in the same way as Theorems 1 or 2. Here we are more concerned with the *non existence* of such solutions of (4.1).

Le us indicate the dependence of u on α and write $u = u(\theta, \alpha)$. Because of the condition $u(\theta + j, \alpha) = u(\theta, \alpha) + j$ it is evident that if $u(\theta, \alpha)$ solves (4.1) for α then also for $\alpha + j$ and we can take $u(\theta, \alpha + j) = u(\theta, \alpha)$. Hence with (4.2) one has also infinitely many invariant curves

$$x = u(\theta, \alpha), \; y = u(\theta, \alpha) - u(\theta - j - \alpha, \alpha) \sim j$$

for large j. Hence we obtain closed invariant curves for arbitrarily large y if we have only one such invariant curve. Conversely, if we have no invariant curves for $0 \le \alpha < 1$ then we have no invariant curve at all. In this respect, the difference equation differs radically from the corresponding differential equation.

From this it follows that, if f is small enough, then the mapping is stable, and $\sup |y_n| < \infty$ for all orbits. In particular, the standard mapping is stable for small $|\lambda|$.

On the other hand from the work of Mather [4] it follows that for f given by (1.10) and for $\lambda > \frac{2}{3\pi}$ there are no such invariant curves, and, moreover, that the mapping is not stable. This means, that if $\Omega_M = \{(x, y) \in \mathbf{R}^2, |y| \le M\}$ is a given strip there exists no strip Ω_N containing $\cup_{n \ge 0} \phi^n \Omega_M$, where ϕ is given by (1.9) with $f = \lambda \sin 2\pi x$.

Indeed, otherwise the set $\cup_{n \ge 0} \phi^n \Omega_M \subset \Omega_N$ for some N would be a set invariant under ϕ and the translation $(x, y) \to (x+1, y)$. By a theorem of Birkhoff for such twist mappings the boundary of such invariant sets contains invariant curves given as the graph $y = w(x) = w(x + 1)$ of a Lipschitz function w. Mather showed [4] that for $\lambda > \frac{2}{3\pi}$ such functions do not exist. To be sure, these invariant curves need not be smooth and are not of the nature described above. But they are related to non-smooth, even discontinuous monotone solutions of (4.1) which always exist. In any event it follows that a) for such f no smooth solutions of (4.1) exist even if α is not a Liouville number, and b) that the mapping ϕ is not stable in contrast to the stable behaviour of the differential equation (1.6). The reason for this discrepancy is that for large α the solution of (1.6) can be approximated by those of $\ddot{x} \sim 0$ while the orbits of ϕ belonging to α have the same behavior as those for $\alpha - j, j \in \mathbf{Z}$, e.g. $\alpha - [\alpha] \in [0, 1]$, $[\alpha]$ being the integral part.

5. Symmetries

The question whether or not the above equations have quasi-periodic solutions is evidently quite subtle and depends on Diophantine conditions. Although the above corollary guarantees the existence of quasi-periodic solutions of (1.1) for sufficiently large α satisfying (3.4) one can construct $f \in C_0^\infty(\mathbf{T}^2)$ such that (1.6) has no quasi-periodic solutions for any α in a given compact set (see [7]). Bangert [1] gave interesting examples of partial differential equations of this nature, however these equations have a more general form than (1.1). Therefore for a given α satisfying (3.4) the set of $f \in C_0^\infty(\mathbf{T}^{n+1})$ admitting a solution of (1.5) is not empty and open but certainly not all of $C_0^\infty(\mathbf{T}^{n+1})$. One can expect that the Diophantine conditon (3.4) is necessary for the openness of the set; for the analogue monotone twist mappings such results were proven recently by Mather [5] but, again his systems are more general than the ones considered here.

Thus the existence of quasi-periodic solutions *for all* α can occur only for special functions f, and we want to describe such situations now. Clearly, if f does not depend on u, i.e. the system in invariant under $u \to u + \lambda$, then (1.1) possesses quasi-periodic solutions for all α. This statement is trivial, since then the differential equation (1.1) is linear and there exists a periodic solution $q(x) \in C^\infty(\mathbf{T}^n)$ of $\Delta q = f(x)$ and

$$U(\overline{x}) = x_{n+1} + q(x)$$

is a solution of (1.5). In fact, in this case all solutions with linear growth have the form

$$u(x) = U(x, \alpha \cdot x + \lambda) = \alpha \cdot x + \lambda + q(x)$$

as one sees from Liouville's Theorem.

We consider another symmetric situation, namely the case where f is independent of x_1. In this case the system is invariant under the translation $\overline{x} \to \overline{x} + \lambda e_1$ and in this case there exist quasi-periodic solutions for all $\alpha \in \mathbb{R}^n$ with $\alpha_1 \neq 0$, regardless of Diophantine conditions.

For $n = 1$ this statement is evident since in this case $\ddot{x} = f(x)$ is autonomous (t corresponds to x_1) and possesses an energy integral. All solutions of this equation with $\alpha \neq 0$ are periodic with period α^{-1}. This argument fails, of course, for partial differential equations, but using the theory on minimal foliations (see [10]) one can argue as follows: Without loss of generality we assume that f is of the form (3.2) so that (1.1) is the Euler equation of the variational problem

$$F(u) = \int \left\{ \frac{1}{2} |u_x|^2 + Q(x, u) \right\} dx.$$

According to that theory there exists for any $\alpha \in \mathbb{R}^n$ a "minimal without self intersection" u of this variational problem for which $\sup |u - \alpha \cdot x| < \infty$. Here "minimals" are functions in C^1_{loc} for which

$$\int_{\mathbb{R}^n} \left\{ \frac{1}{2} |u_x + \phi_x|^2 + Q(x, u + \phi) - \frac{1}{2} |u_x|^2 - Q(x, u) \right\} dx$$

holds for all $\phi \in C^1_{\text{comp}}(\mathbb{R}^n)$, thus minimals form a special class solutions of $\Delta u = Q_u(x, u)$. One says u has no self intersections on T^{n+1} if $u(x+j) - j_{n+1} - u(x)$ does not change sign for any choice of $(j, j_{n+1}) \in \mathbb{Z}^{n+1}$. These minimals without self intersections play a distinguished role here, since every quasi-periodic solution of the form (1.3) with $\alpha \notin \mathbb{Q}^n$ has this property.

In the special case $Q_{x_1} \equiv 0$, $\alpha_1 \neq 0$ one can show that for a minimal u without self intersection with $\sup |u - \alpha \cdot x| < \infty$ one has

$$u(x + \alpha_1^{-1} e_1) = u(x) + 1, \quad u_{x_1} > 0$$

and, moreover, that for all α with $\alpha_1 \neq 0$

$$U(x, x_{n+1}) = u\left(x + \frac{x_{n+1} - \alpha \cdot x}{\alpha_1} e_1 \right)$$

is a solution of (1.5).

The proof, which can not be given here, follows rather simply from the theory referred to above. We mention this result since it illustrates that the destruction

of quasi-periodic solutions is related to the destruction of symmetries.

Appendix

Here we give an alternate proof for the stability of the equation (1.6) by using the existence theorem for invariant curves for area-preserving twist mappings [6]. It is based on the idea that for large velocities $\dot{x} = y$ the system with the Hamiltonian

$$\frac{1}{2} y^2 - Q(t, x)$$

has solutions which are close to those of the integrable system with the Hamiltonian $\frac{1}{2} y^2$. For a differential equation

$$\text{(A.1)} \qquad \ddot{x} = Q_x(t, x); \; Q \in C^\infty(\mathrm{T}^2)$$

one usually considers the Poincaré map of the section $t = 0 \pmod 1$ in the three dimensional phase space $\mathrm{T}^2 \times \mathbb{R}$ with the coordinates $t, x, \dot{x}$. This was indeed done in the paper by Levi [11]. But since we are interested in orbits with large $\dot{x} \sim \alpha$ we choose to use the section $x = 0 \pmod 1$. As the remaining variables we use the rescaled energy and the time:

$$\text{(A.2)} \qquad r = \gamma \left(\frac{1}{2} \dot{x}^2 - Q(t, x) \right), \qquad \theta = t$$

where $\gamma > 0$ is a small parameter to be taken of the order α^{-1}; here α will be the rotation number

$$\lim_{t \to \infty} x(t)/t$$

of the solution to be determined. We restrict ourselves to the fixed annulus

$$\frac{1}{3} \leq r \leq 3, \quad \theta \pmod 1.$$

The first variable r is proportional to the Hamiltonian of (A.1) and $t = \theta$ is its conjugate. Therefore, the equation can be written in canonical form

$$\frac{d\theta}{dx} = K_r(x, \theta, r; \gamma);$$
$$\frac{dr}{dx} = -K_\theta(x, \theta, r; \gamma)$$

where

$$K(x, \theta, r; \gamma) = \gamma\sqrt{2}(r + \gamma^2 Q(\theta, x))^{1/2} = \gamma\sqrt{2r} + O(\gamma^3).$$

This can be verified by a direct calculation.

For small γ we have

$$\theta' = \gamma(2r)^{-1/2} + O(\gamma^3), \; r' = O(\gamma^3)$$

and integration from $x = 0$ to $x = 1$ gives

$$\text{(A.3)} \qquad \theta_1 = \theta_0 + \gamma(2r_0)^{-1/2} + O(\gamma^3); \; r_1 = r_0 + O(\gamma^3)$$

where $\theta_i = \theta(i)$, $r_i = r(i)$ for $i = 0, 1, \ldots$. Moreover, since K is a single valued function the mapping $(\theta_0, r_0) \to (\theta_1, r_1)$ of the section $x = 0 \pmod 1$ is exact symplectic, i.e. preserves

$$\int_0^1 r(\theta)d\theta$$

for any closed curve $r = r(\theta) = r(\theta + 1)$. In particular, it has the intersection property of [6]. We will apply the theorem proved there to find invariant curves of the mapping (A.3) for small values of γ, and with rotation number

$$\omega = \lim_{n\to\infty} \frac{\theta_n}{n} = \alpha^{-1}$$

Note that the change of section leads to a change of rotation number: Since

$$\lim_{t\to\infty} \frac{x(t)}{t} = \alpha$$

it follows that for $t = \theta_n$, $x(\theta_n) = n$

$$\frac{\theta_n}{n} \to \alpha^{-1}.$$

We assume that α is large of order $\gamma^{-1} > 1$ and satisfies (3.4) which reads in this case

(A.4) $$|\alpha p - q| \geq c^{-1} p^{-\tau}; \quad \alpha \in (\gamma^{-1}, 2\gamma^{-1})$$

for all rationals q/p. This condition implies for $\omega = \alpha^{-1}$ and all rationals $p/q, q > 0$

(A.5) $$|\omega q - p| \geq c_2^{-1} |\omega| q^{-\tau} \geq \frac{1}{2c_2} \gamma q^{-\tau}, \quad \omega \in (\tfrac{1}{2}\gamma, \gamma).$$

Indeed, if $|p\alpha| \geq |q| + 1$ one has $|\alpha p - q| \geq 1$ and for $|p\alpha| \leq |q| + 1 \leq 2q$ (A.4) implies $|\alpha p - q| \geq c_2^{-1} q^{-\tau}$ for all $q \geq 1$, if $c_2 \geq c(\frac{2}{|\alpha|})^\tau$. Hence, in both cases (A.4) implies

$$|\alpha p - q| \geq c_2^{-1} q^{-\tau};$$

dividing by $|\alpha|$ we get (A.5).

The condition (A.5) is precisely the small divisor condition required in [6]. Moreover, the interval

$$\{\gamma(2r)^{-1/2}, \ \frac{1}{3} \leq r \leq 3\}$$

over which the twist ranges contains the interval $[\frac{1}{2}\gamma, \gamma]$ in which ω lies. Therefore by the theorem of [6] there exists an invariant curve $r = \phi(\theta) = \phi(\theta + 1)$ in this annulus, provided $\gamma > 0$ is small enough.

The orbits starting on this invariant curve are given by quasi-periodic solutions $\theta = \theta(x)$ with frequencies 1 and ω or by quasi-periodic solutions $x = x(t)$ with

frequencies 1 and $\alpha = \omega^{-1}$. This concludes the alternate proof.

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