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Quantum groups

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Abstract. An elementary introduction to the notions of the quantum Lie Groups and quantum Lie algebras is given. The approach is based on the fundamental commutation relations which appeared first in the quantum inverse scattering method.

1. Introduction

A new mathematical object, called *Quantum Group* appeared as an abstraction of the development of the theory of quantum integrable dynamical systems. In this paper I shall first present the formal motivations and definitions of this notion and after this describe its connections with integrable models. In the end I shall comment on possible new applications of quantum groups.

2. Main Results

The theory of classical Lie Groups is intimately connected with the theory of matrix algebras. Given a family of matrices $T = \|T_{ik}\|$, $i, k = 1, \dots, n$, we have the operation of multiplication

$$(T'T'')_{ik} = \sum_j T'_{ij} T''_{jk} \tag{1}$$

and the action in the linear space of vectors $X = (x_i)$

$$(Tx)_i = \sum_k T_{ik} x_k.$$

The idea of quantization (or in more mathematical terms deformation) consists in taking the matrix elements T_{ik} and vector components x_k as q-numbers (elements of an associative algebra) in such a way that these formulas still make sense.

In more details, let us impose on the generators T_{ik} some commutative relations such that if T'_{ik} and T''_{ik} are two commuting exemplars of these generators, then $(T'T'')_{ik}$ as defined by (1) also satisfy them. Furthermore, the relations are to be imposed on x_k such that $(Tx)_k$ also satisfy them, if T_{ik} and x_k commute.

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We shall deal mainly with the first requirement, which is basic for the definition of the quantum (matrix) group itself. We shall write the relevant relations with the hope, that their naturalness will be evident. No claim will be made that they are the only possible relations suitable to our goal.

The relations look most elegant if one uses matrix notation. Let V denote the vector space in which the classical matrices T act (say, $V = \mathbb{C}^n$). Given a formal matrix T of generators T_{ik} we form two matrices

$$T_1 = T \otimes I, \qquad T_2 = I \otimes T,$$

associated with $V \otimes V$. With these notations the relations look as follows

$$RT_1T_2 = T_2T_1R, \tag{2}$$

where R is a $\mathbb{C}$-number matrix acting in $V \otimes V$.

The main requirement can be easily checked:

$$\begin{aligned} RT_1'T_1''T_2'T_2'' &= RT_1'T_2'T_1''T_2'' \quad \text{(using the commutativity of } T' \text{ and } T'')\\ &= T_2'T_1'T_2''T_1''R \quad \text{(using the main relation)}\\ &= T_2'T_2''T_1'T_1''R \quad \text{(using the commutativity once more)} \end{aligned}$$

The main relations, presented more explicitly in terms of the matrix elements T_{ik}, look as follows

$$\sum_{p,q} R_{ik|pq}T_{pm}T_{qn} = \sum_{p,q} T_{kp}T_{iq}R_{qp|mn}$$

where the structure constants $R_{ik|mn}$ are the matrix elements of the matrix R written in a natural basis in $V \otimes V$.

The main relations, being quadratic in generators, could lead to the new ones of higher order. Indeed, consider the product $T_1T_2T_3$ in $V \otimes V \otimes V$ and compare it with $T_3T_2T_1$ using the main relations. There are two ways to make the necessary commutations in correspondence with the following scheme

$$\begin{array}{ccccc} & \nearrow (213) \longrightarrow (231) \searrow & \\ (123) & & (321) \\ & \searrow (132) \longrightarrow (312) \nearrow & \end{array}$$

More explicitly we get the relations

$$R_{23}R_{13}R_{12}T_1T_2T_3 = T_3T_2T_1R_{23}R_{13}R_{12}$$

and

$$R_{12}R_{13}R_{23}T_1T_2T_3 = T_3T_2T_1R_{12}R_{13}R_{23}$$

with the evident notations: R_{12} is a matrix in $V \otimes V \otimes V$ acting as R in the first two spaces and as a unit matrix in the third one, etc.

The relations obtained could be new independent cubic relations on the generators T_{ik}. However, if we require that the structure constants $R_{ik|mn}$ satisfy the property, written in short notation in the form

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{3}$$

then no new relations appear. Moreover, the general theory of categorical considerations show that the higher commutations also do not produce more relations. So in what follows we shall require that (3) be satisfied, thus supplementing the definition on the quantum matrix algebra.

The relation (3) appears in several physical situations. If we consider the matrix R as a two-body scattering matrix, then relation (3) means that the scattering in the three-body system is multiplicative and does not depend on the time ordering of the two-body collisions. In the vertex models of classical statistical physics $R_{ik|mn}$ can be considered as the Boltzman weights on the vertices. The relation (3) then allows the exact solution of the corresponding model.

However relation (3) also has a well known mathematical meaning. Introduce a matrix $\hat{R}$ as follows

$$\hat{R} = PR,$$

where P is a permutation matrix in $V \otimes V$

$$P(a \otimes b) = b \otimes a,$$

or in terms of the matrix elements

$$\hat{R}_{ik|mn} = R_{ki|mn}.$$

Then relation (3) will take the form

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23},$$

recognizable as the main relation in the theory of braids. The relation between braids and the deformation of classical groups is one of the unexpected aspects of this development.

In our formulation of the quantum inverse scattering method, based on the quantum version of the Lax equation

$$\phi_{n+1} = Q^n\phi_n \tag{4}$$

we impose the commutation relations on the matrix Q^n of its coefficients

$$RQ_1^nQ_2^n = Q_2^nQ_1^nR$$

$$Q_1^nQ_2^m = Q_2^mQ_1^n, \quad n \neq m,$$

with a suitable matrix R. Relation (3) holds for these R. The fundamental commutation relation (4) leads to the same relation for the monodromy matrix

$$T = Q^N \dots Q^1$$

of the equation (4). Thus in the concrete examples of the quantum integrable models we have already used the explicit realizations of the quantum matrix algebras.

In our investigations, revealing the universality of relations (2) and (3), we have called the relation (3) the Yang-Baxter equation, having in mind the most prominent users of it in scattering theory and statistical physics, correspondingly (besides of Mc-Guire, Berezin, Brezin and Zinn-Justin, Zamolodchikov, or Lieb, Gaudin and others). On the other hand, the fundamental commutation relation (2) appeared first in the quantum inverse scattering method and subsequently has led to the invention of the notion of quantum group.

In all physical applications mentioned above, the matrices T and R depended on a complex parameter λ – rapidity in scattering, combination of the thermodinamical parameters in statistical physics, spectral parameter in the inverse scattering method. The Yang-Baxter relation in this case looks as follows

$$R_{12}(\lambda-\mu)R_{13}(\lambda-\sigma)R_{23}(\mu-\sigma)=R_{23}(\mu-\sigma)R_{13}(\lambda-\sigma)R_{12}(\lambda-\mu),$$

or, after a trivial change of variables

$$R_{12}(x)R_{13}(x+y)R_{23}(y)=R_{23}(y)R_{13}(x+y)R_{12}(y).$$

The work on concrete integrable models produces an impressive list of examples of R-matrices. In particular, a series of R-matrices, corresponding to any classical Lie algebra was constructed. The parameters in this list are the Lie algebra, its representation and 0, 1 or 2 the so called anisotropy parameters. Depending on the anisotropy, the variable λ runs through the complex plane $\mathbb{C}$, the cylinder $\mathbf{R}\times\mathbf{S}^1$ or the torus $\mathbf{T}^2$. For the theory of the quantum groups the most interesting case happened to be that of the cylinder. In this case the known R-matrices admit a nontrivial limit when $x\to\pm\infty$. The limits

$$R_\pm=\lim_{x\to\pm\infty}R(x)$$

evidently satisfy the relation (3). Taken as the structure constants for the main relation (2) they produce the examples of quantum deformations of the classical Lie groups. The anisotropy parameter remains in R and define the parameter of deformation.

It is now time to give an illustrative example. For the spin 1/2-representation of the $\mathrm{SL}(2)$ group, the known matrix R looks as follows

(5)
$$R=\begin{pmatrix} q & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q-\frac{1}{q} & 1 & 0\\ 0 & 0 & 0 & q\end{pmatrix}$$

where q is a complex number.

If we use the traditional notation for the matrix elements of a 2×2 matrix

$$T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

the relations (2) will be realized as follows (only 6 out of 16 are independent)

(6)
$$\begin{aligned} ab&=qba\\ ac&=qca\\ db&=\frac{1}{q}bd\\ dc&=\frac{1}{q}cd\\ bc&=cb\\ ad-da&=(q-\frac{1}{q})bc.\end{aligned}$$

It is easy to check that the q-determinant

$$\det_q T=ad-qbc$$

commutes with everything. Imposing the condition

(7)
$$\det_q T=1$$

we obtain the quantum group $\mathrm{SL}_q(2)$. In particular we have the definition of T^{-1}, namely

$$T^{-1}=\begin{pmatrix} d & -\frac{1}{q}b\\ -qc & a\end{pmatrix}.$$

In this case the corresponding quantum linear space is easily introduced. Indeed, the relation

$$x_1x_2=qx_2x_1$$

is conserved after the action of the matrix T with the relations (6) on a vector X with components x_1,x_2 (of course a,b,c,d and x_1,x_2 commute).

Another example is given by the triangular 2×2 matrices. Writing them in the form

$$L^+=\begin{pmatrix} q^{H_+/2} & \left(q-\frac{1}{q}\right)X_+\\ 0 & q^{-H_+/2}\end{pmatrix}$$

or

$$L^-=\begin{pmatrix} q^{-H_-/2} & 0\\ \left(\frac{1}{q}-q\right)X_- & q^{H_-/2}\end{pmatrix},$$

and requiring relation (2) to hold for the matrix (5), we come to the relations

(8)
$$[H_+,X_+]=2X_+$$

and

(9) $$[H_-, X_-] = -2X_-.$$

Less trivial is the fact, that we can impose the condition

(10) $$RL_1^+ L_2^- = L_2^- L_1^+ R$$

after identifying

(11) $$H_+ = H_- = H.$$

The new relation implies

(12) $$[X_+, X_-] = \frac{1}{q - \frac{1}{q}} \left(e^H - e^{-H}\right).$$

The relations (8, 9, 11, 12), evidently give a deformation of the Lie algebra of the group SL(2).

The generators a, b, c, d with relations (6) and (7) and the generators X_+, X_-, H with relations (8, 9, 11, 12), generate the associative algebras $\mathrm{Fun}_q(G)$ and $U_q(g)$ which can be considered as the q-deformation of the algebra of functions $\mathrm{Fun}(G)$ on the group $G = \mathrm{SL}(2)$ and the universal enveloping algebra $U(g)$ of the Lie algebra $g = \mathrm{sl}(2)$. Both these algebras admit a Hopf algebra structure with a formula of type (1) giving the corresponding comultiplication. Moreover the natural duality of $\mathrm{Fun}(G)$ and $U(g)$ is conserved after the deformation and in terms of the generators assumes the form

$$\langle L^{\pm}, T \rangle = R^{(\pm)}$$

where

$$R^+ = PRP \qquad R^{(-)} = R^{-1}.$$

The example presented above can be generalized to all classical groups by means of the relevant R-matrix and Borel structure. The formulas (2), (10), (13) are universal.

The q-deformation lifts some degeneracy of the algebras $\mathrm{Fun}(G)$ and $U(g)$. Indeed while $\mathrm{Fun}(G)$ is commutative and $U(g)$ cocommutative, the algebras $\mathrm{Fun}_q(G)$ and $U_q(g)$ are both noncommutative and noncocommutative. Moreover, the algebras $\mathrm{Fun}(G)$ and $U(g)$ become more similar after quantization. With a natural definition of the quantum homogeneous space one can say that

$$U_q(g) = \mathrm{Fun}_q(G^+ \times G^- / H)$$

where G^+ and G^- are Borel subgroups and H Cartan subgroup.

Let us give several historical comments and a short guide to the existing literature. The relation (2) as the main formal ingredient of the quantum inverse scattering method as devised in Leningrad was introduced in [1], [2]. A survey of the method and its connection with the classical method as presented in [3] can be found in [4]. The relations (8, 9) and (12) first appeared first in [5] (see also [6]) and subsequently were the origin of the general definition of the algebra $U_q(g)$ in [7] and [8]. The term "quantum group", and its Hopf algebra interpretation were introduced in [9]. The list of R-matrices, relevant for the quantization of the classical Lie groups, can be obtained by taking the limit $x \to \pm\infty$ of the trigonometrical R-matrices found in [10], [11].

The approach taken in this note and amplifying the universal role of the relations (2) and (3) was given in [12] and will be systematically presented in [13]. A more general, though maybe less developed, view of quantum groups was recently given in [14].

The theory of quantum groups still has not shown its full power. Its relevance to the theory of knots and links is already clear (see [15] and references therein). Recently, the promising applications to conformal field theory were indicated in [16], [17]. Its usefulness in combinatorics is illustrated by [18], [19]. However we consider all this to be only a beginning.

The deformations of algebraic structures played a most important role in the development of our understanding of the structure of matter in our century. Indeed the passage from Galilei relativity to that of Lorentz as well as the transition from classical to quantum mechanics are nothing but deformations. Moreover these are deformations of unstable structures into stable ones. Both are associated with the dimension parameters C and h. With these historical examples in mind one cannot help speculating that the new deformation, found in the (almost) pure mathematical development will find its main application in the future theory of matter.

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