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# Polyhedrons and pi-stable homotopies from 3-manifolds into the plane

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**Abstract.** For a stable map from a closed 3-manifold into the plane, its Stein Factorization gives useful information. Our main result presents the list of changes in the Stein Factorization of the maps in a generic family of homotopies (the Pi-stable ones).

## 0. Introduction

Ree's Theorem, **[11]**, gives information on the topology of the source manifold M of a stable smooth map $f: M \to \mathbb{R}$ which has only the simplest singularities: maxima and minima; in this case $M$ is a sphere. Morse theory (see **[11]**) also gives results in this direction.

For 3-manifolds, Bürlet and the De Rham [**2**] have studied *special generic* maps into the plane which have the simplest singularities: the definite folds, and in this case they also provide important information on the topology of the source manifolds. In the first section below, we shall present an equivalent form of the Poincaré Conjecture in dimension three; the equivalence is based on [**2**].

The results mentioned above raise the problem of simplification of singularities. The concepts of Stein Factorization and Pi-stable homotopy are studied here as an approach to achieve this simplification. The concept of Pi-stable homotopy was presented in [**3**] and will be discussed in section 2. We now introduce the concept of Stein Factorization.

**Definition.** Let $f: N \to P$ be a map from a topological space $N$ into a set $P$. Consider the equivalence relation on $N$ which identifies any two points of $N$ that belong to the same connected component of a fiber $f^{-1}(p)$ of $f$, $p \in P$. Denote by $W\langle f\rangle$ que quotient space, and let

$$q\langle f\rangle: N \to W\langle f\rangle$$

denote the quotient map. Finally, let $f': W\langle f\rangle \to P$ be the map that makes the

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following a commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\quad f \quad} & P \\ & {\scriptstyle q\langle f\rangle} \searrow \quad \nearrow {\scriptstyle f'} & \\ & W\langle f\rangle & \end{array}$$

This commutative diagram is called the *Stein Factorization* of $f$, but we shall call the quotient space $W\langle f\rangle$ with the same name.

It is known, see [5], [6], [8], [9] or [10], that for a stable map $f$ from a 3-manifold into the plane, its Stein Factorization $W\langle f\rangle$ is a 2-dimensional simplicial complex (a polyhedron), and $f': W\langle f\rangle \to \mathbf{R}^2$ is a stratified immersion.

The main result of this paper concerns the Stein Factorization of Pi-stable homotopies, and can be stated as follows:

**Theorem.** *Let $M^3$ be a closed, orientable 3-manifold and let $F: M^3 \times I \to \mathbf{R}^2$ be a Pi-stable homotopy, $b \in I$ a point of bifurcation, $p \in \mathbf{R}^2$ the critical value of bifurcation at $b$ and $S(F_b)$ the singular set of the map $F_b = F|M^3 \times \{b\}$. Then for $\varepsilon > 0$ small enough:*

(a) *If $S(F_b) \cap F_b^{-1}(p)$ is not contained in a single connected component of $F_b^{-1}(p)$ then $W\langle F_{b-\varepsilon}\rangle$ and $W\langle F_{b+\varepsilon}\rangle$ are homeomorphic.*

(b) *If $S(F_b) \cap F_b^{-1}(p)$ is contained in a single connected component of $F_b^{-1}(p)$ then $W\langle F_{b-\varepsilon}\rangle$ and $W\langle F_{b+\varepsilon}\rangle$ can be gotten one from the other by cutting and pasting. The piece that has to be cut is paired below with the one to be pasted. The twenty two figures (pairs) shown below present all the possible transitions if the two orders are considered for each pair.*

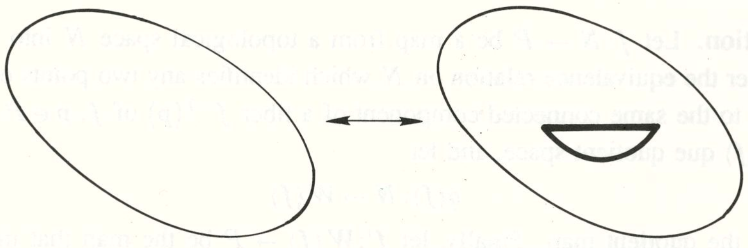


**Fig. 1**



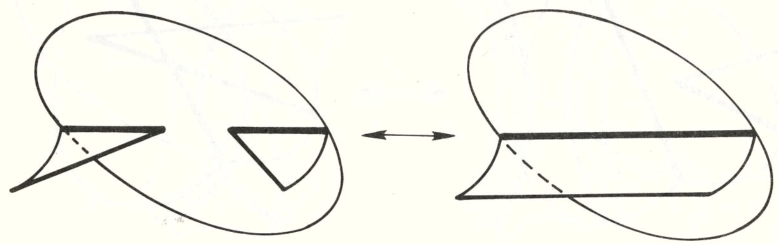


Fig. 2

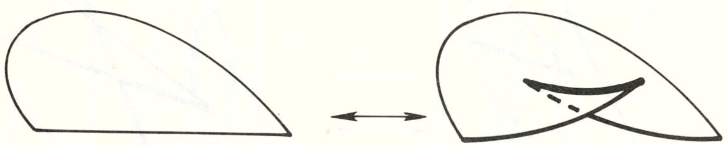


Fig. 3

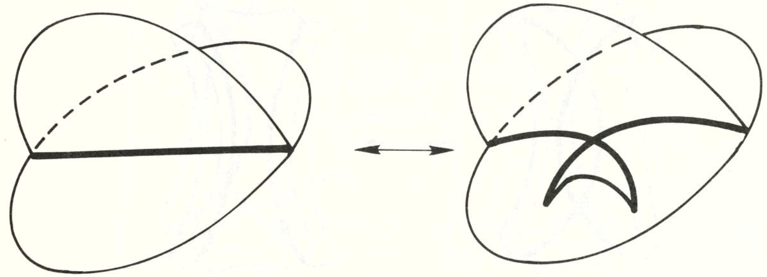


Fig. 4

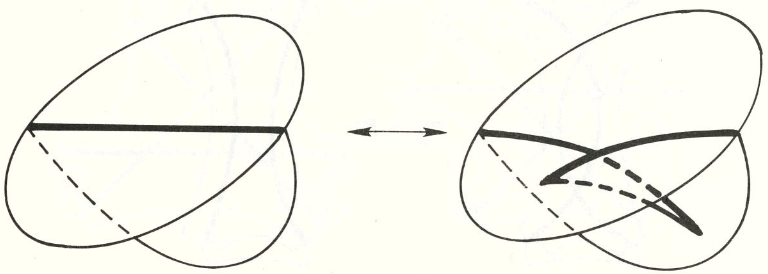


Fig. 5

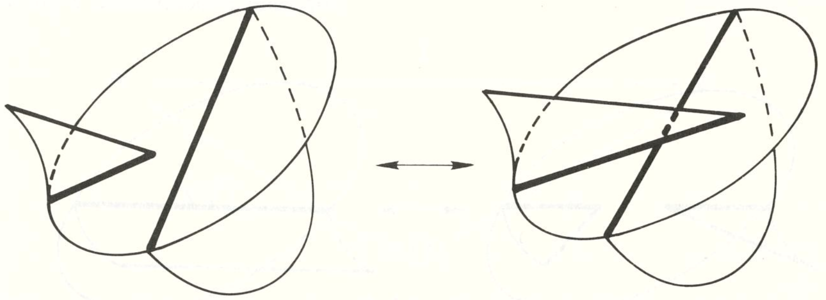


Fig. 6

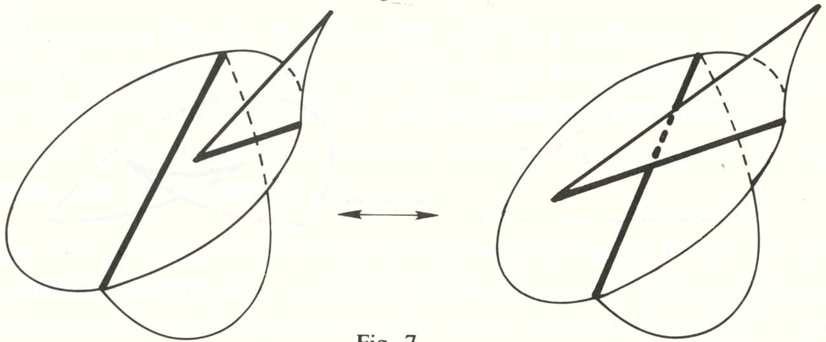


Fig. 7

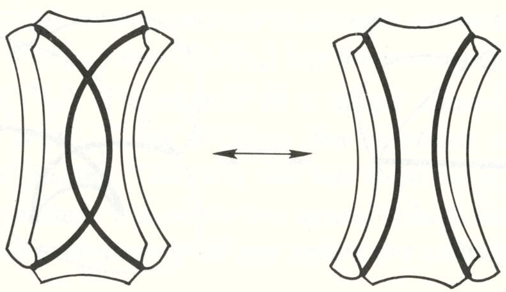


Fig. 8

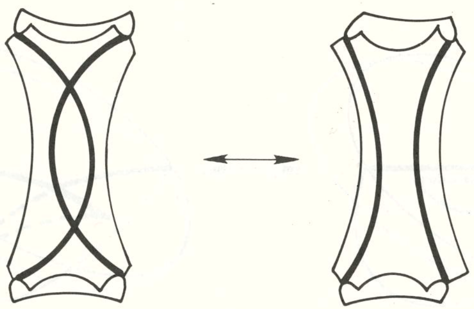


Fig. 9



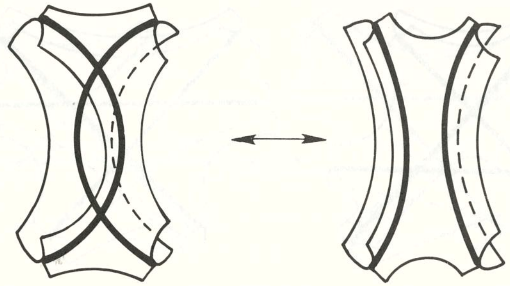


Fig. 10

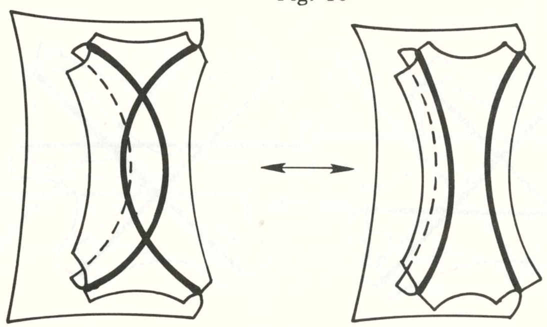


Fig. 11

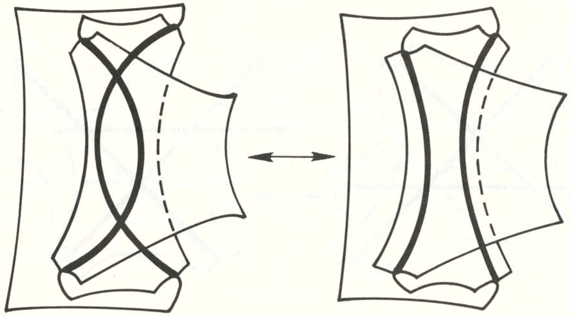


Fig. 12

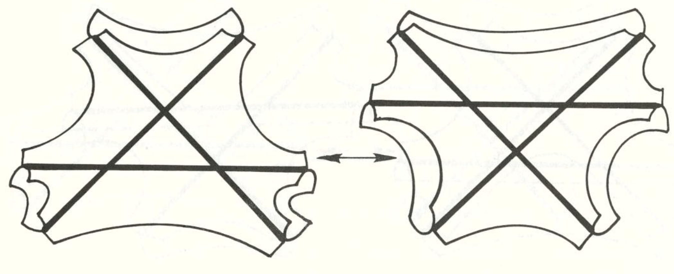


Fig. 13

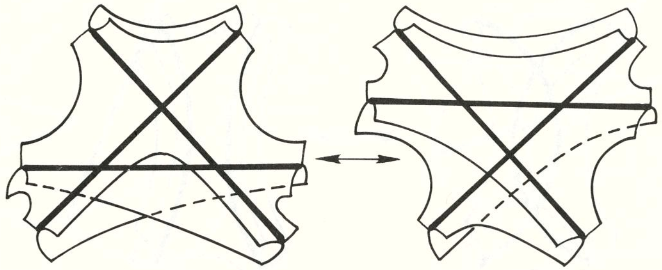


Fig. 14

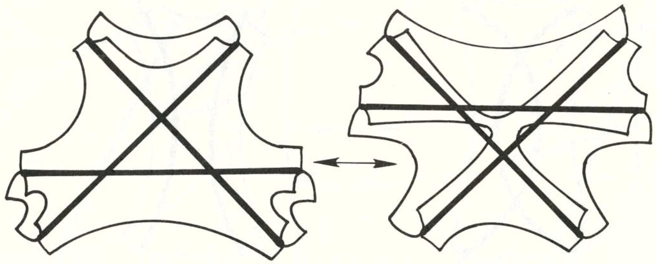


Fig. 15

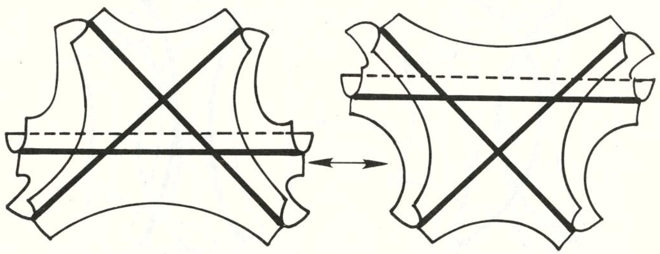


Fig. 16

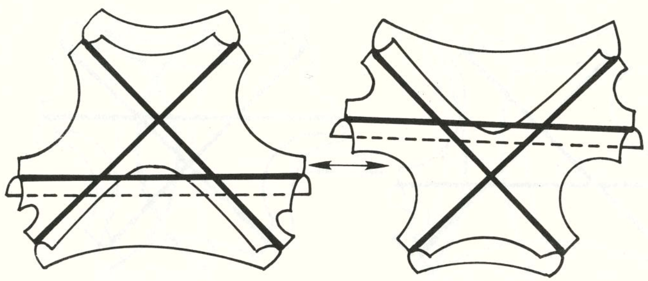


Fig. 17



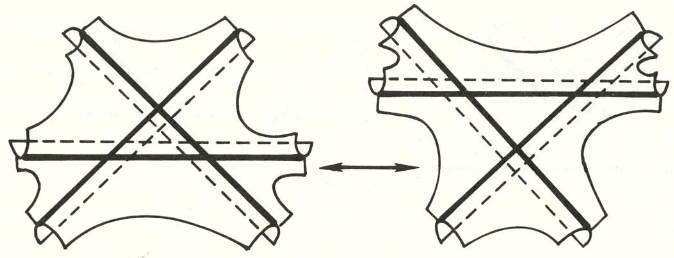


Fig. 18

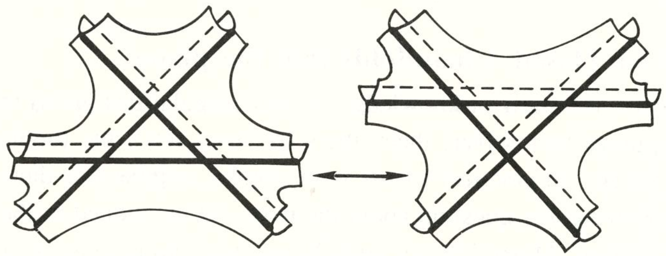


Fig. 19

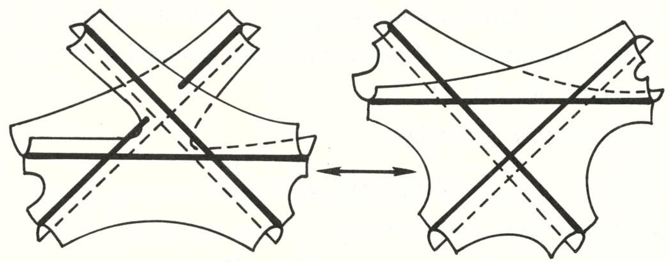


Fig. 20

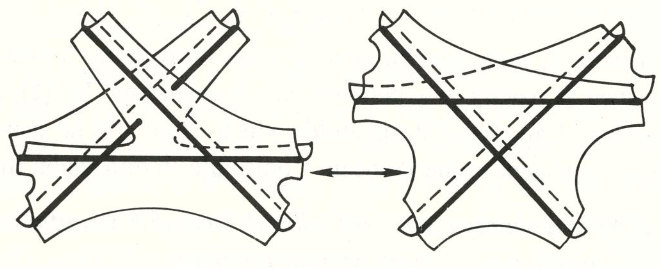


Fig. 21

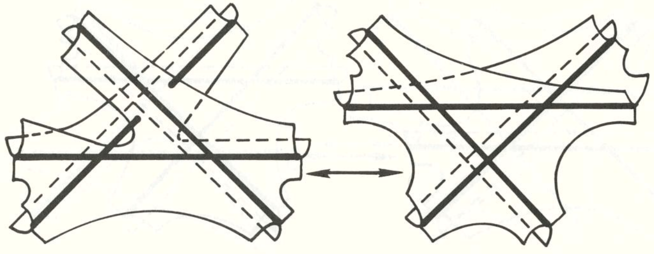


Fig. 22

## 1. Maps from closed 3-manifolds into the plane

For smooth maps, the study or knowledge of their singularities yields information about the mappings themselves, their fibers (or level sets), the topology of the source manifold etc. In practical terms: with simpler singularities there are more chances of answering any question about the map. As mentioned in the introduction, Ree's Theorem, Morse Theory and [2] are interesting consequences of the maps having simple singularities.

In the case of mappings from closed 3-manifolds into the plane, the Stable Maps form an open and dense set in the space of smooth maps (with Whitney $C^\infty$-topology). The singularities of stable maps are grouped in three types up to local coordinates, (see [8], Ch. 1).

*Definite fold singularities*, given by

$$(x,y,z) \mapsto (x, y^2 + z^2)$$

*Indefinite fold singularities*, given by

$$(x,y,z) \mapsto (x, y^2 - z^2)$$

*Cusp singularity* (at zero),

$$(x,y,z) \mapsto (x, xy + y^3 + z^2)$$

Bürlet and De Rham [2] study *special generic* maps from closed 3-manifolds into the plane, described as having only definite folds singularities. In ([2], p. 284), it is presented the list of all closed 3-manifolds which admit special generic maps. The following is equivalent to the Poincaré Conjecture in dimension three:

**Conjecture.** *If $M$ is a simply connected, closed, smooth 3-manifold then there exists a special generic map $f$ from $M$ into the plane.*

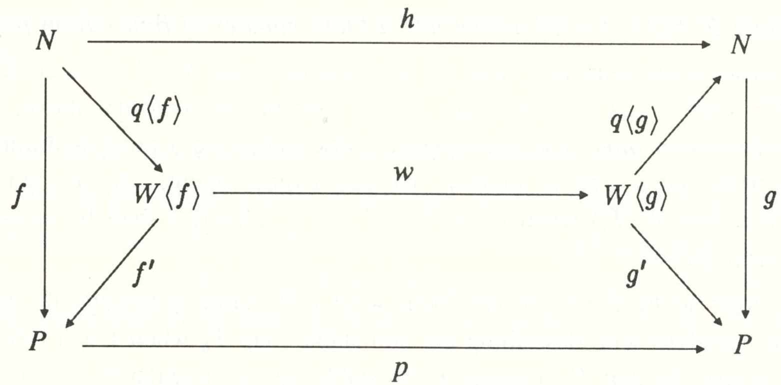
The equivalence with the better known conjecture is immediate from [2], because the only simply connected manifold in the list is $S^3$.



Stable maps have been studied in [5], [6] and [8] using the Stein Factorization. See ([8], Ch. 1) or ([10], Ch. 1-2) as basic reference on stable maps and the Stein Factorization or [9] for a brief note on these.

The Stein Factorization $W\langle f\rangle$ of a stable map $f\colon M^3 \to \mathbf{R}^2$ is a polyhedron (a 2-dimensional simplicial complex). If $M$ is orientable, the lower dimensional strata of $W\langle f\rangle$ provides (see [8], Ch. 1) a complete description of the singularities of $f$. If $f$ is special generic as in [2], then $W\langle f\rangle$ is a surface with boundary. This boundary corresponds to the definite folds.

**Definition.** Given two maps $f, g\colon N \to P$, we say that their Stein Factorizations are equivalent if there are homeomorphisms $h$, $w$ and $p$ such that the diagram in the following figure is commutative:




The fact is that the Stein Factorizations of $f$ and $g$ are equivalent if and only if $f$ and $g$ are conjugated by homeomorphisms, say $f = p^{-1} \circ g \circ h$. In particular if two stable maps $f, g\colon M^3 \to \mathbf{R}^2$ are $\mathcal{A}$-equivalent, i.e. right-left equivalent, then the corresponding $W\langle f\rangle$ and $W\langle g\rangle$ are homeomorphic. If $W\langle f\rangle$ and $W\langle g\rangle$ are not homeomorphic, we may want to compare them to investigate the differences between the singularities of $f$ and $g$.

On the other hand, if $f$ and $g$ are not $\mathcal{A}$-equivalent it may still happen that $W\langle f\rangle$ and $W\langle g\rangle$ are homeomorphic as the following example would show (in these cases their singular sets are in some sense identical). Consider $M^3 = S^3$, the unitary 3-sphere in $\mathbf{R}^4$. Define $f\colon M^3 \to \mathbf{R}^2$ as the orthogonal projection into the first copy or $\mathbf{R}^2$ in $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$. The map $f$ is stable; the factoring map $f'\colon W\langle f\rangle \to \mathbf{R}^2$ is a homeomorphism into its image (the unit 2-disk, $B_2$ in $\mathbf{R}^2$). Next consider a smooth map $d\colon \mathbf{R}^2 - (2,0) \to \mathbf{R}^2 - (2,0)$ without singularities, which maps $B_2$ onto an annulus in $\mathbf{R}^2$. The map $g = d \circ f\colon M^3 \to \mathbf{R}^2$ is stable; the polyhedron $W\langle g\rangle$ is homeomorphic to $B_2$ and $W\langle f\rangle$, but its image is an annulus, hence $g$ *is not* $\mathcal{A}$-equivalent to $f$ because, the *number of connected components of the set of regular values* is an invariant of $\mathcal{A}$-equivalence.

## 2. Pi-stable homotopies

We shall consider homotopies between stable maps. It is desirable to have homotopies which are "simple".

Chincaro presents in [3], a generic family of homotopies, called Pi-stable homotopies, which connect any two (homotopic) stable maps. In our case, for maps into the plane, any two maps are homotopic and the theorem in ([3], p. 117) has the following immediate corollary:

*Any two stable maps $f, g: M^3 \to \mathbb{R}^2$ can be connected with a Pi-stable homotopy $F: M^3 \times I \to \mathbb{R}^2$ which has a finite number of bifurcation points.*

A bifurcation point is a value of the parameter, say $b \in I = [0,1]$ such that $F_b: M^3 \to \mathbb{R}^2$ is not a stable map. In the case of the corollary above, $F_b$ has a $\mathcal{A}$-codimension *one*. The homotopy $F$, when crossing a point of bifurcation, moves from one $\mathcal{A}$-orbit to another. We may expect changes in the topology of the polyhedron $W\langle F_t\rangle$ when $t$ runs "over" $b \in I$. This fact shall be investigated in the next section.

In order to be more precise about what a Pi-stable homotopy is, we shall content ourselves with describing the non-stable map $F_b$ when $b$ is a bifurcation point. It may be that $F_b$ presents a non-stable (germ) singularity, see (i), (ii) or (iii) below, or it may fail to satisfy exactly one global stability condition, see (iv), (v) and (vi). In some sets of local coordinates in $M^3$, about $b$ in $I$ and in $\mathbb{R}^2$, the germ of the homotopy $F$ is given by one of the following (see [3], pp. 115-117), also ([13], p. 75), [1] and [4]):

(i) *Lips.*

$$F(x,y,z,t) = (x, y^3 + yx^2 + z^2 + yt);$$

(ii) *Beak to Beak.*

$$F(x,y,z,t) = (x, y^3 - yx^2 + z^2 + yt);$$

(iii) *Swallowtail* (first and second kind).

$$F(x,y,z,t) = (x, y^4 + yx \pm z^2 + y^2 t);$$

The following are the possible codimension one multigerms

(iv) *Intersection of a Fold and a Cusp.*

$$F_{|U}(x_1,y_1,z_1,t) = (y_1^2 \pm z_1^2 + t, x_1)$$
$$F_{|V}(x_2,y_2,z_2,t) = (x_2, y_2^3 + y_2 x_2 + z_2^2)$$

(v) *Non-transversal intersection of two folds.*

$$F_{|U}(x_1,y_1,z_1,t) = (y_1^2 \pm z_1^2 + t, x_1)$$
$$F_{|V}(x_2,y_2,z_2,t) = (y_2^2 \pm z_2^2 - x_2^2, x_2)$$

(vi) *Intersection of three folds.*

$$F_{|U}(x_1,y_1,z_1,t) = (y_1^2 \pm z_1^2 + t, x_1)$$
$$F_{|V}(x_2,y_2,z_2,t) = (y_2^2 \pm z_2^2 + x_2, x_2)$$
$$F_{|V}(x_3,y_3,z_3,t) = (y_3^2 \pm z_3^2 - x_3, x_3)$$

**Remark.** Only in (i), (ii) and (iii) there is a change in the types of singularities occuring in $F_t$. In the remaining cases the stable singularities cross in a non-stable manner.

## 3. The Stein factorization of pi-stable homotopies

In this section we present a proof of the main theorem stated in the introduction.

Let us consider a Pi-stable homotopy $F: M^3 \times I \to \mathbb{R}^2$ with bifurcation point $b = 0 \in I = [-1,1]$, and critical value of bifurcation the origin in $\mathbb{R}^2$. We want to describe the change in the topology of $W\langle F_t\rangle$ as $t$ goes, from negative to positive values.

**Preliminaries of the proof.** We shall restrict the homotopy $F$ to a smaller interval $[-e,e]$ so that $b = 0$ is the only bifurcation point. This homotopy $F$ represents a universal unfolding of the codimension one map $F_0$. If we consider another universal unfolding of $F_0$, say $h$, then a theorem of G. Lassalle ([7], p. 220) tells us that: $h_{-e}$ and $h_e$ are $\mathcal{A}$-equivalent to say $F_{-e}$ and $F_e$, respectively. Hence, $W\langle h_{-e}\rangle$ is homeomorphic to $W\langle F_{-e}\rangle$, and $W\langle h_e\rangle$ is homeomorphic to $W\langle F_e\rangle$.

We shall look for a universal unfolding $h$ of $F_0$ as the one given by the following lemma,

**Lemma 0.** *Let $F: M^3 \times [-e,e] \to \mathbb{R}^2$ a universal unfolding of the codimension one map $F_0$, as in the statement of the main theorem and with the restriction just made above (specifically consider the cases* (i) *through* (vi) *above). Let $B_r$ be the closed ball of radius $r$ about the origin in $\mathbb{R}^2$. Then,*

(1) *After an appropriate rescaling change of coordinates at the origin in $\mathbb{R}^2$, we can assume that $F_0$ intersects transversally the boundary of $B_4$.*

$\partial B_4$. *It follows that* $N = F_0^{-1}(B_4)$ *is a smooth, orientable, 3-dimensional submanifold of* $M^3$ *(with boundary).*

(2) *By the transversality condition in* (1), $F_0$ *is a product map when restricted to some small product neighbourhood of* $\partial N = F_0^{-1}(\partial B_4)$.

(3) *There exists a universal unfolding* $h$, *of* $F_0$, *such that*

$$h_t(x) = F_0(x), \quad \forall t \in [-e, e] \quad \text{if} \quad F_0(x) \notin B_3 \subseteq \mathbf{R}^2.$$

**Proof.** Parts (1) and (2) follow by inspection of the local canonical forms (i) through (vi) together with a direct application of Ehresmann's Theorem, see ([8], p. 7). For part (3), we shall construct the new universal unfolding $h: M^3 \times [-e, e] \to \mathbf{R}^2$. Let us start by considering a small enough neighborhood $O$, of the singularities of $F_0$ above the origin in $\mathbf{R}^2$ (the critical value of bifurcation). We can assume that in this neighborhood $O$, $F_0$ has the form given by one of the local canonical forms of Chincaro ((i) through (vi) above). Consider a small radius $r > 0$ such that the 3-dimensional disk of radius $2r$ is contained in $O$ and $F_0(x, y, z) \in B_3$ if $x^2 + y^2 + z^2 \leq 4r^2$. Let $\mu: [0, \infty) \to [0, 1]$ be a smooth function such that $\mu(s) = 1$ if $s \leq r^2$, and $\mu(s) = 0$ if $s \geq 4r^2$.

We define $h$ for each of the six cases under consideration as follows.

For cases (i), (ii) and (iii), define

$$h(x, y, z, t) = F(x, y, z, t\mu(x^2 + y^2 + z^2)) \quad \text{if} \quad (x, y, z) \in O, \quad \text{and}$$
$$h_t = F_0 \quad \text{outside of} \quad O.$$

For cases (iv) and (v), define

$$h_{|U}(x_1, y_1, z_1, t) = F_{|U}(x_1, y_1, z_1, t\mu(x_1^2 + y_1^2 + z_1^2)),$$
$$h_{|V} = F_{|V}, \quad \text{if} \quad (x1, y1, z1) \in O, \quad \text{and}$$
$$h_t = F_0 \quad \text{outside of} \quad O.$$

Finally, for case (vi) we define,

$$h_{|U}(x_1, y_1, z_1, t) = F_{|U}(x_1, y_1, z_1, t\mu(x_1^2 + y_1^2 + z_1^2))$$
$$h_{|V} = F_{|V},$$
$$h_{|W} = F_{|W} \quad \text{if} \quad (x_1, y_1, z_1) \in O, \quad \text{and}$$
$$h_t = F_0 \quad \text{outside of} \quad O.$$

The map $h$ is smooth and if $F_0(x) \notin B_3 \subseteq \mathbf{R}^2$, then $h_t(x) = F_0(x)$ by the construction. The map $h$ is a universal unfolding of $F_0$. This is true, because $h$ coincides with $F$ in a neighborhood of the singularities of $F_0$ above the critical value of bifurcation (the origin), hence $h$ is also a universal unfolding of $F_0$, see [7].

Once the existence of $h$ has been granted (satisfying (3) in Lemma 0), it



follows immediately that $h_t$ satisfies (1) and (2) for $t$ small enough. We shall restrict $h$ to $N = h_0^{-1}(B_4)$ like in (1) above. The family $h_t$ is constant, with respect to $t$, on the $h_0$-preimage of the annulus $B_4 - \text{int}(B_3)$; in particular $h_{-e} = h_e$ on a neighborhood of $\partial N$.

According to the cases (i) through (vi) for $h_0$, we have in figures 23 through 28, the pictures of the critical values on $B_4$ of $h_{-e}$, $h_0$ and $h_e$, say going from the left to the right.

**Proof of part (a) of the theorem.** If the singularities of $h_0$ above the origin of $\mathbf{R}^2$ do not belong to the same connected component of the $h_0$-fiber the q-image of those singularities contains at least two points $w_1$ and $w_2$ in $W\langle h_0 \rangle$. In [10] it is shown that $W\langle h_0 \rangle$ is Hausdorff so the points $w_1$ and $w_2$ can be separated by disjoint open neighborhoods $W_1$ and $W_2$ in $W\langle h_0 \rangle$. In fact, by the construction of $h$ above, we can even take $W_1 \cup W_2 = W\langle h_0 \rangle$. We can look now to the disjoint open submanifolds $N_1 = q^{-1}(W_1)$ and $N_2 = q^{-1}(W_2)$ in $N$. In each of these submanifolds the restriction of the unfolding $h$ is a homotopy of stable maps, hence $h_{-e}$ and $h_e$ are $\mathcal{A}$-equivalent when restricted to the $N_i$'s; we conclude then that $W\langle h_{-e} \rangle = W\langle h_{-e|N_1} \rangle \cup W\langle h_{-e|N_2} \rangle$ is homeomorphic to $W\langle h_e \rangle = W\langle h_{e|N_1} \rangle \cup W\langle h_{e|N_2} \rangle$ and part (a) is finished.

**Proof of part (b) of the theorem.** Now we are assuming that the singularities of $h_0$, above the origin in $\mathbf{R}^2$, belong to the same connected component of the $h_0$-fiber. This is always true for cases (i), (ii) and (iii).

If $N = h_0^{-1}(B_4)$ is not connected, we shall replace $N$ by the connected component of $h_0^{-1}(B_4)$ that contains the singularities above the origin. Let's consider the comparison mapping $g$ of $h_e$ and $h_{-e}$ (defined in the appendix of this paper),

$$g = h_e | h_{-e}: 2N \to 2B_4 = \mathbf{S}^2 \quad \text{(2-dim. sphere)}.$$

Notice that $h$ satisfies conditions (1) and (2) of Lemma 0 and, $h_e = h_{-e}$ on some product neighborhood of $\partial N$ as remarked after the lemma's proof; then, sufficient conditions are satisfied to construct the comparison mapping $g$. Notice also that $e > 0$ can be taken arbitrarily small.

The map $g$ is stable and $W\langle g \rangle$ is a polyhedron as in [8], [9], [10] or [12]. Locally, $g$ looks like $h_e$ and/or $h_{-e}$, then the singularities of $g$ are those of $h_e$ and/or $h_{-e}$. The critical values of $g$ are in the image (by the quotient) of the critical values of $h_e$ and/or $h_{-e}$, in $2B_4 = \mathbf{S}^2$.

The reason for introducing the comparison mapping $g$ is that $W\langle g \rangle$ "contains" $W\langle h_e \rangle$ and $W\langle h_{-e} \rangle$. In fact, to get $W\langle h_e \rangle$ and $W\langle h_{-e} \rangle$ we just have to cut (appropriately) the polyhedron $W\langle g \rangle$ in two pieces. On the other hand, $W\langle g \rangle$

$h_{-e}$ $h_0$ $h_e$

$(i)$

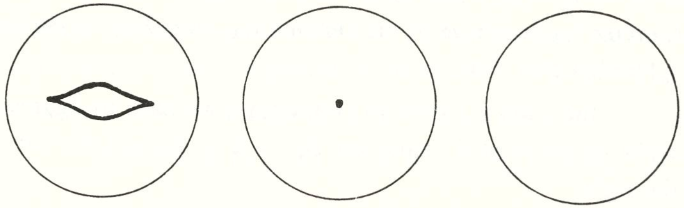


Fig. 23

$(ii)$

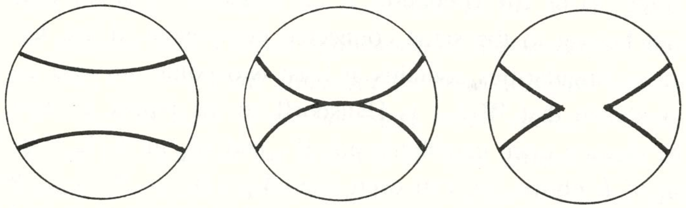


Fig. 24

$(iii)$

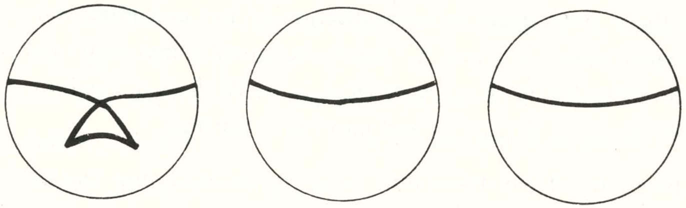


Fig. 25

$(iv)$

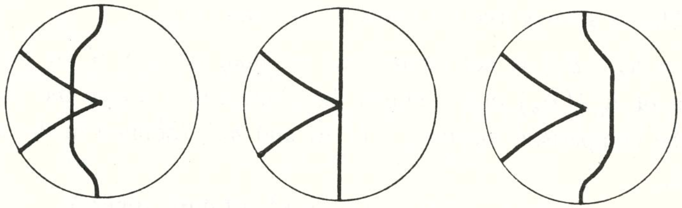


Fig. 26

$(v)$

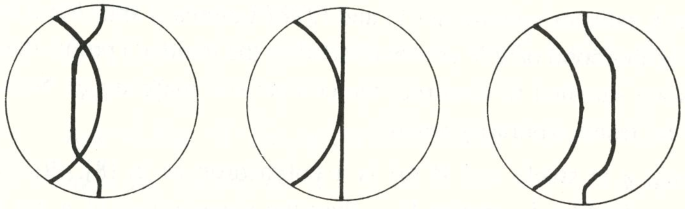


Fig. 27

$(vi)$

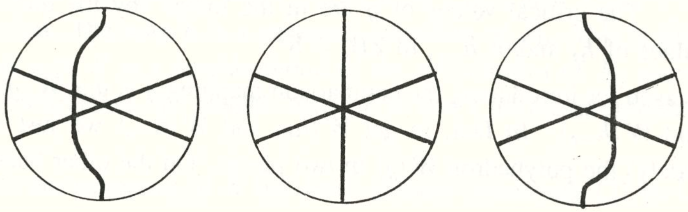


Fig. 28



is simpler to compute than the pair $W\langle h_e\rangle$ and $W\langle h_{-e}\rangle$, essentially because $W\langle g\rangle$ is just one object (with some structure), while $W\langle h_e\rangle$ and $W\langle h_{-e}\rangle$ are two objects (same kind of structure) with some sort of relations between them. I shall mention that the *method of the comparison mapping*, may seem too elaborated for the simple cases (i) through (iv); the fact is that the method does not add any complications. The real usefulness of the comparison mapping is more evident by tackling cases (v) and (vi); this can be seen from the long list of figures (associated to these cases) presented in the statement of the theorem.

The following Lemma is an important step in the analysis that follows.

**Lemma 1.** *Under the hyphotesys of part* (b) *of the theorem, there exists a regular value* $z_0 \in \mathbf{S}^2$ *of the comparison mapping* $g$, *such that* $g^{-1}(z_0)$ *is connected (a copy of* $\mathbf{S}^1$*).*

**Proof.** See the appendix.

We shall be looking at the connected components of the set of regular values of $g$, $RV(g)$. For any connected component $R$ of $RV(g)$, we shall define $a(R)$ =associated number of $R$, as the number of connected components of the $g$-fiber above any regular value in $R$. Clearly, $a(R)$ is a well defined non-negative integer.

Lemma 1 says, that under the hypothesys of part (b), there exists a connected component $R$ of $RV(g)$ with $a(R) = 1$. Notice that this may not be simultaneously true for $h_e$ and $h_{-e}$ (in place of $g$).

The next step will be to consider a diffeomorphism, the stereographic projection $s: \mathbf{S}^2 - \{z_0\} \to \mathbb{R}^2$, from any regular value $z_0$ as in the lemma above, and the map, $G: 2N - g^{-1}(z_0) = L \to \mathbb{R}^2$, given by

$$G(x) = s(g(x)), \quad \forall x \in L = 2N - g^{-1}(z_0).$$

The space $L$ is an orientable (non-compact) 3-manifold. The map $G$ is smooth, proper and stable; its singularities are those of $g$ and its critical values are the image of the critical values of $g$ by the stereographic projection $s$ into $\mathbb{R}^2$. Furthermore, $W\langle g\rangle$ is the one-point compactification of $W\langle G\rangle$, hence, if we know $W\langle G\rangle$ we can get $W\langle g\rangle$. The most compelling reason for considering $G$ (and $W\langle G\rangle$) instead of $g$ (and $W\langle g\rangle$) is that the new one is easier to represent on paper (drawings).

Let $RV(G)$ be the set of regular values of $G$ in $\mathbb{R}^2$. For any connected component $R$ of $RV(G)$, we can talk about the associated number of $R$, $a(R)$. By the construction of $G$ we have that $RV(G)$ has exactly one *unbounded* connected component, call it $E$ (for "exterior"); this connected component $E$ has associated number one, $a(E) = 1$.

**Definition.** We shall call two connected components $R_1$ and $R_2$ of $RV(G)$, *adjacent*, if the intersection of their closures in $\mathbb{R}^2$ contains a (non-empty) interval of critical values of $G$.

**Remark.** From previous work on the Stein Factorization of stable maps from orientable 3-manifolds into the plane, as in [**5**], [**6**] or [**8**], it is known that for any two adjacent connected components $R_1$ and $R_2$ of $RV(G)$, their associated numbers differ exactly by *one*, i.e. $|a(R_1) - a(R_2)| = 1$. This last remark follows explicitly from [**8**], p. 15; it can also be checked that the remark might not be true if the domain of $G$ is non-orientable.

For any map $G$, let us consider $CRV(G)$, the set of connected components of $RV(G)$.

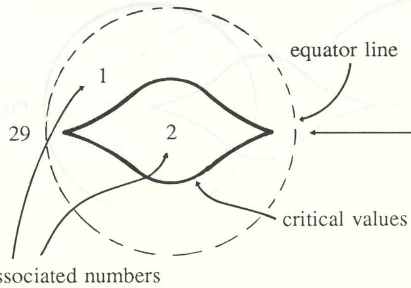
**Definition.** We shall call an *arrangement of associated numbers*, to any map $c: CRV(G) \to \mathbb{N} =$ non-negative integers, such that for adjacent connected components $R_1$ and $R_2$ in $RV(G)$, $|c(R_1) - c(R_2)| = 1$ and $c(E) = 1$ for the unbounded connected component $E$ of $RV(G)$.

There exists only a finite number of such arrangement of associated numbers. We shall make explicit (in a set of pictures) all the possible arrangements of associated numbers. This pictorial list also shows, for each figure, the critical values of $G$ in $\mathbb{R}^2$ (represented with a heavy line), together with some special circles (shown with a broken line), which represent "equator lines of $S^2$" whose meaning will be explained later.

The intention of the pictorial list is, to leave as an exercise for the reader the following task: for each of the figures in the pictorial list that follows, construct all the possible polyhedrons $W\langle G\rangle$ that under the map $G': W\langle G\rangle \to \mathbb{R}^2$, project onto the pictures shown (do not consider the "equator lines"). To accomplish this, the reader should refer and follow the rules in [**8**], p. 20, noticing that the associated number on a connected component $R$ of $RV(G)$, refers to the number of *surface points* in the polyhedron $W\langle G\rangle$ mapped onto each point of $R$.

The outcome of the above exercise is (almost) that each figure has exactly one "realization" $W\langle G\rangle$ (up to homeomorphism). In the case (vi), there are two excepcional arrangements of associated numbers: the first with associated number *four* in the middle triangle, which has *two* (non-homeomorphic) realizations. The second exception is for the arrangement of associated numbers which has associated number *two* for the middle triangle and the three adjacent triangles each with associated number *three*; for this one, it does not exist any realization $W\langle G\rangle$ that fits the picture.



Fig. 29


Generates Fig. 1

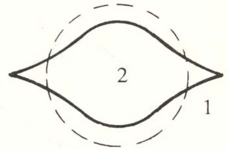
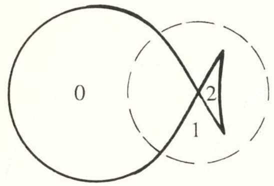
Generates Fig. 2


Fig. 30

Fig. 31


Generates Fig. 3

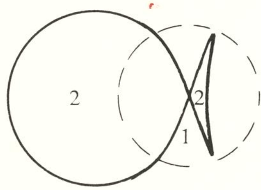
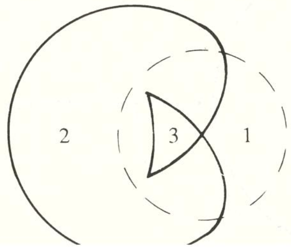
Generates Fig. 4


Fig. 32

Fig. 33


Generates Fig. 5

Generates Fig. 6

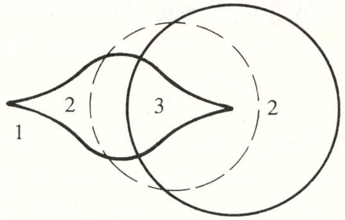
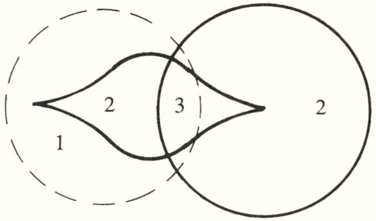



Fig. 34

Fig. 35




Generates Fig. 7

Generates Fig. 8

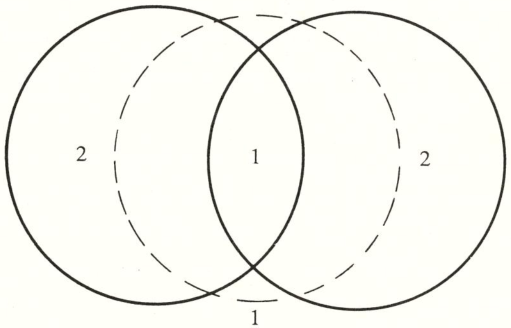
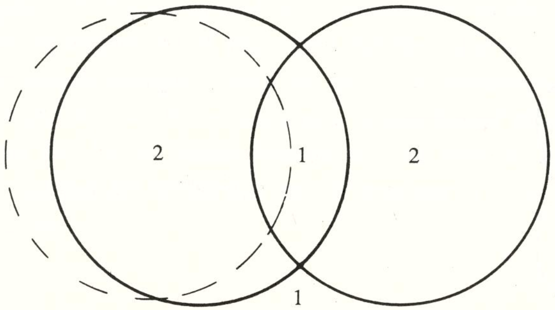



Fig. 36

Fig. 37




Generates Fig. 9

Generates Fig. 10

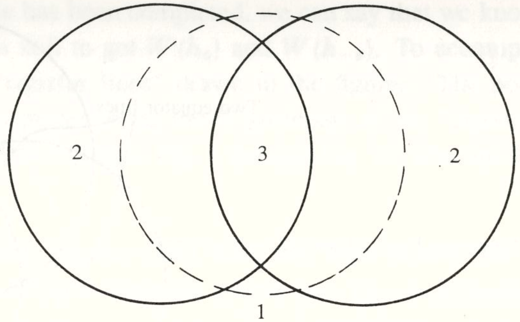
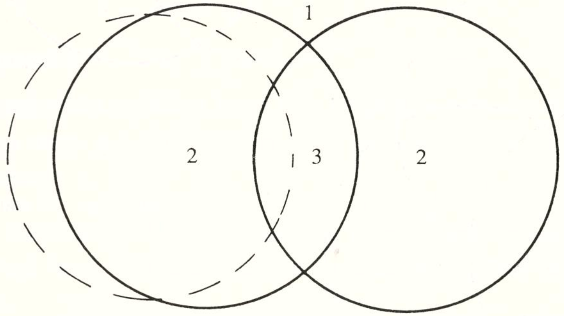



Fig. 38

Fig. 39




Generates Fig. 11

Generates Fig. 12

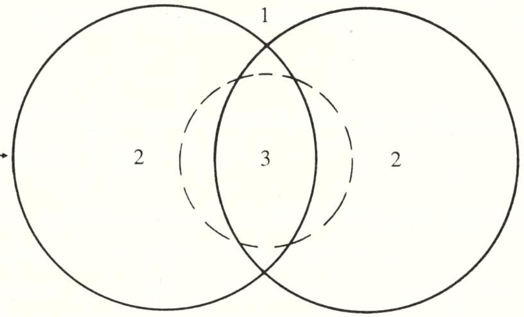
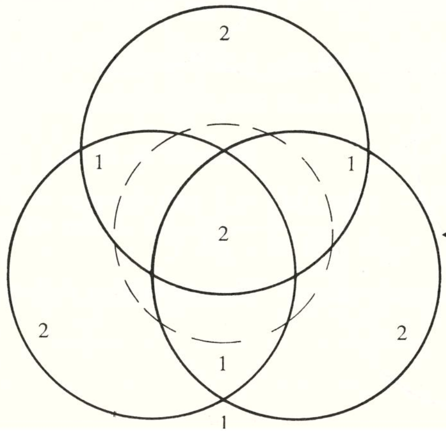



Fig. 40

Fig. 41




Generates Fig. 13

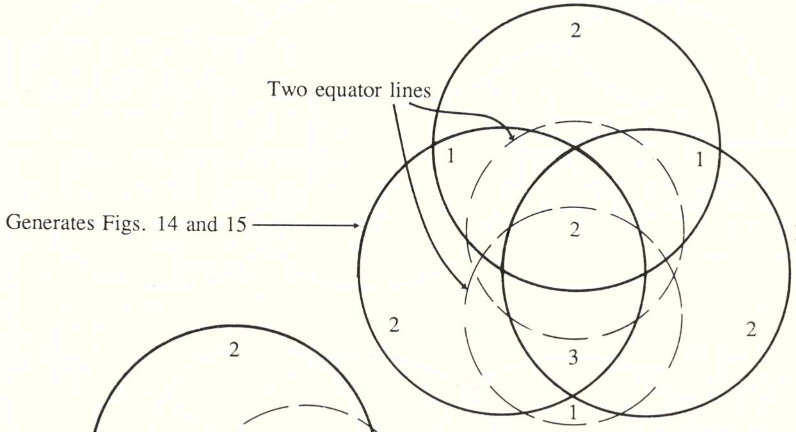


Fig. 42

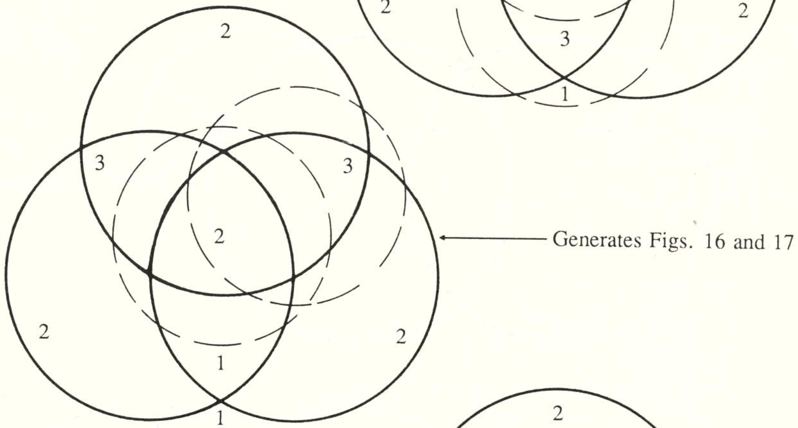


Fig. 43

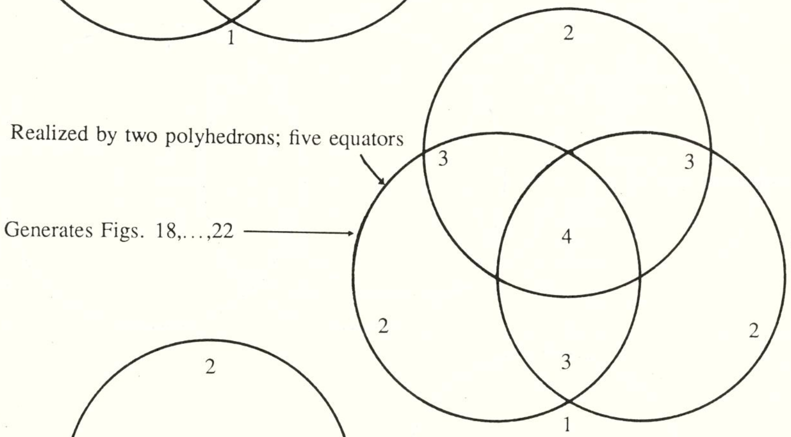


Fig. 44

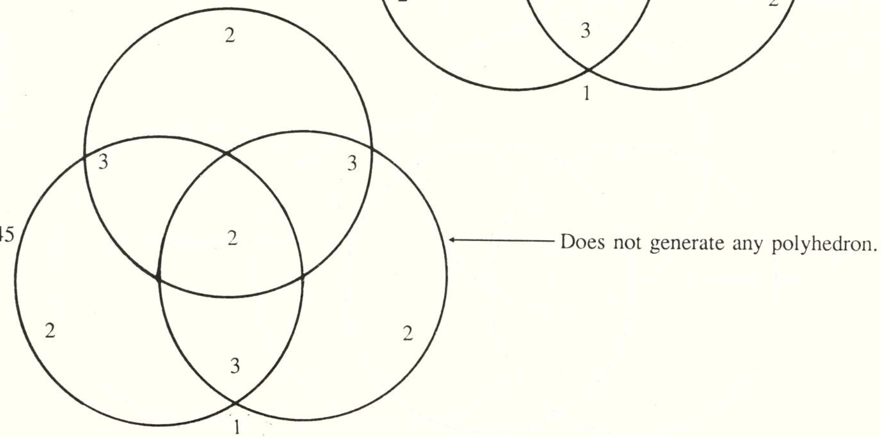


Fig. 45

Once the above exercise has been completed, we can say that we know all the possible $W\langle G\rangle$'s. It remais still to get $W\langle h_e\rangle$ and $W\langle h_{-e}\rangle$. To accomplish this, there come into play the "equator lines" drawn in the figures. The polyhedron $W\langle G\rangle$ has to be cut in two pieces along the $G$-preimage of the "equator lines"; getting as a result the twenty two figures, Fig. 1 through Fig. 22, after the statement of the main theorem.

**Comentary.** Many possible equator lines have not been drawn, for they do not produce new pictures, different from the twenty two figures presented in the statement of the theorem.

## Appendix

This appendix contains in its first part the construction of the comparison mapping of two maps together with its basic properties. The second part contains the proof of Lemma 1.

**The comparison mapping.** The comparison mapping of two maps, to be constructed here, has been useful in finding "structural" differences between two maps.

Let $(M_0, M_1)$ and $(P_0, P_1)$ be in the category of topological pairs, and $f: (M_0, M_1) \to (P_0, P_1)$ be a continuous map between pairs, i.e. $M_1 \subset M_0$ and $f(M_1) \subset P_1$.

**Definition.** A *product neighborhood* of $M_1$ in $M_0$ is a homeomorphism (into) $j: M_1 \times [0,1] \to M_0$ with $j(x,1) = x, \ \forall x \in M_1$.

**Definition.** Given a product neighborhood $j$ of $M_1$ in $M_0$, we shall denote by $2M_0$ the *double of* $M_0$ *through* $j$, as the quotient space of $M_0 \times \{l\} \cup M_0 \times \{r\}$ (disjoint union) by the smallest equivalence relation including $(x,l) \sim (y,r)$, if there exists $(z,t) \in M_1 \times [0,1]$, such that $j(z,t) = x$, and $j(z, 1-t) = y$.

**Claim 1.** *If* $(M, \partial M)$ *is a smooth manifold with boundary and* $j: \partial M \times I \to M$ *is an embedding, then* $2M$ *is a smooth manifold without boundary.*

**Proof.** Left as an exercise.

Now suppose that $f, g: (M_0, M_1) \to (P_0, P_1)$ are continuous and

$$j: M_1 \times [0,1] \to M_0, \quad k: P_1 \times [0,1] \to P_0$$

are product neighborhoods. Assume also that the following diagram is commutative,

$$\begin{array}{ccc} M_1 \times I & \xrightarrow{\quad j \quad} & j(M_1 \times I) \\ {\scriptstyle (f,Id)}\downarrow \ \downarrow{\scriptstyle (g,Id)} & & {\scriptstyle f}\downarrow \ \downarrow{\scriptstyle g} \\ P_1 \times I & \xrightarrow[\quad k \quad]{} & k(P_1 \times I) \end{array}$$

Observe that $f$ and $g$ coincide on $j(M_1 \times I)$ in $M_0$. With the assumptions above we define the map $f|g: 2M_0 \to 2P_0$, called the *comparison mapping of f and g*, given by:

$$f|g[(x,l)] = [f(x),l)], \quad \text{and}$$
$$f|g[(x,r)] = [g(x),r)], \quad \forall x \in M_1,$$

where the brackets denote the corresponding equivalence classes.

**Claim 2.** *$f|g$ is well defined and continuous.*

**Proof.** Immediate.

For smooth manifolds with boundary, we have by Claim 1:

**Claim 3.** *In the smooth case $f|g$ is a smooth mapping.*

**Proof.** Locally $f|g$ looks like $f$ and/or $g$.

**Proof of Lemma 1.**

**Lemma 1.** *Under the hypothesis of part* (b) *of the theorem, there exists a regular value $z_0 \in \mathbf{S}^2$ of the comparison mapping $g$, such that $g^{-1}(z_0)$ is connected ( a copy of $\mathbf{S}^1$).*

**Proof.** We shall consider again the universal unfolding $h$ restricted to $N$, $h: N \times [-e,e] \to \mathbf{R}^2$. Let $H$ be the suspension of $h$,

$$H: N \times [-e,e] \to \mathbf{R}^2 \times [-e,e],$$

defined by

$$H(x,t) = (h_t(x),t), \ \forall x \in N, \ \forall t \in [-e,e].$$

Observe that $(x,t)$ is a singularity of $H$ if and only if $x$ is a singularity of $h_t$. The critical values of $H$, form a finite collection of parametrized (not necessarily regular) surfaces in $\mathbf{R}^2 \times [-e,e]$. To complete the proof, it is enough to find a regular value $(y_0,t_0)$ of $H$, with $t_0 = 0$, such that $H^{-1}(y_0,t_0)(= h_{t0}^{-1}(y_0))$ is



connected, for then, taking $e = |t_0|$, the comparison map $g$ satisfies that $g^{-1}(z_0)$ is connected as desired, where $z_0$ is the image of $y_0$ by the quotient map into $2\dot{B}_4 = S^2$, ($z_0$ is a regular value of $g$).

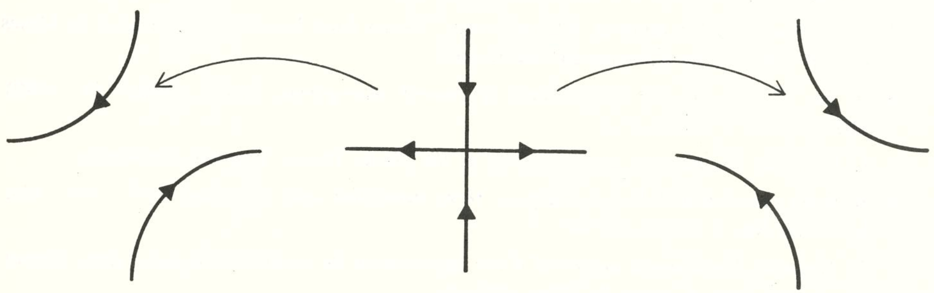
At this point, we shall distinguish the first three cases from the last three.

For the cases (i), (ii) and (iii), it is clear, from the right hand side of the figures 23, 24 and 25, that there are regular values $y_0$ of $h_e$ for which $h_e^{-1}(y_0)$ is connected, say considering $y_0 = (0,1)$ or $(0,-1) \in B_4 \in \mathbf{R}^2$.

For the remaining cases (iv), (v) and (vi), we shall need a longer argument. From our hypothesis in part (b), we know that all the singularities of $h_0$ are indefinite fold singularities (the same for any $h_t$), except in case (iv) were it has a cusp singularity. In fact, if a definite fold singularity where present in the fiber $h_0^{-1}(0,0)$, this alone would form a connected component of that fiber, contradicting the assumption in part (b).

Let $L = h_0^{-1}(0,0)$. This fiber $L$ is not a 1-manifold, for it has one, two or three exceptional points according to the cases (iv), (v) or (vi), respectively. The exceptional points are the indefinite fold singularities present in the fiber $L$. A neighborhood of an exceptional point in $L$ is homeomorphic to the letter "X" (the shape).

If we move onto a regular value $(y_0,t_0)$ of the suspension $H$, we get a fiber $L_0 = H^{-1}(y_0,t_0) = h_{t0}^{-1}(y_0)$ which is a closed 1-manifold. The transition from the fiber $L$ to $L_0$ can be thought as a replacement of each exceptional neighborhood in $L$ by two open segments that join consecutive ends of the "X", as the drawing below shows,



We shall call this process, an *opening* of the exceptional neighborhood. For any given exceptional neighborhood, we have the two possible openings shown in the drawing above. To determine the exact opening that will be produced, we just have to move from one side to the other, of the surface of critical values determined by the singularity in the exceptional neighborhood. Hence, by choosing appropriately the connected component of the set of regular values of $H$, we can produce any (global) opening of the fiber $L$.

It only remains to prove the following claim.

**Claim.** *Any closed, connected, 1-manifold with a finite number n of exceptional points, L, admits a connected opening*

**Proof.** We shall prove the claim by induction on n. If $n = 1$, then $L$ is homeomorphic to the figure "8", which certainly has a connected opening: the letter "○" (a circle!).

Suppose now that $L$ has $k$ exceptional points $P_1, \dots, P_k$. At $P_1$ we make an arbitrary opening, and we get another space $L_1$ which has at most two connected components. Each connected component lf $L_1$ satisfies the hypothesis of the claim, with at most $k-1$ exceptional points, then by induction, each one has a connected opening. If the resulting space is connected, we are done. If this opening of $L_1$ is not connected we undo the opening at $P_1$ that we started with, and as a result we get a space which satisfies the hypothesis of the claim with exactly *one* opening; again this one has a connected opening, which is a connected opening of $L$.

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