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Differentiable Conjugacies of Morse-Smale Diffeomorphisms

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Abstract. We provide in this paper a complete set of invariants for differentiable conjugacies of Morse-Smale diffeomorphisms. In the setting of topological conjugacies, these diffeomorphisms play an important role similar to that of gradients whose singularities are non-degenerate and have their stable and unstable manifolds in general position: they exhibit a simple and yet stable dynamics, and they exist on every manifold **[PS]**. Our main global invariant corresponds to a generalization of Mather's invariant for diffeomorphisms of the interval.

1. Introduction

In many branches of Mathematics, we often aim at obtaining global structures for the objects we are studying, once many examples and results about them are known. A central theme then is the classification problem: to give criteria to determine when objects in the category one is considering are isomorphic and to describe the set of their isomorphisms classes. In particular an important question is to determine the automorphism group of an object.

In (smooth, discrete) dynamical systems, the objects are pairs (M, f) formed by a smooth manifold (usually also compact, boundaryless and connected) and a smooth diffeomorphism f of M. An isomorphism between two objects (M, f), (M', f') called dynamical systems is here a *smooth diffeomorphism* h from M onto M' which conjugates f and f', i.e. that satisfies $hf = f'h$. The automorphism group of an object (M, f) is the *centralizer* of f in the group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M.

A weaker equivalence relation between (M, f) and (M', f') is that of topological equivalence: they are topologically conjugate, i.e. there is a *homeomorphism* $h^*: M \to M'$ such that $h^* f = f' h^*$. We also say in this case that

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(M, f) and (M', f') have the *same dynamics*. It is clearly necessary to have a good understanding of topological equivalence before trying the more difficult task of differentiable conjugacy. The simplest situation with regard to topological equivalence is, for a diffeomorphism $f \in \text{Diff}(M)$, to have an equivalence class which is *open* in $\text{Diff}(M)$: such diffeomorphisms are called *structurally stable*. Although we do not know at the moment how to characterize the C^∞ structurally stable diffeomorphisms of a manifold M, we know (Robbin, Robinson, Mañé [**R1, R2, M**]) what are the C^1 structurally stable diffeomorphisms, i.e the diffeomorphisms whose equivalence class for topological conjugacy is open in the C^1 topology: they are precisely those which satisfy Axiom A and the strong transversality condition, namely the limit set $L(f)$ is hyperbolic and the stable and unstable manifolds of points in $L(f)$ are all in general position. Let $\mathcal{A}(M)$ be the open subset of $\text{Diff}(M)$ formed by the C^1 structurally stable diffeomorphisms. Each equivalence class (for topological conjugacy) contained in $\mathcal{A}(M)$ is the union of a certain number of connected components of $\mathcal{A}(M)$. *The problem we are interested in is to describe the moduli space, for differentiable conjugacy, of such an equivalence class* (this is, for instance, the analogue of Teichmüller theory for compact Riemann surfaces, after a preliminary classification – by the genus, i.e. topological equivalence – has been made).

In this paper we will only consider the differentiable conjugacy problem for the non-empty open subset $\mathcal{MS}(M)$ of $\mathcal{A}(M)$ formed by the Morse-Smale diffeomorphisms of M: those whose limit set is formed by a finite number of hyperbolic periodic orbits, which have stable and unstable manifolds intersecting transversely. When $M = \mathbf{S}^1$ (or [0,1]), one has $\mathcal{MS}(\mathbf{S}^1) = \mathcal{A}(\mathbf{S}^1)$, and the differentiable conjugacy problem was settled by J. Mather (see [**Y**]).

In a forthcoming paper, we will study the differentiable conjugacy problem for *basic sets of diffeomorphisms* of *surfaces* (a basic set for a diffeomorphism f of a manifold M is a compact, invariant, hyperbolic, locally maximal subset of M): for example, horseshoes, D.A or Plykin attractors or Anosov diffeomorphisms of $\mathbf{T}^2$ (see [**PdM**]). This last case has already been considered by de la Llave, Marco and Moriyón ([**LMM**], [**MM**]), who show that the eigenvalue of the derivatives of the map at the periodic orbits form a complete set of invariants for differentiable conjugacy.

The principle of our differentiable classification is the same as in Mather's classification of hyperbolic diffeomorphisms of the interval and may perhaps be traced to Kopell [**K**]; the same principle underlies the Ecalle-Voronin classification of the germs of holomorphic diffeomorphisms at $(\mathbb{C}, 0)$ ([E], [V]), when this classification is considered from a geometrical point of view. What one does is first to classify, up to smooth conjugacy, the germs of diffeomorphisms at their periodic *sources* or *sinks*; this, in fact, gives a classification of the restrictions of these diffeomorphisms to the basins of their periodic sources or sinks. Then, a complete set of invariants for differentiable conjugacy is obtained by describing how to glue the stable basins of the periodic sinks to the unstable basins of the periodic sources.

The paper is organized as follows. In section 2, we introduce basic notations about the groups of jets of germs of diffeomorphisms of $(E, 0)$, E being a finite dimensional vector space. We recall some standard facts about these groups, especially the role of resonances in the (algebraic) conjugacy problem for such groups. In section 3, we recall how the classification of the stable basins of periodic sinks up to smooth conjugacy is seen to be equivalent to the description of the set of conjugacy classes in a finite dimensional Lie group (essentially determined by the resonances between the eigenvalues at the periodic sink). In section 4 we define our global invariant for Morse-Smale diffeomorphisms, which is, as mentioned above, the graph of the glueing map between the stable basins of the periodic sinks and unstable basins of the periodic sources. We then give a precise statement of our classification theorem.

Now comes an interesting point: in the Mather's or Ecalle-Voronin's situation, or even in our situation, when the diffeomorphisms do not have periodic saddles (north pole-south pole dynamics on spheres), the proof that the invariants one has defined form a complete set of invariants is really trivial, because the union of the stable basins of the sinks and the unstable basins of the sources is the whole space. However, for Morse-Smale diffeomorphisms with periodic saddles, this union is only open and dense, the complementary set being the union of the periodic saddles and their heteroclinic orbits. On the other hand, it is not clear at all that our set of invariants determine the differentiable type of the periodic saddles of the diffeomorphism, not even for example the spectra of the derivatives at these periodic saddles. What we need and prove is a kind of removable singularity theorem: in rough terms, the union of the periodic saddles and their heteroclinic orbits is "*too small*" (from the dynamical point of view) for the conjugacy, which is smooth outside this set, to be singular at some of its points. The precise statement is given in section (4.3).

In section 5 the local version of this removable singularity theorem is stated and proved: given two germs of diffeomorphisms near fixed saddle points, any smooth conjugacy, defined on complements of closed proper subsets of their stable manifolds, automatically extends to a full smooth conjugacy between the two germs. *We do not need to assume that the spectra of the derivatives of the diffeomorphisms at their fixed saddle points are equal: this follows from the existence of this partial conjugacy.*

Finally, we use this local version in section 6 to prove the global version stated in (4.3) and conclude that the set of conjugacy invariants defined for Morse-Smale diffeomorphisms is a complete set of invariants. It should also be mentioned that the set of invariants do not only provide necessary and sufficient conditions for two Morse-Smale diffeomorphims to be conjugate, but they also allow us to determine the centralizer of a Morse-Smale diffeomorphism, i.e. its automorphism group in the category of diffeomorphisms. In this direction we recall that in [PY] we proved that the centralizer is usually small: for an open and dense set of Morse-Smale diffeomorphisms the centralizer is reduced to the iterates of the diffeomorphism.

2. Jets

2.1 Let E be a real finite dimensional vector space. We denote by $D(E,0)$ the group of germs of C^∞ diffeomorphisms of E at 0 which fix 0. For $n \in \mathbb{N}$, let D_n be the normal subgroup of $D(E,0)$ whose elements are the germs having a contact of order at least $n+1$ with the identity, i.e. satisfying:

$$f(x) = x + o(\|x\|^n),$$

near 0 (where $\| \ \|$ is some norm on E).

Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and $D_\infty = \bigcap_{n\in\mathbb{N}} D_n$. For $n \in \overline{\mathbb{N}}$, the group $J_n(E)$ of $n-$jets is defined to be the quotient group $D(E,0)/D_n$. One has $D_0 = D(E,0)$, so $J_0(E) = \{1\}$ and $J_1(E)$ is canonically isomorphic to the linear group $\mathrm{GL}(E)$. For $n \in \mathbb{N}$, $J_n(E)$ is a (finite-dimensional) real Lie group; $J_{n+1}(E)$ is obtained from $J_n(E)$ by an abelian extension:

$$1 \to D_n/D_{n+1} \longrightarrow J_{n+1}(E) \longrightarrow J_n(E) \longrightarrow 1,$$

and $J_\infty(E)$ is the projective limit of the $(J_n(E))_{n\in\mathbb{N}}$.

2.2 Canonical representation of $J_n(E)$. Let $S(E)$ be the graded commutative symmetric tensor algebra of E, and $S^+(E)$ be the maximal ideal of $S(E)$; for $n \in \mathbb{N}$, denote by $S^n(E)$ the component of grade n of $S(E)$, so that $S^+(E) = \bigoplus_{n>0} S^n(E)$. For $n \geq 1$, let P_n be the set of maps from $\{1,\dots,n\}$ into itself; for $p \in P_n$, $i \in \{1,\dots,n\}$, let $I(p)$ be the image of p, $n(p)$ the cardinality of $I(p)$, $n_i(p)$ the cardinality of $p^{-1}(i)$.

The derivative of order $n \geq 1$ at 0 of a germ $F \in D(E,0)$ is considered as a linear map $D^nF\colon S^n(E) \to E$. The canonical representation $\Pi\colon D(E,0) \to \mathrm{GL}(S^+(E))$ is defined as follows; for $n \geq 1$, $x_1,\dots,x_n \in E$, $F \in D(E,0)$, let:

$$\widetilde{\prod}^n(F)(x_1,\dots,x_n) = \sum_{p\in P_n} \frac{(n-n(p))!}{n!} \prod_{i\in I(p)} D^{n_i(p)}F\left(\prod_{p(j)=i} x_j\right)$$

(where the products are taken in $S^+(E)$). This defines a $n-$ linear symmetric map $\widetilde{\prod}^n(F)\colon E^n \to S^+(E)$, and we take the associated linear map $\prod^n(F)\colon S^n(E) \to S^+(E)$ as the restriction of $\Pi(F)$ to $S^n(E)$. Formulas for derivatives of composed maps allow to check that Π is indeed a representation. For $n \geq 1$, $x_1,\dots,x_n \in E$ and $F \in D(E,0)$, the component of $\Pi(F)(\Pi x_i)$ in $S^m(E)$ is zero for $m > n$, equal to $\Pi DF(x_i)$ for $m = n$, and equal to $D^nF(\Pi x_i)$ for $m = 1$.

The kernel of Π is exactly D_∞; we denote again by Π the effective representation: $J_\infty(E) \to \mathrm{GL}(S^+(E))$. For $n \geq 1$, let $S_n(E) = \bigoplus_{m=1}^{n} S^m(E)$; this is an invariant subspace of the representation Π, and the kernel of the restriction of Π to this subspace is D_n. We obtain in this way as effective representation $\Pi_n\colon J_n(E) \to \mathrm{GL}(S_n(E))$, which is finite-dimensional. Observe that Π_1 is just the canonical identification: $J_1(E) \to \mathrm{GL}(E)$.

2.3 Semi-simple and unipotent jets. Let $n \geq 1$, $j \in J_n(E)$, then $\Pi_n(j)$ is semi-simple if and only if j is conjugate in $J_n(E)$ to the $n-$jet of a linear semi-simple automorphism of E. On the other hand $\Pi_n(j)$ is unipotent if and only if the image of j in $J_1(E) = \mathrm{GL}(E)$ is an unipotent automorphism.

For $j \in J_n(E)$, we denote by j_s (resp. j_u) the semi-simple (resp. unipotent) component of j in $J_n(E)$, so we have $j = j_s j_u = j_u j_s$. We can still define

$j_s, j_u \in J_\infty(E)$ for $j \in J_\infty(E)$, going to the projective limit. For a germ $F \in D(E,0)$, this means that we can find $H \in D(E,0)$, $A \in \mathrm{GL}(E)$, $F_1 \in D(E,0)$ such that:

(i) $HFH^{-1} = F_1 A$;

(ii) A is semi-simple;

(iii) $DF_1(0)$ is unipotent;

(iv) $F_1 A - AF_1$ is flat at 0.

2.4 Resonances. We have just seen that, up to conjugacy, every jet $j \in J^\infty(E)$ belongs to the centralizer in $J^\infty(E)$ of the ∞–jet of some semi-simple linear automorphism A of E. We denote by $Z(A)$ the centralizer of A in $J^\infty(E)$.

To describe $Z(A)$, we first complexify the objects: we denote by $E_\mathbb{C}$, $A_\mathbb{C}$, $J^\infty_\mathbb{C}(E)$ the complexifications of $E, A, J_\infty(E)$ and by $Z_\mathbb{C}(A)$ the centralizer of $A_\mathbb{C}$ in $J^\infty_\mathbb{C}(E)$; then $Z(A)$ will be equal to the subgroup of $Z_\mathbb{C}(A)$ formed by the real jets (i.e. invariant under complex conjugation). Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of $A_\mathbb{C}$ and $E_1, \dots, E_r$ the corresponding eigenspaces, so that $E_\mathbb{C} = \bigoplus_{i=1}^r E_i$. Associated to this decomposition of $E_\mathbb{C}$, one has a corresponding decomposition of $S^+(E_\mathbb{C})$:

$$S^+(E_\mathbb{C}) = \bigoplus_{n \in \mathbb{N}^r - \{0\}} \prod_{i=1}^r S^{n_i}(E_i),$$

where $S^{n_i}(E_i)$ is considered as a subspace of $S(E_\mathbb{C})$ and $\prod_{i=1}^r S^{n_i}(E_i)$ is the subspace of $S(E_\mathbb{C})$ generated by the products $\prod_{i=1}^r y_i$, $y_i \in S^{n_i}(E_i)$; actually, $\prod_{i=1}^r S^{n_i}(E_i)$ is a subspace of $S^{|n|}(E_\mathbb{C})$, with $|n| = \sum_{i=1}^r n_i$.

Let $p\colon S^+(E_\mathbb{C}) \to E_\mathbb{C}$ be the canonical projection. It is clear from the definition of the representation Π that $p\Pi_\mathbb{C}$ is an homeomorphism from $J^\infty_\mathbb{C}$ onto

$$\mathrm{GL}(E_\mathbb{C}) \times L\left(\bigoplus_{m=2}^\infty S^m(E_\mathbb{C}), E_\mathbb{C}\right).$$

Moreover, we have the decomposition:

$$L\left(\bigoplus_{m\geq 2} S^m(E_\mathbb{C}), E_\mathbb{C}\right) = \bigoplus_{\substack{n\in\mathbb{N}^r\\ |n|\geq 2}} \bigoplus_{j=1}^r L\left(\prod_{i=1}^r S^{n_i}(E_i), E_j\right).$$

The centralizer $Z_\mathbb{C}(A)$ is now determined by its image under $p\Pi_\mathbb{C}$; first, one defines the resonance set R_A of A as formed by the pairs $(n,j) \in \mathbb{N}^r \times \{1, \dots, r\}$ such that $|n| \geq 2$ and:

$$\prod_{i=1}^r \lambda_i^{n_i} = \lambda_j.$$

Then one has:

$$p\Pi_\mathbb{C}(Z_\mathbb{C}(A)) = \prod_{i=1}^r \mathrm{GL}(E_i) \times \bigoplus_{(n,j)\in R_A} L\left(\prod_{i=1}^r S^{n_i}(E_i), E_j\right).$$

As mentioned before, the centralizer $Z(A)$ is obtained by considering the real elements in $Z_\mathbb{C}(A)$ or in $p\Pi_\mathbb{C}(Z_\mathbb{C}(A))$.

The problems of conjugacy or centralizers for jets in $J_\infty(E)$ having semi-simple part A are now reduced to the same problems in the (generally much smaller) group $Z(A)$. More precisely, two jets $j, j' \in Z(A)$, with semi-simple part A, are conjugate in $J_\infty(E)$ if and only if they are conjugate in $Z(A)$; and the centralizer of j in $J_\infty(E)$ is contained in $Z(A)$. Indeed, a conjugacy between j and j' (with $j' = j$ for centralizers) must conjugate their semi-simple parts, hence it belongs to $Z(A)$.

3. Contractions

3.1 Let E be a real finite dimensional vector space, $\mathrm{Diff}(E)$ the group of C^∞ diffeomorphisms of E, and $\mathrm{Diff}(E,0)$ the subgroup formed by the diffeomorphisms which fix 0. We say that $f \in \mathrm{Diff}(E,0)$ is a contraction of E if the following conditions are satisfied:

(i) $\lim_{n\to+\infty} f^n(x) = 0, \quad \forall x \in E$;

(ii) all eigenvalues of Df at 0 have modulus strictly smaller than 1.

We now recall how the conjugacy problem for contractions, and the determination of their centralizers in $\mathrm{Diff}(E)$ are reduced to the same problems in finite dimensional Lie groups.

3.2 Let A be a linear semi-simple automorphism of E, whose eigenvalues (over $\mathbb{C}$) have modulus strictly smaller than 1. The resonance set R_A is finite, hence $Z(A)$ is a finite-dimensional Lie subgroup of $J_\infty(E)$. We embed $Z(A)$ into $\mathrm{Diff}(E,0)$ associating to a jet $j \in Z(A)$ the unique polynomial map from E to E whose jet at 0 is j. We still denote by $Z(A)$ the image subgroup in $\mathrm{Diff}(E,0)$. The main results about contractions are summarized in the following proposition.

Proposition.

1. *Let f be a contraction of E such that the semi-simple part of Df at 0 is equal to A. Then there exists $h \in \mathrm{Diff}(E,0)$ such that $h^{-1} \circ f \circ h$ belongs to $Z(A)$.*
2. *Let F_0, F_1 be two elements of $Z(A)$ having semi-simple part A. Then, any $G \in \mathrm{Diff}(E,0)$ which conjugates F_0 and F_1 in $\mathrm{Diff}(E,0)$ belongs to $Z(A)$.*

As special cases of the second part of the proposition, the centralizer in $\mathrm{Diff}(E)$ of a diffeomorphism $F \in Z(A)$ with semi-simple part A is contained in $Z(A)$; the centralizer of A in $\mathrm{Diff}(E)$ is exactly $Z(A)$.

3.3 We recall briefly the proof of the proposition. Let f be a contraction of E such that $Df(0)$ has semi-simple part A. By Borel's theorem and (2.3) above, we can assume that the ∞-jet of f at 0 belongs to $Z(A)$, hence we can write $f = f_0 + \varphi$, $f_0 \in Z(A)$ and φ flat at 0. With $f_t = f_0 + t\varphi$ for $0 \le t \le 1$, we look for a diffeomorphism h_t satisfying $f_t \circ h_t = h_t \circ f_0$, $h_0 = \mathrm{id}_E$, obtained by integrating a time-dependent vector field X_t flat at 0. The equation for X_t is

$$\varphi = X_t \circ f_t - Df_t X_t$$

with solution:

$$X_t = -\sum_{n=0}^{\infty} Df_t^{-n-1} \circ f_t^n(\varphi \circ f_t^n),$$

and this proves the first part of the proposition.

Under the assumptions in the second part, we know from (2.4) that the ∞-jet of G at 0 belongs to $Z(A)$. We may therefore assume that $F_0 = F_1$ (so G commutes with F_0) and that $G - \mathrm{id}_E = \psi$ is flat at 0. From the commutation equation:

$$\psi \circ f_0 = F_0 \circ G - F_0,$$

we see that for some constant $c > 0$ one has.

$$\|\psi(F_0(x))\| \ge c\,\|\psi(x)\|, \quad x \in E$$

and this is not compatible with the flatness of ψ at 0 unless $\psi \equiv 0$. □

3.4 Stable basins.

Definition. A *stable basin* of dimension n is a pair (V,F), where V is a C^∞ manifold diffeomorphic to the disjoint union of a finite number of copies of $\mathbb{R}^n$, and F is a C^∞ diffeomorphism of V having the following property: for any connected component V_0 of V, if $m \ge 1$ is the smallest integer such that $F^m(V_0) = V_0$, then $F^m \mid V_0$ is smoothly conjugate to a contraction in $\mathbb{R}^n$. There is a natural notion of isomorphism between stable basins, namely an isomorphism between (V,F) and (V',F') is a smooth diffeomorphism $H: V \to V'$ such that $HF = F'H$. A stable basin (V,F) is *irreducible* if F acts transitively on $\Pi_0(V)$, the number m of components of V is then the *period* of (V,F); if $h: \mathbb{R}^n \to V_0$ is a C^∞ diffeomorphism of $\mathbb{R}^n$ onto some component of V, sending 0 to the fixed point of $F^m \mid V_0$, then $h^{-1}(F^m \mid V_0)h$ is a contraction of $\mathbb{R}^n$, and the conjugacy class of $h^{-1}(F^m \mid V_0)h$ in $\mathrm{Diff}(\mathbb{R}^n,0)$ only depends on (V,F); it is called the *type* of (V,F).

A finite disjoint union of stable basins (of the same dimension) is a stable basin. Any stable basin is the disjoint union of its irreducible subspaces.

3.5 The classification of stable basins and their automorphisms groups.

Two isomorphic irreducible stable basins have the same period and type. Conversely, let (V,F), (V',F') be two irreducible stable basins having the same period m and the same type, let V_0 (resp. V_0') be a component of V (resp. V') and h_0 be a diffeomorphism from V_0 onto V_0' such that $h_0(F^m \mid V_0) = (F'^m \mid V_0')h_0$; then we define an isomorphism H of (V,F) onto (V',F') by

$$H \mid F^k(V_0) = F'^k h_0 F^{-k}, \qquad 0 \le k < m.$$

We, therefore, obtain the following proposition.

Proposition 1.

1. *Irreducible stable basins are classified by their period and type.*
2. *Let (V,F) be an irreducible stable basin, m its period, $\mathrm{Aut}(V,F)$ its automorphism group; choose a component V_0 of V and denote by Z_0 the centralizer of F^m/V_0 in $\mathrm{Diff}(V_0)$. Then one has an exact sequence*
$$1 \longrightarrow Z_0 \longrightarrow \mathrm{Aut}(V,F) \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 1,$$
where Z_0 embeds as the subgroup of $\mathrm{Aut}(V,F)$ which leaves V_0 invariant, and $H(V_0) = F^n(V_0)$ if the image of $H \in \mathrm{Aut}(V,F)$ in $\mathbb{Z}/m\mathbb{Z}$ is the class of n.

By the proposition in (3.2), Z_0 is a finite-dimensional real Lie group, hence the same is true for $\mathrm{Aut}(V,F)$. Let Σ be the set of conjugacy classes of contractions in $\mathbb{R}^n$, and $\Sigma^* = \Sigma \times \mathbb{N}^*$; we can consider Σ^* as the set of isomorphism classes of irreducible stable basins of dimension n.

For a (non-necessarily irreducible) stable basin (V,F), and $\sigma \in \Sigma^*$, let V_σ be the disjoint union of the irreducible components of (V,F) which belong to σ and $n(\sigma)$ the number of these irreducible components.

Proposition 2.

1. *The isomorphism type of a stable basin (V,F) is determined by the family $(n(\sigma))_{\sigma\in\Sigma^*}$.*
2. *The automorphism group of a stable basin is the direct product of the automorphism groups of its isotypic components V_σ; namely, any automorphism must preserve each isotypic component.*
3. *Let (V,F) be an isotypic stable basin, isomorphic to the disjoint union of n copies of an irreducible stable basin (V_0,F_0). We have an exact sequence:*
$$1 \longrightarrow [\mathrm{Aut}(V_0,F_0)]^n \longrightarrow \mathrm{Aut}(V,F) \longrightarrow S_n \longrightarrow 1,$$
where S_n is the symmetric group of order n, $[\mathrm{Aut}(V_0,F_0)]^n$ embeds as the subgroup of $\mathrm{Aut}(V,F)$ preserving each irreducible component of (V,F), and an automorphism $H \in \mathrm{Aut}(V,F)$ acts on the set of these irreducible components according to its image in S_n.

Proofs are immediate. Propositions 1 and 2 show that $\mathrm{Aut}(V,F)$ is a finite-dimensional real Lie group, whose determination is reduced to the determination of centralizers in the Lie groups $Z(A)$ introduced in (3.2). The number of connected components of $\mathrm{Aut}(V,F)$ is finite.

4. A complete set of conjugacy invariants for Morse-Smale diffeomorphisms

4.1 The local invariants.

Let (M,f) be a pair formed by a compact connected C^∞ manifold of dimension n and a C^∞ Morse-Smale diffeomorphism f of M. We want to classify these pairs up to C^∞ conjugacies.

For any periodic point p of f, we denote by $W^s(p)$ (resp. $W^u(p)$) the stable (resp. unstable) manifold of p. The union of the stable (resp. unstable) manifolds of all the periodic sinks (resp. sources) of f is an open and dense subset of M which we denote by $V^s(f)$ (resp. $V^u(f)$). This subset is also invariant under f, and $(V^s(f),f)$ (resp. $(V^u(f),f^{-1})$) is a stable basin of dimension n, whose automorphism group is denoted by $Z^s(f)$ (resp. $Z^u(f)$).

The product $J(f) = V^s(f) \times V^u(f)$ is called the *Jacobian space* of f. The product $Z(f) = Z^s(f) \times Z^u(f)$ of the automorphism groups of $(V^s(f),f)$ and $(V^u(f),f^{-1})$ is called the *structural group* of f. It acts effectively on $J(f)$. Considered as an element of $Z(f)$, the diffeomorphism f is called the *special element* of $Z(f)$ (or automorphism of $J(f)$). The isomorphism types, as stable basins, of $(V^s(f),f)$ and $(V^u(f),f^{-1})$ are the local invariants of (M,f). The structural group $Z(f)$ is determined by these invariants; it is a finite-dimensional real Lie group with a finite number of connected components.

4.2 The global invariant.

With (M,f) as above, let $V(f) = V^s(f) \cap V^u(f)$. Then $V(f)$ embeds diagonally in the Jacobian space $J(f)$ and we denote the image by $\Theta(f)$. We observe that $\Theta(f)$ is a n-dimensional submanifold of the $2n$-dimensional manifold $J(f)$, which is invariant under the special automorphism in $Z(f)$. The submanifold $\Theta(f)$, or more precisely the orbit of this submanifold under the action of the structural group $Z(f)$, is the *global invariant* of f.

Theorem 1. *Let (M,f), (M',f') be two pairs as above. Then, there exists a smooth conjugacy $h\colon M \longrightarrow M'$ between f and f' if and only if there* *exist isomorphisms* $H_s: (V^s(f), f) \to (V^s(f'), f')$ *and* $H_u: (V^u(f), f^{-1}) \to (V^u(f'), f'^{-1})$ *of stable basins such that* $H_s \times H_u(\Theta(f)) = \Theta(f')$.

In other words, necessary and sufficient conditions for smooth conjugacy are the following:

(i) the stable basins $(V^s(f), f)$ and $(V^s(f'), f')$ are isomorphic;

(ii) the unstable basins $(V^u(f), f^{-1})$ and $(V^u(f'), f'^{-1})$ are isomorphic;

(iii) choosing some isomorphisms $h_s: (V^s(f), f) \to (V^s(f'), f')$ and $h_u (V^u(f), f) \to (V^u(f'), f')$, there exists an element g in the structural group $Z(f')$ of f' such that:

$$h_s \times h_u(\Theta(f)) = g \cdot \Theta(f').$$

Theorem 2. *Let* (M, f) *be as above. The centralizer of* f *in* $\mathrm{Diff}(M)$ *is canonically isomorphic to the stabilizer of* $\Theta(f)$ *for the action of the structural group* $Z(f)$ *on* $J(f)$.

The stabilizer of $\Theta(f)$ means here the subgroup of those $g \in Z(f)$ such that $g \cdot \Theta(f) = \Theta(f)$. The image of f under this canonical automorphism is the special automorphism in $Z(f)$.

4.3 A removable singularity theorem for conjugacies. The proofs of theorems 1 and 2 are easily reduced to the following result, which will be proved in the next sections.

Let (M, f), (M', f') two pairs as above. We define:

$$U(f) = V^s(f) \cup V^u(f), \quad U(f') = V^s(f') \cap V^u(f'),$$

so that $M - U(f)$ is the nowhere dense, closed, f-invariant subset of M consisting of the saddle periodic points of f and the heteroclinic orbits associated to these saddle periodic points.

Theorem 3. *Let* $h: U(f) \to U(f')$ *be a* C^∞ *diffeomorphism satisfying* $hf = f'h$. *Then* h *extends (uniquely) to a smooth conjugacy* $M \to M'$ *between* f *and* f'.

This theorem explains why we did not have to consider, in our set of conjugacy invariants, the local invariants (such as eigenvalues of the derivative) at the *periodic saddle points*. This type of information is in fact contained in the global invariant $\Theta(f)$.

One half of theorem 1 is trivial: with (M, f), (M', f') as in the theorem, a smooth conjugacy $h: M \to M'$ between f and f' defines, by restriction, isomorphisms H_s, H_u as desired such that $H_s \times H_u(\Theta(f)) = \Theta(f')$. This also gives one half of theorem 2, namely that the centralizer of f in $\mathrm{Diff}(M)$ embeds into the stabilizer of $\Theta(f)$ in $Z(f)$.

Conversely, given isomorphisms $H_s: (V^s(f), f) \to (V^s(f'), f')$ and $H_u: (V^u(f), f^{-1}) \to (V^u(f'), f'^{-1})$, the relation $H_s \times H_u(\Theta(f)) = \Theta(f')$ means that H_s and H_u coincide on the intersection $V(f) = V^s(f) \cap V^u(f)$, so they define a diffeomorphism $h: U(f) \to U(f')$ which conjugates f and f'. Therefore the second halves of theorems 1 and 2 are consequence of theorem 3. The next section is essentially devoted to a local version of theorem 3. The theorem itself will be proved in section 6.

5. Saddles

5.1 Let E be a finite-dimensional real vector space of dimension at least 2, and $E = E^s \bigoplus E^u$ a nontrivial decomposition of E into a direct sum of linear subspaces E^s, E^u.

We consider in this section C^∞ diffeomorphisms f from a neighbourhood of $E^s \cup E^u$ onto a neighbourhood of $E^s \cup E^u$, such that $f(E^s) = E^s$ and $f(E^u) = E^u$. If we identify two diffeomorphisms which coincide in a neighbouhood of $E^s \cup E^u$, we obtain a group of germs of diffeomorphisms which we denote by $D(E^s, E^u)$.

An element f of $D(E^s, E^u)$ is called a saddle if the following properties hold:

(i) $Df(0)$ is hyperbolic, with stable eigenspace equal to E^s and unstable eigenspace equal to E^u;

(ii) $\lim_{k \to +\infty} f^k(x) = 0, \quad \forall x \in E^s$;

(iii) $\lim_{k \to -\infty} f^k(x) = 0, \quad \forall x \in E^u$.

The stable manifold (resp. unstable manifold) of 0 is therefore equal to E^s

(resp. E^u).

The main result concerning the classification of saddles up to C^∞ conjugacy is due to Sternberg [S]:

Theorem. *Two saddles in $D(E^s, E^u)$ are conjugated in $D(E^s, E^u)$ if and only if their ∞-jets at 0 are conjugated in $J_\infty(E)$.*

5.2 Let f be a saddle in $D(E^s, E^u)$, and $A \in \mathrm{GL}(E)$ the semi-simple part of $Df(0)$. According to (2.3), we can assume (after possibly conjugating f in $D(E^s, E^u)$) that the ∞-jet of f at 0 commutes in $J_\infty(E)$ with A. Actually, something stronger is true:

Lemma. *There exists $h \in D(E^s, E^u)$, with $h - \mathrm{id}_E$ flat at 0, such that hfh^{-1} commutes with A in $D(E^s, E^u)$.*

Proof. Let $\psi = f \circ A - A \circ f$, which is flat at 0; one can solve $\psi = A \circ \varphi - \varphi \circ A$, with a C^∞ map φ flat at 0; then $f + \varphi$ belongs to $D(E^s, E^u)$, commutes with A, and is conjugated to f in $D(E^s, E^u)$ by Sternberg's theorem. □

5.3 Let f be a saddle in $D(E^s, E^u)$, commuting in $D(E^s, E^u)$ with the semi-simple part A of $Df(0)$. We denote by $Z(f)$ the centralizer of f in $D(E^s, E^u)$.

The image of $Z(f)$ into $J_\infty(E)$ is equal to the centralizer of the ∞-jet of f at 0 in $J_\infty(E)$; in particular, it is contained in the group $Z(A)$ studied in (2.4). For contractions, this map: $Z(f) \to Z(A)$ is injective; this is no longer true for saddles, the kernel being even always infinite-dimensional. On the other hand, from the study of contractions, one knows that the restriction of any diffeomorphism g to E^s (resp. E^u) belongs to a certain finite-dimensional Lie group of polynomial diffeomorphisms of E^s (resp. E^u) which is completely determined by the ∞-jet of f at 0. Something slightly stronger is actually true:

Lemma. *For any $k \geq 1$, any $g \in Z(f)$, the restrictions of $D^k g$ to E^s and E^u are polynomial maps:*

$$D^k g/E^s: \quad E^s \longrightarrow L(S^k(E), E)$$
$$D^k g/E^u: \quad E^u \longrightarrow L(S^k(E), E)$$

determined by the ∞-jet of g at 0.

In particular, if the ∞-jet of g at 0 is $1_{J_\infty(E)}$, then not only g is the identity on E^s and E^u but the ∞-jet of g at any point of E^s or E^u is the jet of the identity map.

Proof. Let k and g as in the lemma. We will prove that for ℓ a big enough integer (depending only on A and k), we have:

$$D^{k+\ell} g_x \left(\prod_{i=1}^{k} v_i \times \prod_{j=1}^{\ell} w_j \right) = 0,$$

for $x \in E^s$, $v_i \in E$, $w_j \in E^s$. This, together with the corresponding statement for E^u, clearly implies the conclusion of the lemma.

For $h\colon U \to V$ a C^∞ diffeomorphism between open sets in E, and $x_0 \in U$, we define:

$$h_{x_0}(t) = h(x_0 + t) - h(x_0);$$

then $h_{x_0} \in D(E, 0)$ and we denote by $j_n(h, x_0) \in J_n(E)$ $(n \in \overline{\mathbb{N}})$ and $\Pi_n(h, x_0) \in \mathrm{GL}(S_n(E))$ the corresponding n-jets and automorphisms (see (2.2)).

We choose a norm $|\ \ |$ on E and constants $0 < c < 1 < C$ such that:

$$|A(w)| \leq c\,|w|, \quad |A(v)| \leq C\,|v|, \quad \left|A^{-1}(v)\right| \leq C\,|v|,$$

for $v \in E$, $w \in E^s$.

On the other hand, we write $f = Af_1 = f_1 A$, where $\Pi_{k+\ell}(f_1, 0)$ is unipotent for any $\ell \geq 0$; we fix some integer $\ell \geq 0$ to be determined later and choose a norm $\|\ \ \|$ on $S_{k+\ell}(E)$ such that

$$\left\|\Pi_{k+\ell}(f_1^{\pm 1}, 0)\right\| < 2.$$

Next, we choose a neighbourhood U of 0 in E and a constant $K > 1$ such that:

$$\left\|\Pi_{k+\ell}(f_1^{\pm 1}, x)\right\| < 3, \qquad x \in U;$$
$$\left\|\prod_{i=1}^{k+\ell} v_i\right\| \leq K \prod_{i=1}^{k+\ell} |v_i|, \qquad v_i \in E;$$
$$\|\Pi_{k+\ell}(g, x)\| \leq K, \qquad x \in U;$$
$$|p(v)| \leq K\,\|v\|, \qquad v \in S_{k+\ell}(E),$$

where p is the canonical projection $p\colon S_{k+\ell}(E) \to E$. Observe that U, K depend on ℓ but not c, C.

We now write $g = f^{-n} g f^n = A^{-n} f_1^{-n} g f_1^n A^n$, near a point $x \in E^s$, where $n \geq 0$ is big enough to have $f_1^m A^n(x) \in U$ for $0 \leq m \leq n$ and $f_1^{-m} g f_1^n A^n(x) \in U$ for $0 \leq m \leq n$. Then one has, for $v_1, \dots, v_k \in E$, $w_1, \dots, w_\ell \in E^s$:

$$\begin{aligned}\left|D^{k+\ell} g_x(\Pi v_i \times \Pi w_j)\right| &\leq C^n \left|D^{k+\ell}(f_1^{-n} g f_1^n A^n)_x(\Pi v_i \times \Pi w_j)\right| \\ &\leq C^n K \left\|\Pi_{k+\ell}(f_1^{-n} g f_1^n A^n, x)(\Pi v_i \times \Pi w_j)\right\| \\ &\leq C^n K^2 3^{2n} \left\|\Pi_{k+\ell}(A^n, x)(\Pi v_i \times \Pi w_j)\right\| \\ &\leq K^3 (C^{k+1} 3^2 c^\ell)^n \prod_{i=1}^{k} |v_i| \prod_{j=1}^{\ell} |w_j| ;\end{aligned}$$

choosing ℓ such that $9C^{k+1}c^\ell < 1$ and letting n go to infinity, we get the desired conclusion. □

We will not use the lemma until later, but the line of argument used in the proof will now give us a more striking result.

5.4 Removable singularities for conjugacies of saddles. Let $f, f' \in D(E_s, E_u)$: be two saddles. Assume we are given neighbourhoods V, V' of $E^s \cup E^u$, a closed f-invariant (resp. f'-invariant) subset F (resp. F') of E^s, *distinct from* E^s, and a C^∞ diffeomorphism h from $V - F$ onto $V' - F'$ such that:

(i) $h(E^s - F) = E^s - F'$;

(ii) $h(E^u - \{0\}) = E^u - \{0\}$;

(iii) $h \circ f = f' \circ h$ in $W - F$, with W some neighbourhood of $E^s \cup E^u$.

Theorem. *Under these assumptions, h extends (uniquely) in a C^∞ diffeomorphism: $V \to V'$ which conjugates f and f' in $D(E^s, E^u)$.*

Proof. We claim that it is sufficient to show that for any $k \geq 0$, $D^k h | (V - F)$ extends to a continuous function from V into $L(S^k(E), E)$. Indeed, it will then follow from Taylor's formula that h extends to a C^∞ map: $V \to V'$, necessarily unique as $V - F$ dense in V, and conjugating f and f' for the same reason. As we can interchange f, f', replacing h by h^{-1}, the extension will be a C^∞ diffeomorphism, as desired.

Next, to show that $D^k h \mid V - F$ extends continuously, it is sufficient to prove that there exists an integer ℓ such that:

$$D^{k+\ell} h_{x+y} \left(\prod_{i=1}^{k} v_i \times \prod_{i=1}^{\ell} w_j \right)$$

goes to 0 as $y \in E^u - \{0\}$ goes to 0, uniformly when x, w_j belong to bounded subsets of E^s and v_i belong to a bounded subset of E. Indeed, we then get the desired extension by integrating ℓ times $D^{k+\ell} h$ in the E^s–direction from a point $x_0 \in E^s - F$ (in the neighbourhood of which h is smooth).

To prove the last assertion, we proceed in a way similar to the proof of (5.3). We may assume (at the beginning of the proof) that f (resp. f') commutes in $D(E^s, E^u)$ with the semi-simple part A (resp. A') of $Df(0)$ (resp. $Df'(0)$). We choose a norm $|\ \ |$ on E and constant $0 < c < 1 < C$ such that:

$$|A(w)| \leq c\,|w|\,, \quad |A(v)| \leq C\,|v|\,, \quad \left|A'^{-1}(v)\right| \leq C\,|v|\,, \tag{1}$$

for $v \in E$, $w \in E^s$.

Let B be a bounded subset of E^s, we choose $\varepsilon > 0$ and a compact subset G of $V - E^s$ such that for $x \in B$, $y \in E^u$ with $0 < |y| < \varepsilon$, one has $f^n(x+y) \in G$ for some integer $n = n(x+y)$; observe that $n(x+y)$ goes to infinity (uniformly in x) when $|y|$ goes to 0.

We write $f = Af_1 = f_1 A$, $f' = A'f_1' = f_1'A'$ and fix an integer ℓ, to be determined later.

As $\Pi_{k+\ell}(f_1, 0)$ and $\Pi_{k+\ell}(f_1', 0)$ are unipotent, we can find norms $\|\ \ \|$, $\|\ \ \|'$ on $S_{k+\ell}(E)$ such that:

$$\begin{aligned}\|\Pi_{k+\ell}(f_1, 0)\| &\leq 2 \\ \left\|\Pi_{k+\ell}(f_1'^{-1}, 0)\right\|' &\leq 2.\end{aligned}$$

Next, we choose a neighbourhood U of 0 in E and a constant $K > 1$ such that:

$$\|\Pi_{k+\ell}(f_1, x)\| \leq 3, \qquad x \in U; \tag{2}$$

$$\left\|\Pi_{k+\ell}(f_1'^{-1}, x)\right\|' \leq 3, \qquad x \in U; \tag{3}$$

$$\|\Pi_{k+\ell}(A)\| \leq K; \tag{4}$$

$$\left\|\Pi_{k+\ell}(A'^{-1})\right\|' \leq K; \tag{5}$$

$$\|\Pi_{k+\ell}(h,z)\| \leq K, \qquad z \in G; \tag{6}$$

$$\|v\|' \leq K\,\|v\|, \qquad v \in S_{k+\ell}(E); \tag{7}$$

$$\left\|\prod_{i=1}^{k+\ell} v_i\right\| \leq K \prod_{i=1}^{k+\ell} |v_i|, \qquad v_i \in E; \tag{8}$$

$$|p(v)| \leq K\,\|v\|', \qquad v \in S_{k+\ell}(E), \tag{9}$$

where p is the canonical projection: $S_{k+\ell}(E) \to E$.

Let $x \in B$, $y \in E^u - \{0\}$ with $|y| < \varepsilon$, $n = n(x+y)$ and m be the integer part of $(n(x+y))^{1/2}$; if $|y|$ is small enough, the points $f_1^j A^{n-m}(x+y)$ and $f_1'^{-j} A'^{-m} h f^n(x+y)$ belong to U for $0 \leq j \leq n$.

Under this assumption, we write:

$$h = f'^{-n} h f^n = A'^{m-n} f_1'^{-n} A'^{-m} h A^m f_1^n A^{n-m}$$

and with $v_1, \ldots, v_k \in E$, $w_1, \ldots, w_\ell \in E^s$, $|v_i| = |w_j| = 1$ ($1 \leq i \leq k$, $1 \leq j \leq \ell$), we write:

$$V_0 = \prod_{i=1}^{k} v_i \times \prod_{j=1}^{\ell} w_j \in S^{k+\ell}(E) \subset S_{k+\ell}(E),$$

$$V_1 = \Pi_{k+\ell}(A^{n-m})(V_0) \in S^{k+\ell}(E) \subset S_{k+\ell}(E),$$
$$V_2 = \Pi_{k+\ell}(f_1^n, A^{n-m}(x+y))(V_1) \in S_{k+\ell}(E),$$
$$V_3 = \Pi_{k+\ell}(A^m, f^n A^{-m}(x+y))(V_2) \in S_{k+\ell}(E),$$
$$V_4 = \Pi_{k+\ell}(h, f^n(x+y))(V_3) \in S_{k+\ell}(E),$$
$$V_5 = \Pi_{k+\ell}(A'^{-m}, f'^n h(x+y))(V_4) \in S_{k+\ell}(E),$$
$$V_6 = \Pi_{k+\ell}(f_1'^{-n}, f_1'^n A'^{n-m} h(x+y))(V_5) \in S_{k+\ell}(E),$$
$$V_7 = p(V_6) \in E,$$
$$V_8 = A'^{m-n}(V_7) \in E.$$

Then we have the following estimates:

$$\|V_1\| \leq K(C^k c^\ell)^{n-m}, \qquad \text{by (1) and (8)};$$
$$\|V_2\| \leq 3^n\,\|V_1\|, \qquad \text{by (2)};$$
$$\|V_3\| \leq K^m\,\|V_2\|, \qquad \text{by (4)};$$
$$\|V_4\|' \leq K\,\|V_4\| \leq K^2\,\|V_3\|, \qquad \text{by (6) and (7)};$$
$$\|V_5\|' \leq K^m\,\|V_4\|', \qquad \text{by (5)};$$
$$\|V_6\|' \leq 3^n\,\|V_5\|', \qquad \text{by (3)};$$
$$|V_7| \leq K\,\|V_6\|', \qquad \text{by (9)};$$
$$|V_8| \leq C^{n-m}\,|V_7|, \qquad \text{by (1)}.$$

Putting this together, we get:

$$\left|D^{k+\ell} h_{x+y}(V_0)\right| \leq K^4 (9C^{k+1}c^\ell)^n (K^2 C^{-k-1} c^{-\ell})^m.$$

We choose ℓ such that $9C^{k+1}c^\ell < 1$. Then, as $|y|$ goes to 0 (so $n(x+y)$ goes to infinity), the right-hand term in the last estimate goes to zero; this gives the result we were looking for. □

6. Proof of theorem 3

6.1 Correspondence between periodic points.

Let (M,f), (M',f'), $U(f), U(f')$ and $h: U(f) \to U(f')$ be as in the hypothesis of theorem 3.

We choose a multiple N of all the periods of the periodic points of f or f' and let $F = f^N, F' = f'^N$.

We order the set P of fixed points of F by the relation:

$$p \leq q \iff W^s(p) \bigcap W^u(q) \neq \varnothing,$$

and do the same for the set P' of fixed points of F'. For p in P (or P') one has:

$$\overline{W^s(p)} = \bigcup_{q \geq p} W^s(q),$$

$$\overline{W^u(p)} = \bigcup_{q \leq p} W^u(q).$$

Lemma. *There exists an isomorphism $h_*: P \to P'$ of ordered sets such that, for any $p \in P$:*

$$h(W^s(p) \bigcap U(f)) = W^s(h_*(p)) \bigcap U(f'),$$
$$h(W^u(p) \bigcap U(f)) = W^u(h_*(p)) \bigcap U(f').$$

Proof. For $x \in U(f)$, let $A^s(x)$ be the set of $y \in U(f)$ such that for any $\varepsilon > 0$, $k_0 \geq 0$ there exist $k \geq k_0$, x' in an ε−neighbourhood of x and y' in an ε−neighbourhood of y with $y' = F^k(x')$. We define similarly $A^u(x)$ with F^{-1} instead of F and $A^s(x'), A^u(x')$ for $x' \in U(f')$ relatively to F', F'^{-1}.

For $x \in W^s(p)$, $p \in P$, one has:

$$A^s(x) = \overline{W^u(p)} \bigcap U(f) = \left(\bigcup_{q \leq p} W^u(q) \right) \bigcap U(f),$$

hence the set $A^s(x)$ determines to which stable manifold x belongs. On the other hand, as h is a conjugacy between F and F' on $U(f)$, one has:

$$h(A^s(x)) = A^s(h(x)), \qquad x \in U(f).$$

These two relations show that there exists a map h_*^s from P to P' satisfying:

$$h(W^s(p) \bigcap U(f)) \subset W^s(h_*^s(p)) \bigcap U(f'), \qquad p \in P;$$

exchanging F and F', we see that h_*^s is a bijection and that we have equality in the last relation. The formula above for $A^s(x)$ also shows that h_*^s and $(h_*^s)^{-1}$ are order preserving.

In the same way, we may define h_*^u, with similar properties. To see that in fact $h_*^s = h_*^u$, we observe that $h_*^s(p) \geq h_*^u(p)$ for any $p \in P$: indeed, for $x \in W^s(p) \cap U(f)$, $y \in W^u(p) \cap U(f)$, one has $y \in A_s(x)$, $h(x) \in W^s(h_*^s(p))$, $h(y) \in W^u(h_*^u(p))$ and:

$$h(y) \in A^s(h(x)) = U(f') \bigcap \left(\bigcup_{q \leq h_*^s(p)} W^u(q) \right).$$

Similarly $(h_*^s)^{-1}(p') \geq (h_*^u)^{-1}(p')$ for $p' \in P'$. Therefore h_*^s is equal to h_*^u, which we indicate simply by h_*. □

6.2 We enumerate the points $p_1, \ldots, p_{r+s+t}$ of P in such a way that $p_1, \ldots, p_r$ are the sinks, $p_{r+s+1}, \ldots, p_{r+s+t}$ are the sources and $p_i \leq p_j$ happens only for $i \leq j$.

Let $p'_i = h_*(p_i)$ for $1 \leq i \leq r+s+t$ and define, for $0 \leq \ell \leq s$:

$$U_\ell = U(f) \cup \left(\bigcup_{j=1}^{\ell} W^s(p_{r+j}) \right),$$

$$U'_\ell = U(f') \cup \left(\bigcup_{j=1}^{\ell} W^s(p'_{r+j}) \right).$$

Then U_ℓ is open (and dense) in M, it is invariant under F, and we have $U_0 = U(f)$, $U_s = M$. Similar properties hold for U'_ℓ.

Theorem 3 will follow if we show that, for $0 \leq \ell \leq s$, h extends to a smooth diffeomorphism between U_ℓ and U'_ℓ: this extension will be a conjugacy between f and f' by continuity.

The proof is by induction on ℓ, the case $\ell = 0$ being the hypothesis of theorem 3. We now assume that h is a smooth diffeomorphism between $U_{\ell-1}$ and $U'_{\ell-1}$ (with $1 \leq \ell \leq s$). With $p = p_{r+\ell}$, $p' = p'_{r+\ell}$, we will extend h to a smooth diffeomorphism from $U_\ell = U_{\ell-1} \cup W^s(p)$ onto $U'_\ell = U'_{\ell-1} \cup W^s(p')$.

Observe that $W^u(p) - \{p\}$ (resp. $W^u(p') - \{p'\}$) is contained in $\bigcup_{q<p} W^s(q)$ (resp. $\bigcup_{q'<p'} W^s(q')$) and, hence, in $U_{\ell-1}$ (resp. $U'_{\ell-1}$). We claim that the restriction of h to $W^u(p) - \{p\}$ is a smooth diffeomorphism onto $W^u(p') - \{p'\}$. Indeed, one has $h(W^u(p) \cap U(f)) = W^u(p') \cap U(f')$ by definition of p', and this implies, as $W^u(p) \cap U(f)$ is dense in $W^u(p)$, that

$$h(W^u(p) - \{p\}) \subset \overline{W^u(p')} = \bigcup_{q' \leq p'} W^u(q').$$

For the same reasons, for $q' < p'$, one has:

$$h^{-1}(W^u(q')) \subset \bigcup_{q \leq h_*^{-1}(q')} W^u(q).$$

This implies that $h(W^u(p) - \{p\}) \subset W^u(p') - \{p'\}$, and we have in fact equality because we can exchange F and F', proving the claim.

Denote by n the dimension of M and by k the dimension of $W^u(p)$. As U_ℓ is open in M, contains $W^s(p)$ and $W^u(p)$, and these manifolds do not accumulate onto themselves, we can choose a neighbourhood $W \subset U_\ell$ of $W^s(p) \cup W^u(p)$

invariant under F and an embedding K of W into $\mathbb{R}^n = \mathbb{R}^{n-k} \oplus \mathbb{R}^k$ such that:

$$K(W^s(p)) = \mathbb{R}^{n-k},$$
$$K(W^u(p)) = \mathbb{R}^k.$$

We choose similarly W', K' for p', F'. Then, with the notations of (5.1), $\Phi = KFK^{-1}$ and $\Phi' = K'F'K'^{-1}$ are saddles in $D(\mathbb{R}^{n-k}, \mathbb{R}^k)$.

The set $\mathcal{F} = K(U_\ell - U_{\ell-1})$ is a closed subset of $\mathbb{R}^{n-k}$, invariant under Φ, distinct from $\mathbb{R}^{n-k}$ (actually with empty interior in $\mathbb{R}^{n-k}$); the same is true for $\mathcal{F}' = K'(U'_\ell - U'_{\ell-1})$ in relation to Φ'.

Now, there exist open neighbourhoods V, V' of $\mathbb{R}^k \cup \mathbb{R}^{n-k}$ such that $H = K'hK^{-1}$ is a smooth diffeomorphism from $V - \mathcal{F}$ onto $V' - \mathcal{F}'$ and is a (partial) conjugacy between ϕ and ϕ'. We are in position to apply the theorem in (5.4) and conclude that H extends to a smooth diffeomorphism from V onto V'; this in turn implies that h extends to a smooth diffeomorphism from U_ℓ onto U'_ℓ.

This finishes the induction step, and also the proof of theorem 3. □

6.3 Rigidity of centralizers.

We recall that in [PY], we showed that a C^∞ diffeomorphism of a compact connected manifold which commutes with a C^1 structurally stable diffeomorphism, and is equal to the identity on some non-empty open set, must be equal to the identify on the whole manifold.

Using the lemma in (5.3), we are here able to prove a stronger rigidity property when the diffeomorphism commutes with a Morse-Smale diffeomorphism.

Proposition. *Let M be a compact connected manifold, and g a C^∞ diffeomorphism of M. We assume that there exists a fixed point x_0 of g where the ∞−jet of g is the jet of the identity, and that g commutes with a C^∞ Morse-Smale diffeomorphism f of M. Then g is equal to the identity on M.*

Proof We distinguish two cases:

1) *x_0 belongs to the stable manifold of a periodic sink or to the unstable manifold of a periodic source of f.*

Assume for example that $x_0 \in W^s(p)$, with p being a periodic sink of f of period m. We have seen in section 3 that one can find a smooth diffeomorphism H from $W^s(p)$ onto $\mathbb{R}^n$ such that the centralizer of Hf^mH^{-1} in $\mathrm{Diff}(\mathbb{R}^n)$ is a group of polynomial diffeomorphisms. Such diffeomorphisms are determined by their jets at any point. Now $gf^m = f^mg$, hence $g(W^s(p)) = W^s(p)$ and $HgH^{-1} = \mathrm{id}_{\mathbb{R}^n}$. Finally, $g = \mathrm{id}_M$ by our previous rigidity theorem.

2) *x_0 belongs to the stable manifold of a periodic saddle p of f, of period m.*

As $g(x_0) = x_0$, we have $g(p) = p$ and $g(W^s(p)) = W^s(p)$, $g(W^u(p)) = W^u(p)$. Using twice the lemma in (5.3), we first conclude that the ∞−jet of g at p is the jet of the identity, and second that g fixes every point x'_0 in $W^s(p) \cup W^u(p)$, with the supplementary property that the ∞−jet of g at x'_0 is the jet of the identity. Choosing x'_0 in the stable manifold of a periodic sink of f, we are back to case (1).□

References

[E] J. Ecalle, *Théorie des invariants holomorphes*, Public. Math. d'Orsay 67, 7409 (1974).

[K] N. Kopell, *Commuting diffeomorphisms*, Proc. AMS Symp. in Pure Math. 14 (1970), 165-184.

[LMM] R. de la Llave, J.M.Marco and R. Moriyón, *Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation*, Annals of Math. 123 (1986), 537-611.

[M] R. Mañé, *A proof of the C^1 stability conjecture*, Publ. Math. IHES 66 (1987), 161-210

[MM] J. M. Marco and R. Moriyón, *Invariants for smooth conjugacy of hyperbolic dynamical systems, III*. Commun. Math. Phys. 112, 317-333 (1978).

[PdM] J. Palis and W. de Melo, "Geometric Theory of Dynamical Systems", Springer-Verlag, N.Y. (1982).

[PS] J. Palis and S. Smale, *Structural stability theorems*, Proc. AMS Symp. in Pure Math. 14 (1970), 223-231.

[PY] J. Palis and J. C. Yoccoz, *Rigidity of centralizers of diffeomorphisms*, Ann. scient. Ec. Norm. Sup. 22, (1989), 81-98.

[R1] J. Robbin, *A structural stability theorem*, Annals of Math. 94 (1971), 447-493.

[R2] C. Robinson, *Structural stability of C^1 diffeomorphims*, J. Diff. Equations 22 (1976), 28-73.

[S] S. Sternberg, *On the structure of local homeomorphism of Euclidean n−space II*, Amer. J. Math 80 (1958), 623-631.

[V] S. M. Voronin, *Analytic classification of germs of conformal mappings* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *with identity linear part*, Funct. Anal. and Applications 15 (1981), 1-17.

[Y] J.C.Yoccoz, "Centralizateurs et conjugaison differentiable des diffeomorphismees du cercle", Theses Docteur d'Etat, Université Paris Sud, 1985 and to appear in Asterisque.

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