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An abstract version of the Morse index theorem and its application to hypersurfaces of constant mean curvature

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Abstract. In [5] Smale generalized the Morse index theorem (originally proved by Morse in [3]) to elliptic partial differential systems in several independent variables. Smale's result was used by Simons in [4] to obtain the index theorem for minimal submanifolds. The purpose of this paper is to give an abstract version of the Morse index theorem and use it to prove an index theorem for hypersurfaces of constant mean curvature. This was sugested by Barbosa and do Carmo in [1].

1. The Index Theorem for Hypersurfaces of Constant mean Curvature

In [1] the hypersurfaces with constant mean curvature are characterized as critical points of a variational problem. The bilinear form which arises as the second variation of the variational problem is given by

$$I(f,g) = \int_M g(-\Delta f - (\overline{R} + \|B\|^2)f)dM \tag{1.1}$$

where we have an immersion $M^n \to \overline{M}^{n+1}$ with constant mean curvature, dM and Δ are, respectively, the volume form and the Laplacian in the induced metric, $\|B\|$ is the norm of the second fundamental form of the immersion, and $\overline{R}$ is n times the Ricci curvature of $\overline{M}$ in the direction of the normal field given by the immersion.

For reasons inherent to the nature of the variational problem the bilinear form I must be taken restricted to the space $\mathcal{F}(M)$ of the smooth functions $f: M \to \mathbb{R}$ with $f \mid \partial M = 0$ and $\int_M f dM = 0$ (we assume that M is a compact connected C^∞ manifold with nonempty boundary), and the *Jacobi fields* are then defined

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as being those normal fields $V = fN$, $f \in \mathcal{F}(M)$, N a unit normal field to the immersion, such that $f \in \ker I$, i.e., $I(f,g) = 0$, for all $g \in \mathcal{F}(M)$.

Following Smale in [5], we say that M' is *a domain* of M *with smooth boundary* if M' is an open set of M such that the boundary $\partial \overline{M}'$ of the closure of M' is a C^∞ submanifold of M of codimension 1. A deformation of M is a family of domains M_t of M with smooth boundary, $a \le t \le b$, such that $\overline{M}_a = M$, and M_t is properly in M_s if $t > s$. The deformation is smooth if $\partial \overline{M}_t$ depends in a C^∞ manner on $t \in [a,b]$.

Given a smooth deformation M_t of M, $a \le t \le b$, we say that $\partial \overline{M}_t$ is a *conjugate boundary* if $I \mid \mathcal{F}(M_t)$ has a non- trivial Jacobi field. The *nullity* of $I \mid \mathcal{F}(M_t)$, which will be denoted by $n(t)$, is the dimension of the Jacobi fields on M_t. The *index* of $I \mid \mathcal{F}(M_t)$, which will be denoted by $i(t)$, is the dimension of a maximal subspace of $\mathcal{F}(M_t)$ in which I is negative definite.

We will prove the following:

1.1 Theorem. *Let $M^n \to \overline{M}^{n+1}$ be an immersion with constant mean curvature of the C^∞ compact connected manifold M, with non-empty boundary, into the riemannian manifold $\overline{M}$. Let M_t be a deformation of M, $a \le t \le b$. Then the indices $i(a)$ and $i(b)$ are finite and we have the formula:*

$$i(a) - i(b) = \sum_{a<t\le b} n(t).$$

Moreover, there exists an $\varepsilon > 0$ such that if $\mathrm{meas}(M_b) \le \varepsilon$, *then* $i(b) = 0$.

The first part of the above theorem will be obtained as a consequence of the abstract index theorem which is stated below.

2. The Abstract Index Theorem

Let $(E, \langle\ ,\ \rangle_E)$ and $(H, \langle\ ,\ \rangle)$ be two real Hilbert spaces such that $E \subset H$ and the embedding $E \hookrightarrow H$ is *compact and dense*. We denote by E' the dual of E. We use the usual identification of H and H' and then we have the embedding $H \hookrightarrow E'$ which is continuous and dense. In these conditions we say that (E, H, E') is *a triple of Hilbert spaces* (t.H.s).

2.1 Definition. We say that (E_t, H_t, E'_t), $a \le t \le b$, is a *deformation of the triple of Hilbert spaces* (E, H, E') if, for each $t \in [a,b]$, (E_t, H_t, E'_t) is a t.H.s. with $E_a = E$, $H_a = H$, $E'_a = E'$ and the following conditions hold:

2.1.1. if $t > s$, $E_t \subsetneq E_s$, $H_t \subsetneq H_s$;

2.1.2. if $t_0 \in (a,b]$ then

$$E_{t_0} = \bigcap_{s<t_0} E_s;$$

2.1.3. if $t_0 \in [a,b)$ then

$$E_{t_0} = \overline{\bigcup_{s>t_0} E_s},$$

the closure being taken in the topology of E.

Let (E, H, E') be a t.H.s. and consider a real continuous bilinear form $B\colon E \times E \to \mathbb{R}$. We say that B is *symmetric* if $B(e,\tilde e) = B(\tilde e, e)$ for all $(e,\tilde e) \in E \times E$ and that B is *coercive* if there exist $\beta \ge 0$ and $\gamma > 0$ such that

$$B(e,e) \ge \gamma \|e\|_E^2 - \beta \|e\|^2 \tag{2.1}$$

for all $e \in E$. We call β a *constant of coercivity* for B. Let then B be a continuous bilinear form on E which is symmetric and coercive. We say that $L\colon V \subseteq E \to H$ is the *representation* of B if V is the set of the elements $e \in E$ such that $B(e,\ \cdot\ ) = \langle f,\ \cdot\ \rangle$, for some $f \in H$, and we define $Le = f$.

It is clear that L is a well defined operator with domain V dense in E and has an unique extension to E as a bounded operator $E \to E'$.

We can easily prove that if B is a continuous symmetric and coercive bilinear form on E, β a constant of coercivity for B and L its representation, then $L + \beta$ can be extended uniquely to a continuous bijection $E \to E'$. If (E_t, H_t, E'_t), $a \le t \le b$, is a deformation of (E, H, E') we can consider the restrictions $B \mid E_t$ and, denoting by $L(t)$ the representation of $B \mid E_t$, the same thing holds for $L(t)$.

Now, $(L+\beta)^{-1}$ viewed as an operator on H, following the diagram below

$$\begin{array}{ccc} E & \hookrightarrow & H \\ (L+\beta)^{-1} \nwarrow & & \swarrow \\ & E' & \end{array}$$

is a *symmetric compact operator*. Then, we can use the Hilbert-Schmidt theorem to state the existence of an orthonormal basis to H, $\{e_n\}_{n\in\mathbb{N}}$, and a nonincreasing sequence of positive numbers α_n, such that

$$(L+\beta)^{-1}e_n = \alpha_n e_n, \tag{2.2}$$

with $\alpha_n \to 0$. We see by (2.2) that $e_n \in E$, for all $n \in \mathbb{N}$. Hence, we obtain a nondecreasing unbounded sequence of numbers λ_n and an orthonormal basis to H, $\{e_n\}$, whose elements are all in E and such that

$$Le_n = \lambda_n e_n \tag{2.3}$$

where

$$\lambda_n = \frac{1}{\alpha_n} - \beta.$$

Analagously, we have

$$L(t)e_n(t) = \lambda_n(t)e_n(t) \tag{2.4}$$

for a certain orthonormal basis to H_t, $\{e_n(t)\}$, the elements of which are all in E_t, and a certain nondecreasing unbounded sequence $\lambda_n(t)$, for each $t \in [a,b]$.

We will need the following lemma which is an adaptation to the present context of a classical result (see [6]).

2.2 Lemma. (min-max formula.) Let (E,H,E') be a t.H.s. and B be a continuous symmetric coercive bilinear form on E. Let λ_n be the sequence of eigenvalues of L, the representation of B. Then, the following holds:

$$\lambda_n = \min_{\substack{F\subset E\\ \dim F=n}} \max_{\substack{e\in F\\ \|e\|=1}} \langle Le, e\rangle. \tag{2.5}$$

We now introduce the abstract version of the concept of unique continuation (see [2]).

2.3 Definition. Let (E,H,E') be a t.H.s. and B a continuous symmetric coercive bilinear form on E. We say that B has the *property of the unique continuation* if given any deformation of (E,H,E'), (E_t,H_t,E'_t), $a \le t \le b$, the following is satisfied:

"*whenever for some $e \in E_t$, $t \in (a,b]$, there exist $\lambda \in \mathbb{R}$ and $s < t$, such that*

$$L(s)e = \lambda e,$$

then $e = 0$".

We call *Morse index* of B and denote $i(B)$ the dimension of a maximal subspace of E in which B is negative definite.

2.4 Theorem. (abstract Morse index theorem.) Let (E,H,E') be a triple of Hilbert spaces and B a continuous symmetric coercive bilinear form on E which has the property of the unique continuation. Then $i(B)$ is finite and for any deformation (E_t,H_t,E'_t), $a \le t \le b$, of (E,H,E') we have

$$i(B) - i(B \mid E_b) = \sum_{a<t\le b} \dim \operatorname{Ker}(B \mid E_t). \tag{2.6}$$

Proof. First, it is easy to see that $i(B)$ is equal to the number of negative eigenvalues of the representation L of B and so it is finite. We will prove (2.6) by means of three lemmas which will state that the functions $\lambda_n(t)$, $n\in\mathbb{N}$, are continuous and strictly increasing on $[a,b]$. The result will then follows from the elementary properties of this type of function.

2.5 Lemma. *For each $k\in\mathbb{N}$, the function $t \mapsto \lambda_k(t)$, defined on $[a,b]$, is nondecreasing and right continuous.*

Proof. The fact that $\lambda_k(t)$ is nondecreasing follows immediatly by the min-max formula. Let us prove the right continuity. We take $t_0 \in [a,b)$ and a decreasing sequence $\{s_i\}_{i\in\mathbb{N}}$ converging to t_0. For each $k\in\mathbb{N}$, the sequence $\{\lambda_k(s_i)\}_{i\in\mathbb{N}}$ is nonincreasing and bounded below by $\lambda_k(t_0)$. Hence, it converges to a $\lambda \in \mathbb{R}$. Let $e_1(t_0),\dots,e_k(t_0)$ be the orthonormal eigenvectors corresponding to $\lambda_1(t_0),\dots,\lambda_k(t_0)$, respectively, and $F \subset E_{t_0}$ be the space generated by these vectors. We recall that

$$E_{t_0} = \overline{\bigcup_{s>t_0} E_s},$$

so we can obtain sequences $\{e_j(s_i)\}_{i\in\mathbb{N}}$, $j = 1,\dots,k$, such that, for each $i\in\mathbb{N}$, $\{e_1(s_i),\dots,e_k(s_i)\}$ is a set of vectors in E_{s_i}, orthonormal in H, and $e_j(s_i) \to e_j(t_0)$ in E, as $i\to\infty$, for each $j = 1,\dots,k$. Since B is continuous in E, we have

$$
\begin{aligned}
\lambda_k(s_i) &= \min_{\substack{F\subset E_{s_i}\\ \dim F=n}} \max_{\substack{e\in F\\ \|e\|=1}} \langle L(s_i)e, e\rangle \\
&= \min_{\substack{F\subset E_{s_i}\\ \dim F=n}} \max_{\substack{e\in F\\ \|e\|=1}} B(e,e) \\
&\leq \max_{\substack{e\in \operatorname{span}\{e_1(s_i),\dots,e_k(s_i)\}\\ \|e\|=1}} B(e,e) \\
&\longrightarrow \max_{e\in \operatorname{span}\{e_1(t_0),\dots,e_k(t_0)\}} B(e,e) \\
&= \lambda_k(t_0),
\end{aligned}
$$

as $i\to\infty$, by passing to a subsequence if necessary. It follows then that $\lim_{i\to\infty}\lambda_k(s_i)=\lambda_k(t_0)$. □

2.6 Lemma. *For each $k\in\mathbb{N}$, the function $t\mapsto\lambda_k(t)$, defined on $[a,b]$, is continuous.*

Proof. It remains to prove the left continuity. Let $t_0\in(a,b]$ and consider an increasing sequence converging to t_0. The sequences $\{\lambda_j(s_i)\}_{i\in\mathbb{N}}$ are increasing and bounded above by $\lambda_j(t_0)$, $j=1,\dots,k$. So, there exist $\lambda_j\in\mathbb{R}$ such that $\lambda_j(s_i)\to\lambda_j$ as $i\to\infty$, $j=1,\dots,k$. For each $i\in\mathbb{N}$, let $e_1(s_i),\dots,e_k(s_i)$ be the orthonormal eigenvectors associated to $\lambda_1(s_i),\dots,\lambda_k(s_i)$. For each $j=1,\dots,k$ we have a sequence $\{e_j(s_i)\}_{i\in\mathbb{N}}$ which is bounded in H and, hence, is also bounded in E'. By the coercivity, we have

$$\langle(\lambda_j(s_i)+\beta)e_j(s_i),e_j(s_i)\rangle=\langle(L_i+\beta)e_j(s_i),e_j(s_i)\rangle\geq c\,\|e_j(s_i)\|_E^2,$$

so $\{e_j(s_i)\}_{i\in\mathbb{N}}$ is a bounded sequence in E, $j=1,\dots,k$. Hence, there exists a subsequence of $\{e_j(s_i)\}$, which we will denote also by $\{e_j(s_i)\}$, which converges to f_j in H, passing to a subsequence if necessary. Let $V_{t_0}=D(L(t_0))$. For all $\xi\in V_{t_0}$, we have

$$
\begin{aligned}
\langle L(t_0)f_j,\xi\rangle &= \langle Lf_j,\xi\rangle=\langle f_j,L\xi\rangle \\
&= \lim_{i\to\infty}\langle e_j(s_i),L\xi\rangle=\lim_{i\to\infty}\langle\lambda_j(s_i)e_j(s_i),\xi\rangle \\
&= \langle\lambda_j f_j,\xi\rangle.
\end{aligned}
$$

Then, it follows that

$$L(t_0)f_j=\lambda_j f_j \quad,\quad j=1,\dots,k.$$

So $\lambda_1,\dots,\lambda_k$ are k eigenvalues of $L(t_0)$ and, further, $\lambda_j\leq\lambda_j(t_0)$, $j=1,\dots,k$. We then conclude that

$$\lambda_j=\lambda_j(t_0) \quad,\quad j=1,\dots,k. \ \square$$

2.7 Lemma. *For each $k\in\mathbb{N}$, the function $t\mapsto\lambda_k(t)$ is strictly increasing.*

Proof. We will prove this by contradiction. So, we assume that there exist $p\in\mathbb{N}$ and $t_1<t_2$ such that

$$\lambda_p(t_1)=\lambda_p(t_2)=\lambda.$$

Let $\{e_j(t_1)\}_{j\in\mathbb{N}}$ and $\{e_j(t_2)\}_{j\in\mathbb{N}}$ be the orthonormal sets of vectors which satisfy

$$L(t_i)e_j(t_i)=\lambda_j(t_i)e_j(t_i) \quad,\quad j\in\mathbb{N} \quad,\quad i=1,2.$$

Let $F\subset E_{t_2}$ be the subspace generated by $\{e_1(t_2),\dots,e_p(t_2)\}$. We have that there exists $u\in F$, $\|u\|=1$, such that

$$\langle u,e_j(t_1)\rangle=0 \quad,\quad j=1,\dots,p-1.$$

Now, $u\in E_{t_1}$, since $E_{t_2}\subset E_{t_1}$, and, hence, we can write

$$u=\sum_{j=p}^{\infty}\langle u,e_j(t_1)\rangle e_j(t_1).$$

From this, we have

$$\langle Lu,u\rangle=\sum_{j=p}^{\infty}\lambda_j(t_1)\,|\langle u,e_j(t_1)\rangle|^2\geq\lambda_p(t_1)=\lambda. \tag{2.6}$$

On the other hand, $u\in F$, then

$$u=\sum_{j=1}^{p}\langle u,e_j(t_2)\rangle e_j(t_2)$$

and so

$$\langle Lu,u\rangle=\sum_{j=1}^{p}\lambda_j(t_2)\,|\langle u,e_j(t_2)\rangle|^2\leq\lambda_p(t_2)=\lambda.$$

From (2.6) and (2.7) we conclude that

$$\langle Lu, u\rangle = \sum_{j=p}^{\infty} \lambda_j(t_1)\,|\langle u, e_j(t_1)\rangle|^2 = \lambda_p(t_1).$$

But the above equality is possible only if $\lambda_j(t_1) = \lambda_p(t_1)$ for all j such that $\langle u, e_j(t_1)\rangle \neq 0$. Then, we have

$$L(t_1)u = \lambda u.$$

Since $u \in E_{t_2}$, by the property of the unique continuation, we obtain that

$$u = 0.$$

which gives a contradiction and proves the lemma. □

3. Proof of the Index Theorem for Hypersurfaces of Constant mean Curvature

We consider the following Sobolev spaces on M:

$$H^1(M) = \{f \in L^2(M) \quad s.t. \quad |\nabla f| \in L^2(M)\},$$
$$H^2(M) = \{f \in L^2(M) \quad s.t. \quad \Delta f \in L^2(M)\},$$

and let $H^{-1}(M)$ be the dual of $H^1(M)$. We denote by $F(M)$, $F^1(M)$, $F^2(M)$ and $F^{-1}(M)$ the closures of $\mathcal{F}(M)$ in $L^2(M)$, $H^1(M)$, $H^2(M)$ and $H^{-1}(M)$ respectively. It is easy to see that $(F^1(M), F(M), F^{-1}(M))$ is a triple of Hilbert spaces. It is also immediate that the bilinear form I defined in (1.1) can be extended to a continuous bilinear form in $F^1(M)$. Let $L^0\colon H^2(M) \to L^2(M)$ be the operator

$$L^0 f = -\Delta f - (\overline{R} + \|B\|^2)f. \tag{3.1}$$

The representation of I in $F^1(M)$ is the operator $L\colon F^2(M) \to F(M)$ given by

$$Lf = L^0 f - \frac{1}{\text{meas}(M)} \int_M (L^0 f)\,dM. \tag{3.2}$$

The first part of Theorem 1.1 will then follow from Theorem 2.4 if we prove that I is symmetric coercive and has the property of the unique continuation.

That the form I is symmetric is obvious. The coercivity can be proved in the same way that Simons [4] proved this property for the second variation of the area for minimal submanifolds. It remains to prove that the form I has the property

of the unique continuation. So, suppose that there existe $t_1 < t_2$ in $[a,b]$, $\lambda \in \mathbb{R}$ and $\phi \in F^1(M_{t_2})$ such that

$$L(t_1)\phi = \lambda\phi.$$

Since $\phi(x) = 0$ for $x \in M_{t_1} \setminus M_{t_2}$, by (3.2) we see that

$$\int_{M_{t_1}} L^0\phi\, dM_{t_1} = 0.$$

Hence,

$$L^0(t_1)\phi = L(t_1)\phi = \lambda\phi.$$

But L^0 has the property of the unique continuation in the classical sense (see [4]) and then $\phi = 0$, which proves the desired result.

The last part of Theorem 1.1 is a direct consequence of Lemma 3.5 of the paper of Smale [5].

We finally would like to remark that from what we have seen above, the Jacobi fields along hypersurfaces with constant mean curvature are those normal fields $V = fN$, $f \in \mathcal{F}(M)$, which satisfy

$$L^0 f = c,$$

for some constant $c \in \mathbb{R}$.

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