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Centrality properties of some abelian by polycyclic groups

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Abstract. We describe the structure of $\mathbb{Z}\,\Gamma$-modules, where Γ is polycyclic, and use the result to give a direct proof of the result of Lennox and Roseblade that finitely generated abelian by polycyclic groups are sn-stunted and sn-eremitic.

Introduction

In their remarkable paper [1] in 1970, J.C. Lennox and J.E. Roseblade proved that a finitely generated abelian by nilpotent group is stunted and eremitic, and they asked whether finitely generated abelian by polycyclic groups have such properties.

In the same paper, some weaker properties, sn-stuntedness and sn-eremiticity, were shown to hold for finitely generated abelian by polycyclic groups. This is given by Theorem E^* [1]. The proof of this theorem uses the "Fan Out Lemma" which is designed to obtain information about the structure of $\mathbb{Z}\,\Gamma$-modules, where Γ is a finitely generated nilpotent group.

In this paper a modified version of the "Fan Out Lemma" that deals directly with $\mathbb{Z}\,\Gamma$-modules, where Γ is polycyclic is established and used to obtain a simpler proof of Theorem E^*.

In a subsequent paper, this modified version of the Fan Out Lemma together with two parallel results are used to extend Theorems A and B of [1] to the groups that are finitely generated abelian by polycyclic where the polycyclic quotient is either abelian by infinite cyclic or a plinth by abelian group. The proofs of all these results make heavy use of the isolator property for suitable subgroup of a

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polycyclic group as given in Proposition 1 of [2].

1.2. Definitions and Notation

We recall from [1] that a group G is said to have upper central height α if $\xi_\alpha(G) = \xi_{\alpha+1}(G)$ while $\xi_\beta(G) < \xi_{\beta+1}(G)$ for all $\beta < \alpha$, where $\xi_\alpha(G)$ is the αth term of the upper central series of G, and G is said to be *centrally stunted* of *height h* if an integer k exists such that every subgroup of G has central height at most k, and h is the least such k. A group G is called *centrally eremitic* of *eccentricity e* if there exists a positive integer f such that for any $x \in G$ and any $n > 0$,

$$C_G(x^n) \leq C_G(x^f)$$

and e is the least such f, where $C_G(x)$ denotes the centralizer of x in G. Let Γ be any group and M a $\mathbb{Z}\,\Gamma-$ module, the additive notation for M is used here and mx denotes the image of $m \in M$ under the action of $x \in \Gamma$. But occasionally m^x is used to denote this image and the multiplicative notation for M is adopted. Following Lennox and Roseblade in [1], we shall think of the pair (M, Γ) as being embedded in the split extension $G = M\Gamma$ with $M \triangleleft G$ and $M \cap \Gamma = 1$. The commutator $[m, x]$ is identified with $m(x-1)$ for $m \in M$ and $x \in \Gamma$. For any ordinal α and any subgroup H of Γ,

$$M_\alpha(H) = M \cap \zeta_\alpha(MH).$$

Thus for a positive integer n, the subgroup $M_n(H)$ of M consists of all $m \in M$ such that $[m, h_1, h_2, \ldots, h_n] = 0$; $h_i \in H$, $i = 1, \ldots, n$. For a $\mathbb{Z}\,\Gamma$-module M, the pair (M, Γ) is *stunted* if there is an integer $h \geq 0$ such that for all subgroups H of Γ the equality $M_\alpha(H) = M_h(H)$ holds for all $\alpha \geq h$. The least such integer h is called the height of (M, Γ). We shall also say that the pair (M, Γ) is *sn-stunted* of *height h* if the above equality holds for all subnormal subgroups H of Γ.

Likewise the pair (M, Γ) is *eremitic* of *eccentricity e* if there is an integer $f > 0$ such that for all $n > 0$ and all $x \in \Gamma$ the inclusion $M_1(x^n) \leq M_1(x^f)$ holds and e is the least such f. The pair (M, Γ) is *sn-eremitic* of *eccentricity e* if there is an integer $f > 0$ such that for all $n > 0$ and all subnormal subgroups H of Γ, $M_1(H^n) \leq M_1(H^f)$ and e is the least such f. Here H^n is the subgroup $\langle h^n, h \in H \rangle$ generated by all nth powers of elements of H.

We state here for easy reference,

Theorem E*. ([1].) If M is a Noetherian $\mathbb{Z}\,\Gamma$-module, where Γ is a polycyclic by finite group, then (M, Γ) is sn-stunted and sn-eremitic.

2.1. The isolator property

We give here the definition and the main facts of the isolator property and use it to give a simple proof for Theorem F in [1].

Let H be a subgroup of a group G and the set $I(H) = \{x \in G : x^n \in H$ for some $n \neq 0\}$. $I(H)$ is called the isolator of H in G and if it is necessary to emphasize the group G then we write $I_G(H)$. We say G has the isolator property if for every subgroup H of G, $I(H)$ is a subgroup. H is said to be isolated in G if $I_G(H) = H$. Observe that if G is polycyclic and $I(H)$ is a subgroup then $|I(H) : H| < \infty$. Proposition 1 of [2] shows that a polycyclic group G has a normal subgroup G_0 of finite index with the isolator property. Other properties of G_0 from [2] are:

(i) $\mathrm{Fitt}(G_0)$ and all centralizers in G_0 are isolated and

(ii) For any subgroup $H \leq G_0$,

$$I(n_{G_0}(H)) = n_{G_0}(I(H)),$$

where $n_G(H)$ is the normalizer in G of the subgroup H.

These facts will be used in what follows without further mention. We give now an alternate proof of the following:

Theorem F. ([1].) Suppose Δ is a subgroup of a polycyclic group Γ. There is a positive integer $d = d(\Gamma, \Delta)$ such that $|H : H \cap \Delta|$ divides d for all subgroups H of Γ for which $|H : H \cap \Delta|$ is finite.

Proof. If Γ is finite the result is clear, so assume Γ infinite and find a subgroup Γ_0 of finite index in Γ that has the isolator property.
If $d(\Gamma_0, \Gamma_0 \cap \Delta)$ exists then we can take

$$d(\Gamma, \Delta) = |\Gamma : \Gamma_0| d(\Gamma_0, \Gamma_0 \cap \Delta),$$

thus we may assume Γ to have the isolator property. Now given Δ and H subgroups of Γ so that $|H : H \cap \Delta| < \infty$ then $I(\Delta) \cap H = H$, since $I(\Delta) \cap H$

is isolated in H and $\Delta \cap H \leq I(\Delta) \cap H \leq H$. Thus $H \leq I(\Delta)$.

Since $|I(\Delta) : \Delta| < \infty$, we can find an integer $n = n(\Delta, \Gamma)$ so that $I(\Delta)^n \leq \Delta$. Now $|H : H \cap \Delta|$ divides $|H : H \cap I(\Delta)^n|$ hence we only need to show that $|H : H \cap I(\Delta)^n|$ divides some integer $d = d(\Gamma, \Delta)$. But $|H : H \cap I(\Delta)^n| = |I(\Delta)^n H : I(\Delta)^n|$ and the latter number divides $|I(\Delta) : I(\Delta)^n|$ which depends on Γ and Δ only. This completes the proof.

3.1. The "Fan Out Lemma" a modified version

We give in this section the statement of the modified Fan Out Lemma and use it to give a simple and direct proof of Theorem E* [1]. The proof of the Fan Out Lemma will be given in Section 5.

Let Γ be a torsion-free polycyclic group with the isolator property and M any $\mathbb{Z}\,\Gamma$-module. Define the set $\mathcal{X}(M, \Gamma)$ of subgroups of Γ as follows:

A subgroup X of Γ is in $\mathcal{X}(M, \Gamma)$ if X is maximal with respect to having the following properties:

(1) $X\ sn \Gamma$ (X is subnormal in Γ)

(2) $M_1(X) \neq 0$

(3) If $X < H \leq \Gamma$ and H has (1) and (2) then

$$|H : X| < \infty$$

(4) $X = I(X)^n$ for some $n > 0$.

For M and Γ as above we have,

3.2 Lemma. *If Γ has a non-trivial subnormal subgroup H such that $M_1(H) \neq 0$, then $\mathcal{X}(M, \Gamma)$ contains non-trivial elements.*

The proof of this lemma follows the same reasoning as Lemma 2 [1], and is not stated here, it rests on the following.

3.3 Lemma. *If Γ is torsion-free polycyclic with the isolator property and H is any subgroup of Γ then there is a subgroup $K \subseteq H$ such that $K = I(K)^n$ for some $n > 0$ and $n_\Gamma(K)$ is isolated.*

Proof. Since $|I(H) : H| < \infty$, $I(H)^n \leq H$ for some $n > 0$ and $I(H)^n \neq 1$ unless $H = 1$. Let $K = I(H)^n$, then $|H : K| < \infty$ hence $I(H) = I(K)$ and $I(K)^n = I(H)^n = K$.

Now K is fully invariant in $I(K)$, hence $K \lhd n_\Gamma(I(K)) = I(n_\Gamma(K))$. It follows that

$$I(n_\Gamma(K)) \leq n_\Gamma(K).$$

Hence $n_\Gamma(K) = I(n_\Gamma(K))$ as required.

3.4. Some elementary proprties of elements of $\mathcal{X}(M, \Gamma)$ are collected here. Their verification is straightforward and is omitted.

Lemma. *For M and Γ as in (3.1), if $X \in \mathcal{X}(M, \Gamma)$ then*

(1) *$X^a \in \mathcal{X}(M, \Gamma)$ for all $a \in \Gamma$*

(2) *There is a subnormal series from X to Γ*

$$X = X_0 \lhd X_1 \lhd \cdots \lhd X_k = \Gamma, \quad \text{such that}$$

$$X_1, X_2, \ldots, X_k \quad \text{are all isolated in } \Gamma.$$

(3) *If $X \leq \Delta$ and Δ is isolated and subnormal in Γ, then $X \in \mathcal{X}(M, \Delta)$.*

3.5 Lemma. *Let M be any $\mathbb{Z}\,\Gamma$-module, where Γ is a torsion-free polycyclic group with the isolator property. Let $1 \neq X \in \mathcal{X}(M, \Gamma)$ and $N = n_\Gamma(X)$ and T a set of coset representatives of N in Γ. Let $U = M_1(X)$ and $Y = U(\mathbb{Z}\,\Gamma)$ then,*

(i) $Y = \bigoplus_{t \in T} Ut$

(ii) *U is $\mathbb{Z}N$-Noetherian if M is $\mathbb{Z}\,\Gamma$-Noetherian*

(iii) $Y_n(H) = \bigoplus_{t \in T_H} U_n(H)t$, *for all $H \leq \Gamma$ and all $n \geq 0$.*

Here $T_H = \{t \in T : H \leq N^t\}$

(iv) *The pair (M, Γ) is sn-stunted and sn-eremitic whenever (U, N) is so.*

This is a variation of the Fan Out Lemma in [1] and as in [1], it plays a key role in the proofs of the other results.

4.1. Proof of Theorem E* ([1])

By induction on the Hirsch length $h(\Gamma)$. If $h(\Gamma) = 0$ then Γ is finite and since $\varepsilon(M\Gamma) < \infty$ where $\varepsilon(G)$ is the least upper bound of orders of all torsion elements of G, the pair (M, Γ) is actually stunted and eremitic by Corollaries 16 and 17 of [1].

Thus assume $h(\Gamma) > 0$ and the obvious induction hypothesis. Let Γ_0 be a subgroup of finite index in Γ which is torsion free and has the isolator property then $M\Gamma_0$ has max $-n$ the maximal condition on normal subgroups (Theorem 15.3.1 [3]), and by Lemma 13 [1], $\varepsilon(M\Gamma_0) < \infty$. By analogues of Corollaries 16 and 17 [1], the pair (M, Γ) will be sn-stunted and sn-eremitic if (M, Γ_0) is so, therefore assume Γ a torsion-free polycyclic group with the isolator property. If (M, Γ) is not sn-stunted and not sn-eremitic then let D be a submodule of M maximal such that $(M/D, \Gamma)$ is not sn-stunted or not sn-eremitic. We assume $D = 0$ and will obtain a contradiciton. In other words, we are assuming that for any nonzero Γ-submodule Y of M, the pair $(M/Y, \Gamma)$ is both sn-stunted and sn-eremitic. So, to reach a contradiction, it is enough by Lemma 4 [1], to show that there is a nonzero submodule Y of M such that (Y, Γ) is both sn-stunted and sn-eremitic. By assumption, there is a nontrivial subnormal subgroup H of Γ such that $M_1(H) \neq 0$, thus by Lemma 3.2 the set $\mathcal{X}(M, \Gamma)$ contains a nontrivial element X say.

Let $N = n_\Gamma(X)$ and T a set of coset representatives of T in Γ. Let $U = M_1(X)$ and $Y = U(\mathbb{Z}\,\Gamma)$.

By Lemma 3.5 (ii), U is N-Noetherian, and since X centralizes U, we have U is N/X-Noetherian. Now (U, N) is sn-stunted and sn-eremitic if and only if $(U, N/X)$ is so. However $h(N/X) < h(\Gamma)$, therefore by the induction hypothesis $(U, N/X)$ and hence (U, N) is sn-stunted and sn-eremitic. By Lemma 3.5 (iv) the pair (Y, Γ) is sn-stunted and sn-eremitic, completing the proof.

5.1. Proof of Lemma 3.5

Part (ii) follows from (i) in the same way as Lemma 5 [1] follows from F.1 [1]. Part (iii) also follows from (i) as F.3 [1] follows from F.1 [1]. Part (iv) follows from

(iii) in the same way as shown in the last paragraph of the proof of Theorem A* and B* on p. 411 of [1]. Thus we only have to establish part (i). We follow the scheme used by Lennox and Roseblade in [1] and shall restate Lemma 3.5 with some more items that are not used directly in the proof of Theorem E*.

3.5.1 Lemma. *If M is any $\mathbb{Z}\,\Gamma$-module, where Γ is a torsion-free polycyclic group with the isolator property then,*

FOL (1): *If $X \in \mathcal{X}(M, \Gamma)$ and T is a set of coset representatives of $N = n_\Gamma(X)$ then*

$$M_1(X) = \bigoplus_{t \in T} M_1(X)t.$$

FOL (2): *If $X \in \mathcal{X}(M, \Gamma)$ and B is a submodule of M then*

$$B \cap M_1(X)(\mathbb{Z}\,\Gamma) \neq 0 \text{ only if } B_1(X) \neq 0.$$

We shall use $X, X_1, X_2, \ldots$ to denote elements of $\mathcal{X}(M, \Gamma)$ and $U = M_1(X)$,

$$U_1 = M_1(X_1), \quad U_2 = M_1(X_2), \ldots \quad \text{and}$$
$$Y = U(\mathbb{Z}\,\Gamma), \quad Y_1 = U_1(\mathbb{Z}\,\Gamma), \quad Y_2 = U_2(\mathbb{Z}\,\Gamma), \ldots.$$

Also $N = n_\Gamma(X)$ and T is a set of coset representatives of N in Γ and $T_H = \{t \in T : H \leq N^t\}$.

To prove FOL (1) and FOL (2) we need some of its consequences which we state as:

5.2 Lemma. *If FOL (1) and FOL (2) hold then so does:*

FOL (3): *For $X \in \mathcal{X}(M, \Gamma)$*

$$Y_n(H) = \bigoplus_{t \in T_H} U_n(H)t, \quad \text{for all} \quad n \geq 0$$

and all $H \leq \Gamma$.

FOL (4): *If $X_\lambda (\lambda \in \Lambda)$ are elements of $\mathcal{X}(M, \Gamma)$ mutually inconjugate in Γ then,*

$$\langle Y_\lambda,\ \lambda \in \Lambda \rangle = \bigoplus_{\lambda \in \Lambda} Y_\lambda;$$

FOL (5): *If $X_1,\ldots,X_r$ are elements of $\mathcal{X}(M,\Gamma)$ mutually inconjugate in Γ and if B is a submodule of M such that $B\cap\left(\bigoplus_{i=1}^{r} Y_i\right)\neq 0$ then $B_1(X_i)\neq 0$ for some $1\le i\le r$.*

The proof of this lemma is analogous to that of Lemma 10 [1] with a small modification in the proof of FOL (4), which we record in

5.3. Proof of FOL (4) from FOL (1) and FOL (2)

By induction on r, if $r=1$ then there is nothing to prove, so assume $r>1$ and $B=\sum_{i=2}^{r} Y_i=\bigoplus_{i=2}^{r} Y_i$ but $B\cap Y_1\neq 0$.
By FOL (2) $B_1(X_1)\neq 0$ and since B is a direct sum, $Y_{i,1}(X)1)\neq 0$ for some $2\le i\le r$. To simplify the notation we write X for X_i and Y for Y_i.

For $i\ge 2$ we have X and X_1 are inconjugate in Γ and $Y_1(X_1)\neq 0$. Since FOL (1) is assumed to hold, FOL (3) holds too, thus

$$Y_1(X_1)=\bigoplus_{t\in T_{X_1}} U_1(X_1)t$$

and therefore

$$Ut\cap M_1(X_1)=U_1(X_1)t\neq 0$$

for some $t\in T_{X_1}$. In other words, $M_1(X^t)\cap M_1(X_1)\neq 0$ and $M_1(\langle X^t,X_1\rangle)\neq 0$. Let $H=\langle X^t,X_1\rangle$ then $X^t<H$ and $X_1<H$ since both X^t and X_1 are in $\mathcal{X}(M,\Gamma)$. (Note: up to this point the proof is as in F4, p. 415 [1]).

Now H $sn\Gamma$ by Theorem 13.1.9 [3], so H has properties (1) and (2) in the definiton of $\mathcal{X}(M,\Gamma)$, hence $|H:X^t|$ and $|H:X_1|$ are both finite. It follows that $I(H)=I(X^t)=I(X_1)$. Let m and m_1 be positive integers such that $I(X^t)^m=X^t$ and $I(X_1)^{m_1}=X_1$ and let $d=g.c.d.(m,m_1)$. Then $X_1=I(X_1)^{m_1}\le I(X_1)^d=I(H)^d$, and similarly $X^t\le I(H)^d$, hence $H\le I(H)^d$. On the other hand, if $d=sm+tm$, s and t integers, then

$$I(H)^d=\langle g^d,\quad g\in I(H)\rangle$$
$$=\langle g^{sm}g^{tm},\quad g\in I(H)\rangle$$

$$\le\langle I(H)^m,\quad I(H)^{m_1}\rangle$$
$$=\langle I(X^t)^m,\quad I(X_1)^{m_1}\rangle$$
$$=\langle X^t,X_1\rangle=H.$$

Thus $H=I(H)^d$, $d>0$ and $H\in\mathcal{X}(M,\Gamma)$ contradicting the maximality of X_1 with respect to being an element in $\mathcal{X}(M,\Gamma)$. This establishes FOL (4).

5.4. To prove FOL (1) and FOL (2) we note here that much of the following proof is analogous to that of F.1 and F.2 in [1]. The proof is by induction on the Hirsch length $h(\Gamma)$.

If $h(\Gamma)=1$ then $h(X)=h(\Gamma)$ for any $1\neq X\in\mathcal{X}(M,\Gamma)$, hence $|\Gamma:X|<\infty$ and $\Gamma=N$ since N is isolated in Γ by Lemma 3.3. Thus $X\triangleleft\Gamma$ and FOL (1) and FOL (2) follow trivially, so assume $h(\Gamma)>1$ and the obvious induction hypothesis. Let $X\in\mathcal{X}(M,\Gamma)$, by Lemma 3.4, there is a subnormal series from X to Γ consisting of isolated subgroups,

$$X=X_0\triangleleft X_1\triangleleft X_2\triangleleft\cdots\triangleleft X_k=\Gamma.$$

If $k=1$ then $X\triangleleft\Gamma$ and the result follows so assume $k>1$ and let $\Delta=X_{k-1}$. Then $X\le\Delta$ and Δ is isolated and subnormal in Γ. Hence $X^a\in\mathcal{X}(M,\Delta)$ for all $a\in\Gamma$ by Lemma 3.4.

Since $h(\Delta)<h(\Gamma)$, by the induction hypothesis FOL (1) and FOL (2) hold for the pair (M,Δ), so if R is a set of coset representatives of $N\cap\Delta$ in Δ then,

$$(*)\qquad M_1(X)\Delta=\bigoplus_{r\in R} M_1(X)r\ldots.$$

Let S be a set of coset representatives of $N\Delta$ in Γ, then it is easy to see that

(**) the elements $\{X^s, s\in S\}$ of $\mathcal{X}(M,\Delta)$ are all mutually inconjugate in Δ

Now since FOL (1) and FOL (2) hold for (M,Δ) so does FOL (4), thus

$$(***)\qquad\langle M_1(X^s)\Delta, s\in S\rangle=\bigoplus_{s\in S} M_1(X^s)\Delta.$$

Let $T=RS$ then T is a set of coset representations of N in Γ and $\Gamma=NT$. It follows that,

$$\begin{aligned} M_1(X)Z\,\Gamma &= M_1(X)NT = M_1(X)NRS \\ &= M_1(X)\Delta S \\ &= \langle M_1(X)\Delta s, \quad s \in S \rangle \\ &= \langle M_1(X^s)\Delta, \quad s \in S \rangle \quad \text{since } \Delta \text{ is normal in } \Gamma \\ &= \bigoplus_{s\in S} M_1(X^s)\Delta \quad \text{by} \quad (***) \\ &= \bigoplus_{s\in S} (M_1(X)\Delta)^s \quad \text{since } \Delta \text{ is normal in } \Gamma \\ &= \bigoplus_{s\in S} (\bigoplus_{r\in R} M_1((X)r)^s \quad \text{by} \quad (*) \\ &= \bigoplus_{\substack{s\in S \\ r\in R}} M_1(X)(rs) = \bigoplus_{t\in T} M_1(X)t \end{aligned}$$

This proves FOL (1). The rest of the proof of FOL (2) is analogous to that in the proof of F.2 in [1] and is omitted.

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