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On the Perron-Frobenius-Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions

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Abstract. Let $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function on the Riemann sphere, φ be a Hölder continuous function on the Julia set $\mathcal{J}$, $\mathcal{P}_\varphi: C(\mathcal{J}) \to C(\mathcal{J})$ denote the Perron-Frobenius-Ruelle operator on the space of continuous functions:

$$\mathcal{P}_\varphi(\psi)(x) = \sum_{y \in f^{-1}(x)} \psi(y) \exp \varphi(y).$$

Suppose that topological pressure $P = P(f, \varphi)$ satisfies $P > \sup \varphi$. Then for every $\psi \in C(\mathcal{J})$ the family $(\exp P)^{-n} \mathcal{P}_\varphi^n(\psi)$ is equicontinuous and there exists a probability measure η on $\mathcal{J}$ and a function $\psi_0 \in C(\mathcal{J})$ such that $\psi_0 > 0$ and for every $\psi \in C(\mathcal{J})$, $\int \mathcal{P}_\varphi(\psi) d\eta = (\exp P) \int \psi d\eta$ and $(\exp P)^{-n} \mathcal{P}_\varphi^n(\psi) \to \psi_0 \cdot \int \psi d\eta$. The measure $\psi_0 \cdot \eta$ is unique equilibrium (Gibbs) state for φ.

This theorem was proved recently by M. Denker and M. Urbański. We give here a significantly different proof of it, less ergodic but going deeper into holormophic dynamics.

We discuss also modulus of continuity of ψ_0, in particular we prove, it is bounded by

$$C(N) \left(\frac{1}{\log(1/\varepsilon)} \right)^N$$

for arbitrary N and a respective constant $C(N)$.

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Appendix A. Proof of Theorem 1 under the assumption $\log\lambda > \sup\varphi$.
Appendix B. On the assumption $P > \sup\varphi$.

0. Introduction, basic ideas and notation

Let f be a rational mapping of the Riemann sphere $\hat{\mathbb{C}}$ and $J = J(f)$ be its Julia set. Let φ be a real continuous function on J. Define the operator $\mathcal{P}_\varphi : C(J) \to C(J)$ on the space of continuous functions by

$$\mathcal{P}_\varphi(\psi)(x) = \sum_{y\in f^{-1}(x)} \psi(y)\exp\varphi(y)$$

(if y is a critical point we repeat it as many times as its multiplicity as the preimage of x).

The main aim of the paper is to give a new proof of the following theorem proved recently by M. Denker and M. Urbański[**DU2**]

Theorem 1. (Denker, Urbański.) *Suppose* $\sup\varphi < P \equiv P(f,\varphi)$ *(the topological pressure) and suppose that* φ *is Hölder continuous. Then* $\mathcal{P}_{\varphi-P}$ *is almost periodic, i.e. for every* $\psi \in C(J)$ *the sequence of functions* $\mathcal{P}^n_{\varphi-P}(\psi)$ *is uniformly bounded and equicontinuous. There exists a positive fixed point for* $\mathcal{P}_{\varphi-P}$ *namely a function* $\psi_0 > 0$ *such that* $\mathcal{P}_{\varphi-P}(\psi_0) = \psi_0$ *and there exists a probability measure* η *on* J *such that for every* $\psi \in C(J)$ *we have* $\int \psi d\eta = \int \mathcal{P}_{\varphi-P}(\psi)d\eta$ *and* $\mathcal{P}^n_{\varphi-P}(\psi) \to \psi_0 \cdot \int \psi d\eta$ *as* $n \to \infty$.

Note that by functional analysis reasons there exist $\lambda > 0$ and a probability measure η such that

$$\mathcal{P}^*_\varphi(\eta) = \lambda\eta \tag{1}$$

From Theorem 1 it follows immediately that $\log\lambda = P$ and η satisfying (1) must be the same as η in Theorem 1 (where it is obviously unique because of its properties). In particular we deduce uniqueness of the probability measure satisfying (1).

As a corollary we obtain in a rather standard way

Theorem 2. (Denker, Urbański.) $\nu = \psi_0 \cdot \eta$ *is the unique equilibrium state for* φ *(i.e. an* f*-invariant probability measure on* J *such that* $h_\nu(f) + \int_J \varphi d\nu = P(f,\varphi)$*, where* h_ν *stands for measure-theoretic entropy).*

Let us mention that we do not know whether ψ_0 must be Hölder continuous. Nevertheless we prove in Section 4 that the modulus of continuity of ψ_0 is bounded by $C(N)\left(\frac{1}{\log(1/t)}\right)^N$ for every $N > 0$ and a constant $C(N) > 0$.

In Sections 1 and 2 we give a technical preparation. In Section 3 we prove the main lemma that $\mathcal{P}^n_\varphi(x)/\mathcal{P}^n_\varphi(1)(y)$ is uniformly bounded over all $n \geq 0$ and $x, y \in J$, after which everything gets easy.

In Section **4** we deduce Theorems 1 and 2.

A proof of Theorem 1 is easy if $\varphi \equiv 0$, see [**Lju2**] or if at least

$$\sum_{y\in f^{-1}(x)} \exp\varphi(y)$$

is constant.

Let us comment now Denker-Urbański's Proof. It consists of two parts: First part proves that $\log\lambda \geq P$, this seems to be a harder part of their paper which uses the conformal measures technique [**DU1**].

Then the assumption $P > \sup\varphi$ yields $\log\lambda > \sup\varphi$, consequently the second part of Denker-Urbański's Proof of Theorem 1 is the proof of its assertions under the assumption $\log\lambda > \sup\varphi$ (a posteriori it occurs $P = \log\lambda$). This is an easier, but nice and tricky part of their paper. For completeness I will give my variant of the proof in Appendix A. I will give some discussion of the assumptions in Appendix B.

All [**DU2**] relies on Mañé's technique [**M3**]. Instead we rely on simple Lemma 1 which roots can be found in [**Lju1**] and which is used also in [**LP**].

The main point is to know that for every two points most of their backward trajectories approach each other. The authors of [**DU2**] ignore backward branches f^{-1} (on discs) meeting critical values. We consider in Lemma 1 and in the sequel, all branches, keeping some control on what happens when we meet a critical value. Especially Lemma 5 ("telescope" lemma) is devoted to it. This is crucial Lemma of the paper.

Commenting again Section 1 let us mention that throughout it we play with ideas virtually present in [**Lju1**] and [**Lju2**] and also in [**MP**]. In particular we obtain slightly modified proofs of $h_{\text{top}}(f) = \deg(f)$ and that f is asymptotically h-expansive. "Telescope" lemma in Section 2 contains however an idea not present in Ljubich papers and this it is just this lemma which allows to make a step from measures with maximal entropy investigated by Ljubich, to a larger class of measures.

Basic notation. Crit or Crit(f) is the set of all critical points of f, i.e. $\{x \in \hat{\mathbb{C}}: f'(x) = 0\}$.

$\mathrm{Critv}(f^n) = \bigcup_{i=1}^n f^i$ (Crit(f)), for $n = 1, 2, \dots$

$S'(f)$ is the set of all periodic sinks for f containing a critical point in the orbit.

For every $\varepsilon > 0$ and a continuous function $g: J \to \mathbb{R}$ we write

$$\mathrm{var}_\varepsilon(g) = \sup\{|g(x) - g(y)|: \mathrm{dist}(x, y) \le \varepsilon\},$$

dist in a fixed standard metric ϱ on $\hat{\mathbb{C}}$.

For $x, y \in \hat{\mathbb{C}}$ we define $\varrho_n(x, y) = \max_{i=0,\dots,n}(\mathrm{dist}(f^i(x), f^i(y)))$. We write $B_n(x, \varepsilon)$ to denote the open ball with the origin at x and radius ε in metric ϱ_n. We call x, y being (n, ε)-close if $y \in B(x, \varepsilon)$ and (n, ε)-separated if $y \notin B(x, \varepsilon)$.

A set A is called (n, ε)-separated if each two points of it are (n, ε)-separated.

Given a function φ on J we write $E_n(y) = \exp \sum_{i=0}^{n-1} \varphi(f^i(y))$.

Let us recall that topological pressure $P = P(f, \varphi)$ for any continuous function φ is defined by

$$P = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup\{\sum_{x \in A} E_n(x)\}$$

supremum being taken over all (n, ε)-separated sets $A \subset J$. If one replaces $\limsup_{n\to\infty}$ by $\liminf_{n\to\infty}$ here, one obtains the same P (see [W] Th. 9.4).

If φ is Hölder continuous we fix numbers $\kappa = \kappa(\varphi)$ and $C(\varphi)$ such that $|\varphi(x) - \varphi(y)| \le C(\varphi)(\mathrm{dist}(x, y))^\kappa$.

If μ is an arbitrary probability measure on J then a general theorem [Pa] says that there exists Jacobian $J_\mu = J_\mu(f)$ on a set of full measure μ. It means that there exists a set Y with $\mu(J \backslash Y) = 0$ and an integrable function J_μ such that for every $E \subset Y$ on which f is 1-to-1 onto an image we have $\mu(f(E)) = \int_E J_\mu d\mu$. If $Y = J$ we say: there exists the Jacobian of the measure μ on J or just there is a Jacobian.

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1. Introductory estimates

We begin with a simple fact (Lemma 0) which appeared already in several papers, not separated into a lemma. We leave its proof to the reader.

Lemma 0. *For every $\alpha > 0$ and every neighbourhood $\mathcal{V}$ of $S'(f)$ there exists $r = r(\alpha) > 0$ and $n_0 > 0$ such that for every $n \ge n_0$ and $x \in \hat{\mathbb{C}}$ such that $f^i(x) \in \hat{\mathbb{C}} \backslash \mathcal{V}$ for each $i = 0, 1, \dots, n$, we have:*

$$\mathrm{Card}\{i \in \{0, \dots, n-1\}: \mathrm{dist}(f^i(x), \mathrm{Crit}(f)) < r\}/n < \alpha.$$

Now, it comes the crucial:

Lemma 1. *For every $\varepsilon, \delta_1, \delta_2 > 0$ there exists $K > 0$ such that for every $n \ge 0$ there exists U_n a covering of a neighbourhood $\mathcal{A}$ of J by discs (in a standard metric on $\hat{\mathbb{C}}$) such that*

(i) Card $U_n < K \exp n\delta_1$; *If for every $B \in U_n$ we denote by B_n the family of components of $f^{-n}(B)$, each repeated as many times as its multiplicity, then*

(ii) *for every $V \in B_n$ and $0 \le i \le n$, $\mathrm{diam}\, f^i(V) < \varepsilon$;*

(iii) $\mathrm{Card}\{V \in B_n: \mathrm{diam}(V) \ge \exp -n\delta_2\} \le K \exp 2n\delta_2$.

$\mathcal{A}$ can be taken independent of n(and $\varepsilon, \delta_1, \delta_2$) for example

$$\mathcal{A} = \hat{\mathbb{C}} \backslash \bigcup \{B(c, r_0): c \in S'(f)\},$$

with $r_0 > 0$ arbitrarily small.

Proof. Fixed n, consider as an element of U_n a disc around each $a \in \mathrm{Critv}(f^n) \cap \mathcal{A}$ of radius $\eta_n = \exp(-\exp(n\delta_1/2))$.

Obviously $\mathrm{Card}(\mathrm{Critv}(f^n)) \le \mathrm{Const} \cdot n$.

Given an arbitrary C, say $C \ge 3$, we can cover the rest of $\mathcal{A}$ by discs

$B(x_j, r_j), j = 1, \dots, k_n$, such that $B(x_j, Cr_j) \cap \mathrm{Critv}(f^n) = \varnothing$ and

$$k_n \le \mathrm{Const} \cdot \mathrm{Card}(\mathrm{Critv}(f^n)) \cdot \frac{C}{\log C - \log(C-1)} \cdot \log \frac{1}{\eta_n} \qquad (1)$$

So for a constant depending on C

$$k_n \le \mathrm{Const} \cdot n \cdot \exp n\delta_1/2 \qquad (2)$$

We prove (1) (it is similar to lemma in **[Lju1]** or lemma in Section 6 of **[Lju2]**). For each $a \in \mathrm{Critv}(f^n)$ we cover $B(a,1)\backslash B(a,\eta_n)$ by annuli

$$B\left(a, (1-\frac{1}{C})^{k-1}\right) \backslash B\left(a, (1-\frac{1}{C})^{k}\right),$$

than cover each annulus (after subtracting discs of radii $(1-\frac{1}{C})^k$ and origins at other critical values) by discs of radii $(1-\frac{1}{C})^k$ and origins in the exterior circle of the annulus. The number of discs needed to cover each annulus is at most Const $\cdot C$ if they are chosen carefully enough. We consider these discs as elements of U_n. A simple computation gives (1).

Of course (2) implies (i).

To prove (ii) and (iii) let us make the following simple observations (we leave the proofs to the reader):

1. There exist $\beta > 0$ and $\xi > 1$ such that for every $x \in \widehat{\mathbb{C}}, \hat{r} > 0$ and every component V of $f^{-1}(B(x,\hat{r}))$ it holds

$$\mathrm{diam}(V) < \xi \hat{r}^{\beta}$$

2. For every $r > 0$ there exists $\xi > 1$ such that for every $x \in \widehat{\mathbb{C}}$ for which $\mathrm{dist}(x, \mathrm{Crit}) \ge r$, for every $\hat{r}$ and for every component V of $f^{-1}(B(f(x),\hat{r}))$

$$\mathrm{diam}(V) < \xi \hat{r}.$$

Fix now $B = B(a,\eta_n) \in U_n$ for $a \in \mathrm{Critv}(f^n)$ and consider components V_i of $f^{-i}(B)$ such that $f(V_i) = V_{i-1}$. From the above observations and from Lemma 0 we conclude that

$$\begin{aligned}\mathrm{diam}\, V_i &\le \eta_n^{\beta^{\alpha i}} \cdot \xi^i \\ &\le \exp(-\exp(n\delta_1/2 + \alpha i \log \beta) + i \log \xi) \\ &\le \exp(-\exp(n\delta_1/3))\end{aligned}$$

for every $i < n$ provided α is small enough, n large enough.

This inequality yields (ii) and (iii) for $B = B(a,\eta_n), a \in \mathrm{Critv}(f^n)$.

For each other $B = B(x_j, r_j) \in U_n$, each branch of f^{-i} for every $i = 0, \dots, n$ is a univalent function on $B(x_j, Cr_j)$. It has distortion bounded by a universal constant C_0 on $\widehat{B} = B(x_j, Cr_j/2)$, i.e. $\left|(f^{-i})'(z_1)\right| / \left|(f^{-i})'(z_2)\right| < C_0$ for $z_1, z_2 \in B(x_j, Cr_j/2)$ (We rely on Koebe's Distortion Theorem, **[H]** Th. 17.4.6). which in Euclidean metric on $\mathbb{C}$ would give $C_0 \le 256$. Here it must be corrected as we deal in $\widehat{\mathbb{C}}$ in another metric.) This implies that length $(f^{-i}(\widehat{x_j\varsigma})) < C_0 \,\mathrm{diam}\, \widehat{\mathbb{C}}$ for every geodesic $(\widehat{x_j\varsigma}) \subset \widehat{B}$ joining x_j with $\varsigma \in \partial \widehat{B}$. So $\mathrm{diam}\, f^{-i}(B) \le \frac{C_0}{C/2} C_0 \,\mathrm{diam}\, \widehat{\mathbb{C}}$. This is less than ε if $C > 2C_0^2 \,\mathrm{diam}\, \widehat{\mathbb{C}}/\varepsilon$.

Now obviously

$$\mathrm{Card}\left\{V \in B_n \colon \mathrm{Vol}\, V \ge \frac{1}{A} \exp -2n\delta_2\right\} \le A \cdot \mathrm{Vol}\, \widehat{\mathbb{C}} \cdot \exp 2n\delta_2$$

for every $A > 0$. By the distortion estimate $\mathrm{Vol}\, V \le \frac{1}{A} \exp -2n\delta_2$ implies $\mathrm{diam}\, V \le \frac{C_0}{\sqrt{A}} \exp -n\delta_2 \le \exp -n\delta_2$ if $C_0 \le \sqrt{A}$. This gives (iii). $\square$

Lemma 2. *Given any continuous function φ on $\mathbb{J}$, for every $\delta > 0$ there exists $C > 0$ such that for every integer $n > 0$ there exists $x \in \mathbb{J}$ with the property*

$$\mathcal{P}_\varphi^n(1)(x) \ge C \exp n(P - \delta).$$

(Note that we do not assume Hölder continuity of φ here and in the rest of this section.)

Proof. By the definition of pressure P, given an arbitrary $\delta_0 > 0$ there exists $\varepsilon_0, C_0 > 0$ such that for every $n \ge 0$ there exists an (n, ε_0)-separated set $\{y_t\} \subset \mathbb{J}$ such that

$$\sum_t E_n(y_t) \ge C_0 \exp n(P - \delta_0).$$

For every fixed n and $B \in U_n$ denote $T_B = \{t \colon f^n(y_t) \in B\}$. By Lemma 1 there exists B_0 such that

$$\sum_{t \in T_{B_0}} E(y_t) \ge \frac{C_0}{K} \exp n(P - \delta_0 - \delta_1),$$

(U_n, K and δ_1 from Lemma 1).

Let x be an arbitrary point in $B_0 \cap \mathbb{J}$. Denote by $\hat{y}_t$ an arbitrary point of the set $f^{-n}(\{x\})$ in the same component of $f^{-n}(B_0)$ as y_t. By (ii), Lemma 1 we have $\mathrm{dist}(f^i(y_t), f^i(\hat{y}_t)) \le \varepsilon$ for every $i = 0, \dots, n, \varepsilon$ from Lemma 1). (This implies that provided $\varepsilon < \varepsilon_0/2$ the points $\hat{y}_t$ are pairwise distinct and that

$$\mathcal{P}_\varphi^n(1)(x) \geq \frac{C_0}{K} \exp n(P - \delta_0 - \delta_1 - \mathrm{var}_\varepsilon\, \varphi). \qquad \square$$

Remarks.

1. If $\varphi \equiv 0$ we obtain

$$(\deg f)^n \geq C \exp n(h_{\mathrm{top}}(f|_J) - \delta),$$

so $\log(\deg f) \geq h_{\mathrm{top}}(f|_J)$. With some more effort one could prove it with $h_{\mathrm{top}}(f)$ on $\hat{\mathbb{C}}$. This is a theorem proved independently by Gromov [G] and Ljubich ([**Lju1**], [**Lju2**]). Our proof is in fact a variant of Ljubich's proof.

Given an arbitrary $\varepsilon_0 > 0$ take in Lemma 1

$$\mathcal{A} = \hat{\mathbb{C}} \setminus U_{p \in S'(f)} B(p, \varepsilon_0/2).$$

Every (n, ε_0)-separated set $\{y_t\}$ can be divided into $(n+2) \cdot \mathrm{Card}(S'(f))$ families $Y_{i,p}$ depending on the last $i: n \geq i \geq 0$ with $f^i(y_t) \in \mathcal{A}$ and to which $B(p, \varepsilon_0/2)$ the point $f^{i+1}(y_t)$ belongs, except for $i = n$. We define $Y_{-1,p} = \{y_t\} \cap B(p, \varepsilon_0/2)$. Each $Y_{i,p}$ must be (i, ε_0)-separated. In Proof of Lemma 2 we could consider y_t with $f^n(y_t) \in \mathcal{A}$ rather than J. For i playing the role of n in Lemma 2 we get

$$\mathrm{Card}\, Y_{i,p} \leq K \exp n\delta_1 (\deg f)^i.$$

Summing over i, p gives $\mathrm{Card}\{y_t\} \leq \mathrm{Const} \cdot (\exp n\delta_1)(\deg f)^n$.

2. If "for every x" could take the place of "there exists x" in the statement of Lemma 2, then it would follow that $\log \lambda \geq P$, and therefore $\log \lambda > \sup \varphi$. In this case the easy part of the proof of the paper of Denker-Urbański could be used and simplify the argument (see Appendix A).

Now let us discuss a lemma which idea comes form [**M-P**].

Lemma 3. *For every δ there exists $C, \varepsilon > 0$ such that for every $x_0 \in J$ and every $n > 0$ and $y_0 \in f^{-n}(x_0)$ the number of the points $z \in f^{-n}(x_0)$ which are (n, ε)-close to y_0 (i.e. such that $\mathrm{dist}(f^i(z), f^i(y_0)) \leq \varepsilon$ for every $i: 0 \leq i \leq n$) is less than $C \exp n\delta$.*

Proof. Define a non-oriented graph $\mathcal{T} = \mathcal{T}(x_0)$ as follows: suppose first that $x_0 \notin \mathrm{Critv}(f^n)$. Then the vertices are pairs (z, i) where $i = 0, \ldots, n$, $f^i(z) = x_0$ and $\mathrm{dist}(f^j(z), f^{n-i+j}(y_0)) < \varepsilon$ for every $j: 0 \leq j \leq i$, (ε will be specified later on).

We join (z_1, i_1), with (z_2, i_2) with an edge iff

$$i_2 = i_1 - 1 \quad \text{and} \quad f(z_1) = z_2.$$

If $x_0 \in \mathrm{Critv}(f^n)$ then we take into account each (z, i) as many times as the degree of f^i at z. We find $x_0' \notin \mathrm{Critv}(f^n)$ so close to x_0 that there is a 1-to-1 correspondence of each $z \in f^{-i}(x_0')$ to the closest point of $f^{-i}(x_0)$. Then the edges of $\mathcal{T}(x_0)$ are determined by the edges of $\mathcal{T}(x_0')$.

For every vertex (z, n) denote by $\mathcal{T}(z, n)$ the subgraph containing the vertices $e_i = (f^i(z), n-i)$ for every $i = 0, \ldots, n$ and the edges joining them. By Lemma 0 for every α we find ε and n_0 such that

$$\mathrm{Card}\{i: e_i \ \text{ is a branch vertex of } \mathcal{T}\}/n < \alpha,$$

if $n > n_0$. A branch vertex means that at least 3 edges meet at it. (ε is not precisely that one from Lemma 0. But if

$$f(x) = f^{i+1}(z), \quad x \neq f^i(z), \quad \text{and} \quad \mathrm{dist}(x, f^i(z)) < 2\varepsilon,$$

then there is a critical point c with $\mathrm{dist}(c, f^i(z)) < K2\varepsilon$ for a constant K. So we first find for α a number ε_1 which plays the role of ε from Lemma 0, then define $\varepsilon = \varepsilon_1/2K$.)

We conclude (as in [**MP**]) that for the set V_n of all the vertices of $\mathcal{T}$ of the form (z, n), for $n > n_0$

$$\mathrm{Card}(V_n) \leq (\deg f)^{\alpha n}.$$

So if α were chosen so that $(\deg f)^\alpha < \exp \delta$ and $C = (\deg f)^{n_0}$ we get $\mathrm{Card}(V_n) \leq C \exp n\delta$. $\square$

Lemma 4. *For any continuous function φ on J and for every $\delta > 0$ there exists $C > 0$ such that for every $x \in J$ and $n \geq 0$*

$$\mathcal{P}_\varphi^n(1)(x) \leq C \exp n(P + \delta).$$

Proof. By Lemma 3 for every x the set $f^{-n}(\{x\})$ can be divided into $C \exp n\delta$ (n, ε)-separated sets (c, δ, ε from Lemma 3). So

$$\mathcal{P}_\varphi^n(1)(x) \leq C \exp n\delta \cdot C_1 \exp n(P + \delta_1),$$

where C_1, δ_1 are defined by

$$\sum_{y \in Y} E_n(y) \leq C_1 \exp n(P + \delta_1)$$

for every n and (n, ε)-separated set Y (δ_1 is arbitrarily small by the definition of pressure). This proves Lemma 4. □

Remarks.

4. For $\varphi \equiv 0$ we obtain the **[MP]** estimate $h_{\text{top}}(f|_J) \geq \deg f$.

5. Modifying Proof of Lemma 2, with the help of Lemma 3 we easily deduce the so-called asymptotic h-expansiveness for f (we recall the definition below). It is again a modification of Ljubich's proof **[Lju 2]**.

We work with $\varphi \equiv 0$. Given $\delta > 0$, take C, ε from Lemma 3. Then for every $\varepsilon' > 0$, for every n, if $\{y_t\}$ is a family of (n, ε')-separated points such that for every $t, y_t \in B_n(x, \frac{\varepsilon}{2})$, we can find $B_0 \in U_n$ with Card $T_{B_0} \geq \frac{1}{C_0 \exp n\delta_1}$ Card$\{y_t\}$ (C_0, δ_1, U_n from Lemma 1, $T_B = \{t\colon f^n(y_t) \in B\}$, but for ε in Lemma 1 we take $\min\left(\frac{\varepsilon'}{2}, \frac{\varepsilon}{2}\right)$.) We obtain pairwise different $\hat{y}_t \in B_n(x, \varepsilon)$ all with the same $f^n(\hat{y}_t) \in B_0$ as in Proof of Lemma 2. So by Lemma 3, Card $T_{B_0} \leq C \exp n\delta$, hence Card$\{y_t\} \leq C_0 C \exp n(\delta + \delta_1)$. So

$$\limsup_{\varepsilon \to 0} \lim_{x \in J} \lim_{\varepsilon' \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(\sup(\text{Card}\{y_t\})) = 0,$$

the latter supremum being taken over all $\{y_t\}$, (n, ε')-separated families in $B_n(x, \frac{\varepsilon}{2})$. (This equality is just the definition of asymptotic h-expansiveness with f a continuous mapping on a metric space J.)

Recall that asymptotic h-expansiveness implies the existence of an equilibrium state for any continuous function **[Mi]** (but for φ Hölder, in our case, we shall prove the existence independently).

2. Telescope Lemma

Lemma 5. *Given a rational map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $0 < \lambda < \mu < 1$ there exists $\varepsilon > 0$ such that for every $N > 0$ and a sequence of curves $\gamma = \gamma_0, \dots, \gamma_N$ for which:*

f maps γ_n 1-to-1 to γ_{n-1} for $n = 1, 2, \dots, N$,

$\operatorname{diam} \gamma_0 < \varepsilon$ *(diameter in a standard conformal metric ϱ on $\hat{\mathbb{C}}$)*

$\operatorname{diam} \gamma_n \leq \gamma^n \operatorname{diam} \gamma_0$, *for every $n = 0, 1, \dots, N$*

the following is satisfied:

There exists sets $V_n \subset \hat{\mathbb{C}}, n = 0, 1, \dots, N$ such that $V_n \supset \gamma_n, V_n$ is a component of $f^{-n}(V_0), V_0 = B(x_0, \varepsilon)$ for some $x_0 \in \gamma_0$, finally $\operatorname{diam}(V_n) \leq$ *μ^n.*

Remarks.

1. Another version of this lemma is that for every $0 < \gamma < \mu < 1$ and $K > 0$, there exists $\varepsilon > 0$ such that if points $x_n, n = 0, 1, \dots, N$, satisfy $f(x_n) = x_{n-1}$ and

$$|(f^n)'(x_n)| \geq K/\lambda^n, \tag{3}$$

then there exists a sequence (V_n) satisfying the assertions above.

2. If one assumes that dist(Crit $f, \gamma_n) \geq$ Const > 0 or dist(Crit $f, x_n) \geq$ Const > 0) or at least dist(Crit $f, \gamma_n) \geq K(\lambda + \theta)^n$ for $\theta > 0$ arbitrarily small, then the assertion of the lemma is well-known. One of the possible proofs is to build a "telescope" in a Lyapunov metric. (The term "telescope" in the expanding case has been introduced by D. Sullivan, see [S].) One gets $f|_{V_n}$ 1-to-1. We shall build a "telescope" also in our case.

3. Of course it is not true in our situation that $\{V_n\}$ can be found so that $f|_{V_n}$ is 1-to-1 for every n. For example, let $c = c_0$ be a critical, non-periodic point with $f^m(c) = y$ being a periodic source, for some $m > 0$. Let $c_n, n \geq 0$, be a backward trajectory of $c, f(c_n) = c_{n-1}$, with the property $|(f^n)'(c_n)| \geq$ Const $\cdot \xi^n$, for $\xi > 1$ and every $n \geq 0$. We can take $x = x_0$ arbitrarily close to y and its backward trajectory x_n following the periodic trajectory of y for an arbitrarily long time, then the trajectory $f^m(c), f^{m-1}(c), \dots, c$, finally c_n. It is easy to prove that (3) holds. Yet $B(x, \varepsilon)$ contains y so some V_k contains c.

4. The following model situation provides an idea of the proof of Lemma 5: Let $(f_n)_{n>0}$ be a sequence of maps $f_n\colon \mathbb{C} \to \mathbb{C}$ such that $f_n(0) = 0, f_n(z) = z^d$ if $N|n$ and $f_n(z) = \lambda_n^{-1} z$ otherwise, for a sequence $\lambda_n > 0, n = 1, 2, \dots$ and integers $N > 0, d > 1$. Let $w_n \in \mathbb{C}$ be a sequence of points such that $f_n(w_n) = w_{n-1}$ and $f_n'(w_n) = \lambda_n^{-1}$ and V_n be a sequence of connected sets such that each V_n is a component of $f_n^{-1}(V_{n-1})$ and each V_n contains 0 and w_n. Then for $n = kN$

$$\operatorname{diam} V_n \leq \operatorname{diam}(V_n \cap f_n^{-1}(B(0, |w_{n-1}|))) +$$
$$+ \operatorname{diam}(V_n \cap f_n^{-1}(V_{n-1} \setminus B(0, |w_{n-1}|))$$

$$\leq 2|w_{n-1}|^{1/d} + \left|(f_n^{-1})'(w_{n-1})\right| \operatorname{diam}(V_{n-1})$$
$$= I + II.$$

The estimate by the summand II follows from the concavity of $r \mapsto r^{1/d}$. As

$$I \leq 2|w_{n-1}|^{1/d-1} \operatorname{diam}(V_{n-1}) = 2d\left|(f_n^{-1})'(w_{n-1})\right| \cdot \operatorname{diam}(V_{n-1}),$$

we obtain

$$\operatorname{diam}(V_n)/\operatorname{diam}(V_{n-1}) \leq 3d\lambda_n.$$

If $n \neq kN$ we have $\operatorname{diam}(V_n)/\operatorname{diam}(V_{n-1}) = \lambda_n$. Thus

$$\operatorname{diam}(V_n) \leq \left(\prod_{j=1}^{n} \lambda_j\right) (3d)^{[n/N]+1} \cdot \operatorname{diam}(V_0).$$

If $\prod_{j=1}^n \lambda_j \leq \lambda^n$ for some $\lambda < 1$ and every $n > 0$, then $\operatorname{diam}(V_n)$ converges exponentially to 0 if $N > \log 3d/\log \lambda^{-1}$.

5. Observe that an assertion analogous to that of Lemma 5 but for forward iteration is false. Concavity of $r^{1/d}$ is replaced by convexity of r^d unfortunately.

Proof of Lemma 5. Choose μ_1 such that $\lambda < \mu_1 < \mu$. Take ε_0 such that for every $x \in \widehat{\mathbb{C}}$ there exists a conformal chart $h_x: B(x, 2\varepsilon_0) \to \mathbb{C}$ with the property:

$$|h_x'|, \left|(h_x^{-1})'\right| \leq \min(\sqrt[3]{\mu_1/\lambda}, 2). \tag{4}$$

(Another condition for ε_0 will be given later when we define another coordinates: H.)

Observe that there exists $K_1 > 1$ such that if $f(x) = f(y)$ then there exists a critical point c $(f'(c) = 0)$ such that

$$\varrho(c, x) \leq K_1 \varrho(x, y) \tag{5}$$

(Check it in a neighbourhood of each critical point, where the map is roughly $z \to z^d$.)

Let $K_2 > 1$ be a constant such that for every univalent map $F: \mathbb{D} \to \mathbb{C}$ ($\mathbb{D}$-the unit disc) the distortion:

$$\sup_{|x|,|y| \leq 1/K_2} |F'(x)/F'(y)| < \sqrt[3]{\mu_1/\lambda}. \tag{6}$$

(K_2 exists by Koebe's Distortion Theorem, [**H**] Th. 17.4.6.)

Take now $\gamma_n, n = 0, 1, \ldots, N$ satisfying the assumptions of our Lemma, (ε will be specified only at the end of the proof). Choose a trajectory $x_n \in$

$\gamma_n, f(x_n) = x_{n-1}$. Define $g_n: \mathbb{C} \to \mathbb{C}, g_n(0) = 0$, by

$$g_n(x) = \tau_{n-1}^{-1}(h_{x_{n-1}} \circ f \circ h_{x_n}^{-1}(\tau_n x)),$$

where

$$\tau_n/\tau_{n-1} = \begin{cases} (\operatorname{diam}\gamma_n/\operatorname{diam}\gamma_{n-1})(\mu_1/\lambda), & \text{if } (*_n) \text{ holds} \\ (\operatorname{diam}\gamma_n/\operatorname{diam}\gamma_{n-1})(\mu_1/\lambda) \cdot C, & \text{otherwise,} \end{cases} \tag{7}$$

where $(*_n)$ denotes $\operatorname{Crit}(f) \cap B(x_n, \varepsilon_1) = \emptyset$. (Each number τ_n is identified with the mapping $z \to \tau_n z$.) The constant $C > 1$ will be specified later on.

Now we will specify τ_0 and ε_1. Fix α such that

$$\mu_1 C^\alpha < \mu.$$

Set $\varepsilon_1 = \min(\varepsilon_0, r(\alpha), 1)$ where $r(\alpha)$ satisfies Lemma 0. Take n_0 also from Lemma 0. Finally take τ_0 such that

$$4 \sup|f'| \tau_0 C^{n_0} \leq \varepsilon_1.$$

We obtain for every n

$$4 \sup|f'| \mu_1^n \tau_0 C^{n\alpha + n_0} < \mu^n \varepsilon_1.$$

By (7) and Lemma 0 this gives for every n

$$4 \sup|f'| \tau_n < \mu^n \varepsilon_1 \tag{8}$$

Write $\mathbb{D}_r = \{|z| < r\}$. In particular (8) implies that each g_n is defined on $\mathbb{D}_1$. Set now $\beta = 1/16K_1K_2$. Suppose that

$$\operatorname{diam}\gamma_0 \leq \beta\tau_0/2. \tag{9}$$

Then by (7) $\operatorname{diam}\gamma_n \leq \operatorname{diam}\gamma_0 \cdot \tau_n/\tau_0 \leq \beta\tau_n/2$.

Denote $\widehat{\gamma}_n = \tau_n^{-1} h_{x_n}(\gamma_n)$. We conclude that

$$\operatorname{diam}\widehat{\gamma}_n \leq \beta.$$

Moreover $\operatorname{diam}\gamma_{n-1}/\operatorname{diam}\gamma_n \geq (\tau_{n-1}/\tau_n)(\mu_1/\lambda)$ and by (4)

$$\operatorname{diam}\widehat{\gamma}_{n-1}/\operatorname{diam}\widehat{\gamma}_n \geq \sqrt[3]{\mu_1/\lambda},$$

so there is $w \in \mathbb{D}_\beta$ with $|g_n'(w)| \geq \sqrt[3]{\mu_1/\lambda}$.

Consider now the case $g_n' \neq 0$ on $\mathbb{D}_1$. Then g_n is 1-to-1 on $\mathbb{D}_{1/8K_1}$, by (5). Finally by (6)

$$g_n(\mathbb{D}_\beta) \supset \operatorname{cl}\mathbb{D}_\beta. \tag{10}$$

Now we consider the case of n such that there exists $\widehat{z}_0 \in \mathbb{D}_1$ where $g_n'(\widehat{z}_0) =$

0. Denote $z_0 = h_{x_n}^{-1}(\tau_n \hat{z}_0)$. In particular $z_0 \in B(x_n, \varepsilon_1)$ i.e. the second alternative in (7) takes place.

Let us supply now the missing condition on ε_0. Suppose that ε_0 is small enough that for every $c \in \mathrm{Crit}(f)$ there exist holomorphic charts

$$H_{c,1}: (B(c, 2\varepsilon_0), c) \to (\mathbb{C}, 0)$$

and

$$H_{c,2}: (B(f(c), 2\varepsilon_0), f(c)) \to (\mathbb{C}, 0)$$

such that

$$\left|H'_{c,i}\right|, \left|(H_{c,i}^{-1})'\right| \leq 2, \quad i = 1, 2$$

and $H_{c,2} \circ f \circ H_{c,i}^{-1}(z) = a_c z^d$, with a coefficient $a_c \in \mathbb{C}$ and $d = d(c)$ degree of f at c.

For

$$G_1 = \tau_n^{-1} \circ H_{z_0,1} \circ h_{x_n}^{-1} \circ \tau_n \quad \text{and} \quad G_2 = \tau_{n-1}^{-1} \circ H_{z_0,2} \circ h_{x_{n-1}}^{-1} \circ \tau_{n-1} \tag{11}$$

we have

$$\Phi_n(z) = G_2 \circ g_n \circ G_1^{-1}(z) = \tau_{n-1}^{-1} a_{z_0} (\tau_n z)^d = (\tau_n^d \tau_{n-1}^{-1} a_{z_0}) z^d.$$

We denote the latter coefficient by T_n, so $\Phi_n(z) = T_n z^d$.

Observe that G_1, G_1^{-1} are well-defined on $\mathbb{D}_2$, G_2, G_2^{-1} are well-defined on $g_n(\mathbb{D}_2) \cup \mathbb{D}_2$ and $\Phi_n(\mathbb{D}_2)$ respectively and

$$|G'_i|, \left|(G_i^{-1})'\right| < 4, \quad i = 1, 2. \tag{12}$$

Similarly as in the non-singular case we prove that there exists $w \in \mathbb{D}_1$ such that $|g'_n(w)| > C$. So by (12) $|\Phi'_n(G_1(w))| > C/16$. But $\Phi'_n(G_1(w)) = d \cdot T_n (G_1(w))^{d-1}$ and by (12), $|G_1(w)| < 8$. So

$$T_n > C/16 \cdot 8^{d-1} \cdot d.$$

By (12) we have $\mathrm{diam}(G_2(\mathbb{D}_\beta)) < 4\beta$. So each component C of $\Phi_n^{-1}(\mathrm{cl}\, G_2(\mathbb{D}_\beta))$ has diameter less than $\sqrt[d]{4\beta \cdot 16 \cdot 8^{d-1} d/C}$. To assure

$$G_1^{-1}(C) \subset \mathbb{D}_\beta \quad \text{for the component } C \text{ such that} G_1^{-1}(C) \ni 0, \tag{13}$$

it is enough to assume (again in view of (12)) that

$$\sqrt[d]{4\beta \cdot 16 \cdot 8^{d-1} d/C} < \beta/4, \quad \text{here with } d = \deg f. \tag{14}$$

This is the missing condition for C.

The inclusions (10) and (13) prove Lemma 5. We have $\mathrm{diam}\, h_{x_n}^{-1} \tau_n(\mathbb{D}_\beta) \leq 2\tau_n \beta \leq \mu^n \varepsilon_1$ and $h_{x_0}^{-1} \tau_0(\mathbb{D}_\beta)$ contains the ball $B(x_0, \tau_0 \beta/2)$, which we pick as V_0. Our Lemma holds for $\varepsilon = \tau_0 \beta/2$. □

3.. The main estimate $\mathcal{P}_\varphi^n(1)(x)/\mathcal{P}_\varphi^n(1)(y) <$ Const.

Now we are able to give the key improvement of Lemma 2 (cf. Remark 2 after its statement) and in consequence the estimate mentioned in the title of this section.

Lemma 6. *For every $\delta > 0$, there exists $C > 0$ such that for every $n > 0$ and $x \in \mathbb{J}$*

$$\mathcal{P}_\varphi^n(1)(x) \geq C \exp n(P - \delta). \tag{1}$$

Proof. Step 1. By Lemma 2 we know that fixed an arbitrary n, there exists $x_0 \in \mathbb{J}$ satisfying (1) (even with $\delta/2$ in place of δ). Take an arbitrary $x \in \mathbb{J}$ and join x with x_0 by a curve $\gamma \subset \mathcal{A}$ hitting at most once each element of U_i for every $i = 0, \dots, n$ (cf. Lemma 1). A curve γ has these properties if it is close to a shortest geodesic joining x with x_0 and the balls of U_n and balls complementary to $\mathcal{A}$ are small enough.

For every $j: 0 \leq j \leq n$ define Γ_j a family of curves such that for the union we have $\cup \Gamma_j = f^{-j}(\gamma)$, f^j maps each $\hat{\gamma} \in \Gamma_j$ to γ 1-to-1 and the families are compatible, i.e., for each $\hat{\gamma} \in \Gamma_j$, $f(\hat{\gamma}) \in \Gamma_{j-1}$. ($\Gamma_j$ can be constructed by induction. Having $\hat{\gamma} \in \Gamma_{j-1}$ we consider lifts by f^{-1}. Meeting a critical value for f in $\hat{\gamma}$ we have a choice of lifting.) For each $\hat{\gamma} \in \Gamma_j$ we denote by $y(\hat{\gamma})$ the point in $\hat{\gamma}$ mapped by f^j to x_0.

Let us order the family Γ_n by $\Gamma_n = (\gamma^t)_{t=1,\dots,T}$, where $T = (\deg f)^n$. For each t the point $z \in \gamma^t$ such that $f^n(z) = x_0$ or x, is denoted by x_0^t or x^t respectively.

Fix $\delta_0 > 0$ (we shall specify it in Step 3). For each $j = 0, \dots, n$ denote

$$A_j = \{t \in \{1, \dots, T\}: \mathrm{diam}(f^j(\gamma^t)) > \exp -(n-j)\delta_0$$

and

$$\mathrm{diam}(f^i(\gamma^t)) \leq \exp -(n-i)\delta_0 \quad \text{for every } i: 0 \leq i < j\}.$$

$$A_{n+1} = \{1, \dots, T\} \setminus \bigcup_{j=0}^{n} A_j.$$

For each $\widehat{\gamma} \in \Gamma_{n-j}$ we denote $A(j,\widehat{\gamma}) = \{t \in A_j : f^j(\gamma^t) = \widehat{\gamma}\}$.

We shall prove now that there exists $C_1 > 0$ depending only on δ_0 (and f, φ) such for each j and each $\widehat{\gamma} \in \Gamma_{n-j}$ and $y = y(\widehat{\gamma})$

$$\frac{\sum_{t\in A(j,\widehat{\gamma})} E_n(x_0^t)}{\mathcal{P}_\varphi^n(1)(x_0)} \le C_1 \frac{E_{n-j}(y)}{\mathcal{P}_\varphi^{n-j}(1)(x_0)}. \tag{2}$$

It is sufficient to consider n and $n-j$ large. Then for $t \in A_j$,

$$\operatorname{diam}(f^j(\gamma^t)) \le \sup|f'| \cdot \exp(-(n-j+1)\delta_0)$$

is small. So Lemma 5 is applicable to the curve $\gamma_0 = \widehat{\gamma}$ and each sequence of its pre-images ending with $\gamma^t, t \in A(j,\widehat{\gamma})$. Consider ε, μ and V_0 from Lemma 5. Observe that there exists C_2 depending only on ε such that for $A = \{w \in f^{-(n-j)}(\{x_0\}) \cap V_0\}$, where we repeat each w as many times as degree of f^{n-j} at w, we have

$$\mathcal{P}_\varphi^{n-j}(1)(x_0) \le C_2 \sum_{w\in A} E_{n-j}(w) \tag{3}$$

This is so because for n_0 such that $f^{n_0}(V_0) \supset \mathcal{J}(f)$ for every $z \in \mathcal{J}$ we have

$$\sum_{w\in A} E_{n-j}(w) \ge \exp\left(n_0(\inf_{v\in\mathcal{J}} \varphi(v))\right) \cdot (\mathcal{P}_\varphi^{n-j-n_0}(1)(x_0)) \ge$$

$$\ge \exp\left(n_0(\inf_{v\in\mathcal{J}} \varphi(v))\right) (\mathcal{P}_\varphi^{n-j}(1)(x_0)) / \sup_{v\in\mathcal{J}} \mathcal{P}_\varphi^{n_0}(1)(v).$$

Now by Lemma 5 for every $w \in A$ and every $t \in A(j,\widehat{\gamma})$ there exists $w^t \in \mathcal{J}$ such that $f^j(w^t) = w$ and for every $i : 0 \le i \le j$ $\operatorname{dist}(f^i(x_0^t), f^i(w^t)) < \mu^{j-i}$. (Some points w^t may coincide but we take care that w^t are taken with right multiplicities when we sum over them in the definition of $\mathcal{P}_\varphi^j(1)(w)$.)

So $E_j(x_0^t)/E_j(w^t) < C_3$, where $C_3 = C_4/(1-\mu^\kappa)$, where C_4 and κ are Hölder continuity constants for φ.

Hence

$$\frac{E_{n-j}(w)}{E_{n-j}(y)} < C_3 \frac{\sum_{t\in A(j,\widehat{\gamma})} E_n(w^t)}{\sum_{t\in A(j,\widehat{\gamma})} E_n(x_0^t)}.$$

Summing over $w \in A$, taking inverses and using (3) we obtain (2) with $C_1 = C_2 \cdot C_3$.

Step 2. For $0 \le k \le n$ we have

$$C_5 \exp n(P-\delta_0) \le \mathcal{P}_\varphi^n(1)(x_0) \le \mathcal{P}_\varphi^k(1)(x_0) \cdot \sup \mathcal{P}_\varphi^{n-k}(1) \le$$

$$\le \mathcal{P}_\varphi^k(1)(x_0) \cdot C_6 \exp(n-k)(P+\delta_0)$$

with $C_5 = C$ from Lemma 2, $C_6 = C$ from Lemma 4 and δ_0 common for both Lemmas, so

$$\mathcal{P}_\varphi^k(1)(x_0) \ge (C_5/C_6) \cdot \exp k\left(P - \delta_0\left(\frac{2n}{k} - 1\right)\right) \tag{4}$$

Step 3. Denote $G_j = \{\widehat{\gamma} \in \Gamma_j : \operatorname{diam}(\widehat{\gamma}) \ge \exp -j\delta_0\}$. To estimate $\operatorname{Card} G_j$ we use Lemma 1 with $\delta_1 < \delta_2$. By (iii) for every $B \in U_j$ intersecting γ, the number of "bad" $\widehat{\gamma}$'s in Γ_j i.e. such that the component of $f^{-j}(B)$ (more precisely element of B_j) intersecting $\widehat{\gamma}$ has diameter $\ge \exp -j\delta_2$, is at most $K \exp 2j\delta_2$. So the number of "bad" $\widehat{\gamma}$'s for at least one B is by (ii) at most $K \exp 2j\delta_2 \cdot K \exp j\delta_1$.

The others, "good" $\widehat{\gamma}$'s, have diameter at most $(\exp -j\delta_2) \cdot K \exp j\delta_1$. So for $\delta_0 < \delta_2 - \delta_1$ and a constant K_1 depending on δ_0 we have

$$\operatorname{Card} G_j \le K_1 \exp j(2\delta_2 + \delta_1).$$

Then for every j:

$$\sum_{\widehat{\gamma}\in G_{n-j}} E_{n-j}(y(\widehat{\gamma})) \le \operatorname{Card}(G_{n-j}) \cdot \exp((n-j)\sup\varphi) \le$$

$$\le K_1 \exp(n-j)(2\delta_2 + \delta_1 + \sup\varphi).$$

Fix an arbitrary $\alpha : 0 < \alpha < 1$. Then by (2), the above and by (4) we obtain (square brackets stand for "integer part")

$$\frac{\sum_{j=0}^{[\alpha n]} \sum_{t\in A_j} E_n(x_0^t)}{\mathcal{P}_\varphi^n(1)(x_0)} \le C_1 \sum_{j=0}^{[\alpha n]} \frac{K_1 \exp(n-j)(2\delta_2+\delta_1+\sup\varphi)}{\mathcal{P}_\varphi^{n-j}(1)(x_0)} \le$$

$$\le C_1 \sum_{j=0}^{[\alpha n]} \frac{K_1 \exp(n-j)(2\delta_2+\delta_1+\sup\varphi)}{(C_5/C_6)\exp(n-j)\left(P - \delta_0\left(\frac{2n}{n-j} - 1\right)\right)} \le$$

$$\le C_7 \sum_{j=0}^{[\alpha n]} \exp(n-j)\left(\sup\varphi - P + 2\delta_2 + \delta_1 + \delta_0\left(\frac{2n}{n-[\alpha n]} - 1\right)\right),$$

for $C_7 = C_1 K_1 C_6 / C_5$.

If δ_1, δ_2 and δ_0 are chosen small enough (given α) then due to the assumption $P > \sup\varphi$ the last expression is less than

$$C_7 \sum_{j=0}^{[\alpha n]} \exp(n-j)\left(\frac{\sup\varphi - P}{2}\right) \le \frac{1}{2} \tag{5}$$

for n large enough.

Step 4. For each $t \notin \bigcup_{j=0}^{[\alpha n]} A_j$ we have

$$\frac{E_n(x_0^t)}{E_n(x^t)} \leq \exp(I + II), \tag{6}$$

where

$$I = \sum_{i=0}^{[\alpha n]} \left(\varphi(f^i(x_0^t)) - \varphi(f^i(x^t))\right),$$

and

$$II = \sum_{i=[\alpha n]+1}^{n-1} \left(\varphi(f^i(x_0^t)) - \varphi(f^i(x^t))\right).$$

The summand I is bounded by a constant, say by 1, for n large enough. This is because $\operatorname{dist}(f^i(x^t), f^i(x_0^t)) \leq \exp -(n-i)\delta_0$ for $i \leq [\alpha n]$ and φ is Hölder continuous.

The summand II is bounded by $\exp((1-\alpha)n \operatorname{var} \varphi)$.

Step 5. (Conclusion). By Step 3 and then by Step 4:

$$\begin{aligned} \mathcal{P}_\varphi^n(1)(x_0) &\leq 2 \sum_{j=[n\alpha]+1}^{n+1} \sum_{t \in A_j} E_n(x_0^t) \\ &\leq 2\exp(1 + (1-\alpha)n \operatorname{var} \varphi) \cdot \mathcal{P}_\varphi^n(1)(x). \end{aligned}$$

Taking α sufficiently close to 1 we get (1). □

Remark. Lemmas 4 and 6 imply that for every $x \in J$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{P}_\varphi^n(1)(x) = P(f, \varphi),$$

in particular the limit exists. This generalizes the equality $\log(\deg f) = h_{\mathrm{top}}(f|_J)$ (cf. Remark 1 in Section 1).

Lemma 7. *There exists $C > 0$ such that for every $x_0, x_1 \in J$ and every $n \geq 0$*

$$\mathcal{P}_\varphi^n(1)(x_0)/\mathcal{P}_\varphi^n(1)(x_1) < C$$

Proof. Repeat the proof of Lemma 6. Instead of (4) we can use now, due to Lemma 6, the better estimate:

$$\mathcal{P}_\varphi^k(1)(x_0) \geq C \exp k(P - \delta).$$

So in Step 3 we can consider $\sum_{j=0}^{n-k_0}$ for a constant k_0 large enough, instead of $\sum_{j=0}^{[\alpha n]}$. So in Step 4 both summands are bounded by constants. □

4. Proofs of Theorems 1 and 2. Modulus of continuity of the density function ψ_0.

Consider the operator $\mathcal{P}_\varphi^*$ conjugate to $\mathcal{P}_\varphi$ on $C(J)$. As mentioned in Introduction there exists a number $\lambda > 0$ and a probability measure η on J such that

$$\mathcal{P}_\varphi^*(\eta) = \lambda\eta, \quad \text{(cf.[B]).} \tag{1}$$

Proposition 1. $\lambda = \exp P$. *The measure η is $\lambda\exp(-\varphi)$-conformal (i.e. $J_\eta = \lambda\exp(-\varphi)$, where J_η is Jacobian on J namely $\int_E J_\eta d_\eta = \eta(f(E))$ for every Borel set $E \subset J$ on which f is 1-to-1 to the image).*

Proof. We have by (1) $\int \mathcal{P}_\varphi^n(1) d\eta = \lambda^n$. So by Lemma 7 and Remark preceding it

$$\log \lambda = \lim_{n\to\infty} \frac{1}{n} \log \int \mathcal{P}_\varphi^n(1) d\eta = P.$$

Now if f is 1-to-1 on E, then

$$\begin{aligned} \int \chi_E(\lambda \exp(-\varphi)) d\eta &= \int \mathcal{P}_\varphi(\chi_E \cdot \exp(-\varphi)) d\eta \\ &= \int \exp(\varphi \circ (f|_E)^{-1}) \cdot \exp(-\varphi \circ (f|_E)^{-1}) \cdot \chi_{f(E)} d\eta \\ &= \eta(f(E)), \end{aligned}$$

so $\lambda \exp -\varphi$ is Jacobian for f, η. □

Proof of Theorem 1. (Stated in Introduction.) Write $\widehat{\mathcal{P}} = \mathcal{P}_{\varphi - P}$ ($= \mathcal{P}_\varphi/\lambda$ by Proposition 1). For every $\psi \in C(J)$ we have $\int \psi d\eta = \int \widehat{\mathcal{P}}(\psi) d\eta$. Observe that for $\psi \in C(J)$ and $n \geq 0$

$$\sup \left|\widehat{\mathcal{P}}^n(\psi)\right| \leq \sup |\psi| \cdot \sup \widehat{\mathcal{P}}^n(1)$$

But we have

$$\int \widehat{\mathcal{P}}^n(1) d\eta = \int 1 d\eta = 1.$$

This and Lemma 7 imply that all $\widehat{\mathcal{P}}^n(\psi)$ are uniformly bounded.

Now we shall prove equicontinuity. We shall again repeat the considerations from Proof of Lemma 6, there will be a simplification however, no need to apply Lemma 5, due to the possibility to use the result of Lemma 7.

Given $x_0, x_1 \in J$ let $\gamma = \gamma(x_0, x_1)$ be a curve joining x_0 with x_1, similar to the curve joining x_0 with x in Proof of Lemma 6. We keep the notation Γ_j as in Lemma 6, Step 1 (here n is not fixed however, so j goes from 0 to ∞).

Let $a: \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary positive, increasing function with $a(\varepsilon) \to 0$ and $a(\varepsilon) \geq \varepsilon^\alpha$ for every $\alpha > 0$ and ε small enough (depending on α).

Let for every $\varepsilon > 0, k(\varepsilon)$ be the maximal integer such that for every x_0, x_1 with $\operatorname{dist}(x_0, x_1) \leq \varepsilon$ and every $\widehat{\gamma} \in \Gamma_j$ for $j \leq k(\varepsilon)$ we have $\operatorname{diam}(\widehat{\gamma}) \leq a(\varepsilon)$. These definitions clearly guarantee $k(\varepsilon) \to \infty$ for $\varepsilon \to 0$.

After these preparations consider a continuous function $\psi \in C(J)$. We can assume it is positive as every ψ is with ψ^+, ψ^- positive. Observe that if $\operatorname{dist}(x, y) < \varepsilon$ then

$$\frac{\psi(x)}{\psi(y)} = 1 + \frac{\psi(x) - \psi(y)}{\psi(y)} \leq 1 + (\operatorname{var}_\varepsilon \psi)/\inf \psi.$$

Fix n, x_0 and x_1 with $\operatorname{dist}(x_0, x_1) = \varepsilon$ and use the notation from Lemma 6: $\gamma^t, x_0^t, x_1^t, A_j$ for $j = 0, \dots, n+1$. Write for $q = 0, 1$:

$$L_j(x_q) = \sum_{t \in A_j} \psi(x_q^t) E_n(x_q^t).$$

We shall estimate in the sequel

$$\frac{\widehat{P}^n(\psi)(x_0)}{\widehat{P}^n(\psi)(x_1)} = \frac{P_\varphi^n(\psi)(x_0)}{P_\varphi^n(\psi)(x_1)} = \frac{\sum_{j=0}^{n+1} L_j(x_0)}{\sum_{j=0}^{n+1} L_j(x_1)} \tag{2}$$

First, show as in Step 3 of Proof of Lemma 6 that $L_j(x_q)$ are neglectible, for $j = 0, 1, \dots$, but j not too close to n. As in Proof of Lemma 7 we do not need to stop at $j = [\alpha n]$. We do the estimates for each j separately. (The first of the sequence of three inequalities in Step 3 uses (2) section 3, which is trivial now, when one can refer to Lemma 7.) We get

$$L_j(x_q) \leq C(\exp -(n-j)\delta)(P_\varphi^n(\psi)(x_q)) \frac{\sup \psi}{\inf \psi} \tag{3}$$

for constants $C, \delta > 0$ and every $j = 0, \dots, n+1$.

Finally for every $j \geq n - k(\varepsilon)$ and every $l: j - 1 \geq l \geq n - k(\varepsilon) - 1, l \geq 0$

$$\frac{L_j(x_0)}{L_j(x_1)} \leq Q_1 \cdot \exp(Q_2 + Q_3), \tag{4}$$

where

$$Q_1 = 1 + \frac{(\operatorname{var}_{b(\varepsilon)} \psi)}{\inf \psi},$$

$$Q_2 = \sum_{i=0}^{l} \operatorname{var}_{\exp -(n-i)\delta}(\varphi),$$

$$Q_3 = (n - l) \operatorname{var}_{a(\varepsilon)} \varphi,$$

and

$$b(\varepsilon) = \begin{cases} \exp -\delta n & \text{if } n > k(\varepsilon), \\ a(\varepsilon) & \text{if } n \leq k(\varepsilon). \end{cases}$$

The factor Q_1 is responsable for the variation of ψ from x_0^t to x_1^t. The summand Q_2 corresponds to I in (6) section 3, Q_3 corresponds to II.

Take now any function $l: \mathbb{R}^+ \to \mathbb{Z}^+$ such that $l(\varepsilon) \leq k(\varepsilon)$ $l(\varepsilon) \to \infty$ and $l(\varepsilon) \cdot \operatorname{var}_{a(\varepsilon)} \varphi \to 0$ for $\varepsilon \to 0$.

When $\varepsilon \to 0$ we estimate (2) by using (3) for all $j \leq n - l(\varepsilon)$, just getting rid of $L_j(x_q)$. For $j > n - l(\varepsilon)$ we use (4) with $l = n - l(\varepsilon)$. We conclude that the ratio in (2) is bounded by $1 + c_1(\varepsilon)$ for all x_0, x_1, n with some $c_1(\varepsilon) \to 0$ for $\varepsilon \to 0$, what proves the equicontinuity, (for more concrete estimates see Proof of Proposition 3).

The operator $\widehat{P}$ satisfies also another property: it is primitive namely for every $\psi \geq 0$ there exists $n > 0$ such that $\widehat{P}^n(\psi) > 0$.

Now we can refer to a general theorem about positive operators, almost periodic and primitive (see **[B]** and **[LL]** th. 8.3):

For every such operator not contracting to 0 there is an eigenfunction ψ_0 with the eigenvalue 1. We have $\widehat{P} = P_1 + P_2$ where P_1 is the projector to span ψ_0, $\ker P_1$ is invariant for $\widehat{P}$ and $\widehat{P}^n(\psi) \to 0$ for every $\psi \in \ker P_1$.

As $P_1 = F \cdot \psi_0$ for a continuous functional F and $\widehat{P}$ is η-invariant (i.e. $\int \psi d\eta = \int \widehat{P}(\psi) d\eta$), we have $\ker P_1 = \ker \eta$. As $F(\psi_0) = \eta(\psi_0) = 1$, we have $F = \eta$.

So for every $\psi \in C(J)$, $\widehat{P}(\psi) = \eta(\psi) \cdot \psi_0 + \widehat{P}^n(\psi - \eta(\psi) \cdot \psi_0) \to \eta(\psi) \cdot \psi_0$. (By the way, $F = \eta$ proves the uniqueness of the probability measure satisfying (1).) □

Proposition 2. *The measure η satisfying* (1) *is the only probability $\lambda \exp(-\varphi)$–conformal measure on J.*

Proof. Let η_1 be any probability measure on J with Jacobian $J_{\eta_1} = \lambda \exp -\varphi$. Let f be 1-to-1 on a Borel set E. Then by the definition of Jacobian, for every integrable function h on $f(E)$, in particular for $h = \exp\varphi \circ (f|_E)^{-1}$ one has $\int_{f(E)} h d\eta_1 = \int_E (h \circ f) \cdot J_{\eta_1} d\eta_1$. So

$$\mathcal{P}_\varphi^*(\eta_1)(E) = \int \mathcal{P}_\varphi(\chi_E) d\eta_1 = \int_{f(E)} \exp\varphi \circ (f|_E)^{-1} d\eta_1$$
$$= \int_E (\exp\varphi \circ (f|_E)^{-1} \circ f) \cdot \lambda \exp -\varphi d\eta_1 = \lambda\eta_1(E).$$

It is easy to see that $\mathbb{J}$ can be decomposed into a finite number ($= \deg f$) of Borel sets E_i on each of which f is 1-to-1 to its image. Then for each E

$$\mathcal{P}_\varphi^*(\eta_1)(E) = \sum_i \mathcal{P}_\varphi^*(\eta_1)(E \cap E_i)$$
$$= \sum_i \lambda\eta_1(E \cap E_i) = \lambda\eta_1(E). \quad \square$$

Proof of Theorem 2. f_*-invariance ot $\nu = \psi_0 \cdot \eta$ is standard [**B**]. We recall the proof: for every $h \in C(\mathbb{J})$

$$\eta(\psi_0 \cdot h) = \eta(((\mathcal{P}_\varphi/\lambda)(\psi_0)) \cdot h) = \eta((\mathcal{P}_\varphi/\lambda)(\psi_0 \cdot (h \circ f))) = \eta(\psi_0 \cdot (h \circ f)).$$

Then we estimate measure entropy:

$$h_\nu(f) \geq \int \log J_\nu(f) d\nu = \int \log J_\eta(f) d\nu + \int \log(\psi_0 \circ f) d\nu - \int \log\psi_0 d\nu$$
$$= \int \log J_\eta d\nu = \log\lambda - \int \varphi d\nu = P(f,\varphi) - \int \varphi d\nu$$

for Jacobians $J_\nu(f)$, $J_\eta(f)$ for the measures ν and η respectively.

The first inequality follows from

$$h_\nu(f) \geq H_\nu(\varepsilon|f^{-1}(\varepsilon)) = \int \log J_\nu(f) d\nu$$

where ε is the partition into points.

So $h_\nu(f) + \int \varphi d\nu \geq P(f,\varphi)$. Actually we have equality here due to the inequality to the other side: the so-called variational principle [**B**].

Now we prove uniqueness of the equilibrium state for φ. We rely on the following claim told to us by M. Ljubich (the claim seems to be known in various versions from a long time, see for example [**Le**]).

Claim. Let u be a continuous function on $\mathbb{J}$ such that for every $x \in \mathbb{J}$

$$\sum_{y \in f^{-1}(x)} \exp u(y) = 1. \tag{5}$$

If a measure ν is an equilibrium state for u and there is a finite entropy generating partition (i.e. a countable partition $\mathcal{A}$ such that $\bigvee_{n=0}^\infty f^{-n}(\mathcal{A}) = \varepsilon, \nu$-a.e.) then

ν has Jacobian on $\mathbb{J}$ satisfying: $J_\nu = \exp -u$.

Proof of the claim. For every $y \in \mathbb{J}$ denote $A(y) = f^{-1}(f(\{y\}))$. There is a system of conditional measures for the partition $f^{-1}(\varepsilon)$, [**Ro**]. A conditional measure of this system on ν-a.e. $A(y)$ is denoted by $\nu_{A(y)}$. We have

$$P(f,u) = h_\nu(f) + \int u d\nu = H_\nu(\varepsilon|f^{-1}(\varepsilon)) + \int u d\nu$$
$$= \int \left(\sum_{z \in A(y)} \nu_{A(y)}(\{z\})(-\log(\nu_{A(y)}\{z\}) + u(z)) \right) d\nu(y).$$

THe latter expression is always negative except for the case $\nu_{A(y)}(z) = \exp u(z)$ ν-a.e. (this is a crucial calculus lemma in the theory of equilibrium states, see [**B**] L.1.1). But $P(f,u) \geq 0$ by Lemma 4 because $\mathcal{P}_u^n(1) = 1$. So for a set $Y = f^{-1}(f(Y))$ of full measure ν, every $y \in Y$ and every $z \in A(y)$ we have

$$\nu_{A(y)}(z) = \exp u(z).$$

So for every Borel set $E \subset Y$ such that f is 1-to-1 on it

$$\nu(f(E)) = \nu(f^{-1}(f(E))) = \int_{f^{-1}(f(E))} \left(\int_{A(y)} 1 d\nu_{A(y)} \right) d\nu(y)$$
$$= \int_{f^{-1}(f(E))} \left(\int \chi_E/\nu_{A(y)}(E \cap A(y)) d\nu_{A(y)} \right) d\nu(y)$$
$$= \int_E \exp -u d\nu.$$

Finally $\nu(f(\mathbb{J}\setminus Y)) = 0$ because $\mathbb{J}\setminus Y = f^{-1}(f(\mathbb{J}\setminus Y))$ and ν is f_*-invariant. The claim is proved.

Proof of uniqueness. Let ν be any equilibrium measure for φ. We set $u(x) = \varphi(x) - \log\psi_0(f(x)) + \log\psi_0(x) - \log\lambda$. The property (5) results from an easy computation with the use of the equality $(\mathcal{P}_\varphi/\lambda)(\psi_0) = \psi_0$. The measure ν is also an equilibrium state for u. Observe that

$$h_\nu(f) = P(f,\varphi) - \int \varphi d\nu \geq P(f,\varphi) - \sup\varphi > 0.$$

Now the existence of a finite entropy generating partition follows from $h_\nu(f) > 0$ by Mañé's construction [**M1**], [**M2**], see also [**P**] section 3. (The partition is countable, maybe not finite).

From the claim we conclude that

$$J_\nu = \lambda(\exp -\varphi)\frac{\psi_0 \circ f}{\psi_0}.$$

So $J_{(1/\psi_0)\cdot\nu} = \lambda \exp -\varphi$, hence $(1/\psi_0)\nu = \eta$ by Proposition 2. So $\nu = \psi_0 \cdot \eta$. □

Proposition 3. *Modulus of continuity of ψ_0, namely the function $\mathcal{M}(\psi_0)\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by*

$$\mathcal{M}(\psi_0)(\varepsilon) = \sup_{x,y\in J}\{|\psi_0(x) - \psi_0(y)| \colon \operatorname{dist}(x,y) \le \varepsilon\}$$

satisfies the inequality

$$\mathcal{M}(\psi_0)(\varepsilon) < C(N)\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^N,$$

for N arbitrarily large and a constant $C(N)$ depending on N.

Proof. Let us proceed with more precision in Proof of Theorem 1. Take for $a(\varepsilon)$ considered there, the function $a(\varepsilon) = \left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^M$ for an arbitrarily large integer $M > 0$. Now we shall precise $l(\varepsilon)$. To have

$$l(\varepsilon) \le k(\varepsilon) \quad \text{and} \quad l(\varepsilon)\cdot \operatorname{var}_{a(\varepsilon)}(\varphi) \to 0,$$

l cannot be too large. To assure the latter convergence it is enough to assume for ε small enough

$$l(\varepsilon) < \left(\log\frac{1}{\varepsilon}\right)^{M\kappa/2}, \tag{6}$$

where $\kappa = \kappa(\varphi)$ is the Hölder continuity exponent for φ. Indeed (6) gives

$$l(\varepsilon)\operatorname{var}_{a(\varepsilon)}(\varphi) < \left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^{M\kappa/3},$$

(3 rather than 2 to swallow the Hölder coefficient $C(\varphi)$ for ε small)).

Let us estimate now $k(\varepsilon)$ from below. We have

$$\varepsilon^{(1/\deg f)^{\alpha k(\varepsilon)}} \ge \left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^M$$

for $\alpha > 0$ arbitrarily small and for ε small enough. This uses Lemma 0, diameters of curves in Γ_j with growing j multiply by a constant or root with degree $\le \deg f$ in α-th part of time. So

$$k(\varepsilon) \ge \frac{1}{\alpha\log(\deg f)}\left(\log\log\frac{1}{\varepsilon} - \log\log\left((\log\frac{1}{\varepsilon})^M\right)\right) \ge A\log\log\frac{1}{\varepsilon},$$

for A arbitrarily large. Thus to have (6) satisfied we just take $l(\varepsilon) = A\log\log\frac{1}{\varepsilon}$.

Compute finally that for every $\widehat{\delta} > 0$

$$\exp -l(\varepsilon)\widehat{\delta} = \left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^{A\widehat{\delta}}$$

We conclude from this preparation using (3) and (4) that if $\operatorname{dist}(x_0, x_1) \le \varepsilon$ then

$$\frac{\widehat{\mathcal{P}}^n(1)(x_0)}{\widehat{\mathcal{P}}^n(1)(x_1)} \le (1 + C\exp -l(\varepsilon)\delta) \times (1 + C\exp -l(\varepsilon)\delta\kappa)\times \tag{7}$$
$$\times\,(1 + l(\varepsilon)\operatorname{var}_{a(\varepsilon)}(\varphi)) = X,$$

for a constant $C > 0$, (the first factor due to (3), two next related to Q_2 and Q_3 in (4)). We continue the estimates:

$$X \le \left(1 + \left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^{\frac{\min(A\delta, A\kappa\delta, M\kappa/3)}{2}}\right).$$

Because A and M can be chosen arbitrarily large this proves that all functions $\widehat{\mathcal{P}}^n(1)$ have moduli of continuity bounded by the same $C(N)(\frac{1}{\log\frac{1}{\varepsilon}})^N$ for arbitrary N. So the same expression bounds also the modulus of continuity for the limit function ψ_0. □

Remarks.

1. It is easy to see that the space

$$C_M = \{g \in C(J(f))\colon (\sup_{x,y\in J(f)}|g(x) - g(y)|)/\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)^M = \|g\|_{C_M} < \infty\}$$

is invariant under $\widehat{\mathcal{P}}$. Observe that if ψ is Hölder continuous then the sequence $\widehat{\mathcal{P}}^n(\psi)$ is uniformly bounded in the norm $\|\cdot\|'_{C_M} = \max(\sup|\cdot|, \|\cdot\|_{C_M})$. Indeed in (7) one should consider additionally the terms corresponding to Q_1 in (4),

$$(1 + C\operatorname{var}_{a(\varepsilon)}\psi)(1 + C\exp -l(\varepsilon)\kappa(\psi)\delta),$$

which can be coped with as corresponding terms for φ. However I see no reason that the sequence $\widehat{\mathcal{P}}^n(\psi)$ is bounded in $\|\cdot\|'_{C_M}$ for each $\psi \in C_M$. In particular maybe the spectrum of $\widehat{\mathcal{P}} - \mathcal{P}_1$ ($\mathcal{P}_1$ is the projection to span ψ_0, see Proof of Theorem 1.) intersects $\mathbf{S}^1$ (unlike for hyperbolic f and the space of Hölder continuous functions with a given exponent)?

2. Theorems 1 and 2 hold for every function φ satisfying

$$\sum_{n=0}^{\infty} \mathrm{var}_{\exp -n\delta}(\varphi) < \infty \quad \text{for} \quad \delta < \frac{1}{2}(P(f,\varphi) - \sup \varphi).$$

In particular it is sufficient to assume about the modulus of continuity that

$$\mathcal{M}(\varphi)(\varepsilon) \le \Big(\frac{1}{\log \frac{1}{\varepsilon}}\Big)^M,$$

for an arbitrary $M > 1$.

3. Hölder continuity of ψ_0 would follow from the following property:

(8) There exists $L > 1$ and $\tau, \varepsilon_0 > 0$ such that for every $x \in \mathbb{J}, \varepsilon < \varepsilon_0, n \ge 0$ and every component V of $f^{-n}(B(x,\varepsilon)), \operatorname{diam}(V) \le L^n \varepsilon^\tau$.

Indeed we can take then $a(\varepsilon) = \varepsilon^\alpha (\alpha < \tau$ will be specified later). The property (8) gives $k(\varepsilon) \ge \frac{1}{\log L}(\tau - \alpha)\log\frac{1}{\varepsilon}$. We take the latter expression as $l(\varepsilon)$, what gives

$$l(\varepsilon)a(\varepsilon) \le \varepsilon^{\alpha - \vartheta} (\vartheta > 0 \text{ arbitrarily small})$$

and

$$\exp -\delta \cdot l(\varepsilon) \le \varepsilon^{(\tau-\alpha)\delta/\log L}.$$

So

$$\mathcal{M}(\psi_0) \le \varepsilon^{\beta\kappa}$$

where β is an arbitrary number satisfying $\beta < \min(\alpha, (\tau-\alpha)\delta/\log L)$. So we can take any $\beta < \beta_0$ where $\beta_0 = \dfrac{\tau\widehat{\delta}/\log L}{1 + \widehat{\delta}/\log L}$ ($\beta_0 = \alpha$: the solution of the equation $\alpha = (\tau - \alpha)\widehat{\delta}/\log L$ with $\widehat{\delta} = \frac{1}{2}(P - \sup\varphi)$).

For example ψ_0 is Hölder continuous ((8) is satisfied) if $\omega(\mathrm{Crit}) \cap \mathrm{Crit} \cap \mathbb{J} = \varnothing$ (ω denotes the limit set under forward iterations for f).

Indeed for a backward trajectory of components $\{V_n\}$, where V_n is a component of $f^{-n}(B(x,s)), f(V_n) = V_{n-1}$, we have $\operatorname{diam}(V_{n+1})/\operatorname{diam}(V_n) < L$ for a constant L until for some $n = n_1$ the set V_{n_1} is close to a critical point. Then after a few steps V_{n_2} is contained in a large disc D such that all branches f^{-m} on say twice larger disc are univalent. Using Koebe's Distortion Theorem one deduces that $\operatorname{diam}(f^{-n_2-m}(V_{n_2}))/\operatorname{diam}(V_{n_2})$ stays bounded (with a universal bound) for $m \to \infty$.

4. All the results of the paper hold for any map of the circle $f: \mathbf{S}^1 \to \mathbf{S}^1$ which

is strictly increasing, $\deg f \ge 2$, of class C^1 and each critical point is not flat. The function φ is defined on $\mathbb{J}$ which can be defined as $\mathbf{S}^1 \setminus \{x: f^n(x) \to S'(f)$, as $n \to \infty\}$ or any smaller set X such that $f^{-1}(X) = X$.

Here we do not need to use bounded distortion to prove Lemma 1. We denote $S = \mathrm{Crit}(f) \cup \{e^{2\pi i k\varepsilon}: k = 0, \ldots, [\frac{1}{\varepsilon}]\}$ and U_n is the family of (closed) arcs joining consecutive points of $\bigcup_{i=0}^n f^i(S)$, (the idea is taken from **[LP]**).

(We do not know only whether the result of Remark 3 is true in this situation, because our proof that $\omega(\mathrm{Crit}) \cap \mathrm{Crit} \cap \mathbb{J} = \varnothing$ yields (8), hence Hölder continuity of ψ_0, uses "bounded distortion property".)

Appendix A.

Proof of Theorem 1 under the assumption $\log\lambda > \sup\varphi$.

Step 1 We claim the following:

Given $\delta > 0$ there exists $C > 0$ such that there exists a dense, full measure η, subset $\Lambda \subset \mathbb{J}$ with the following properties:

1. For every $n > 0$ there exists a set of continuous branches of f^{-n} on Λ, $\mathcal{T}_n = \{f_t^{-n}: t \in \{1, \ldots, d^n\}\}, d = \deg f$, such that for $k > l$ $f_t^{-k} \in \mathcal{T}_k \Rightarrow f^{k-l} \circ f_t^{-k} \in \mathcal{T}_l$ (i.e. the branches are compatible over n).

2. If we set $\mathcal{T}'_n = \{f_t^{-n} \in \mathcal{T}_n: \operatorname{diam} f_t^{-n}(\Lambda) \ge \exp -n\delta\}$ then

$$\mathrm{Card}\, \mathcal{T}'_n < C \exp 3n\delta \tag{2}$$

Proof of the claim. Order all critical values of f into $p_1, \ldots, p_m$. Join p_1 with p_2 by a shortest geodesic γ_1^1 (or a line close to a geodesic). Then join p_3 with γ_1^1 by a shortest geodesic γ_1^2, etc. Denote $\gamma_1 = \bigcup_{i=1}^{m-1} \gamma_1^i$ and $W = \widehat{\mathbb{C}} \setminus \gamma_1$. Now construct γ_n and W_n for every $n \ge 1$ by induction. Given W_{n-1} order all critical values of f^n not being critical values of f^{n-1} into $p_1^n, \ldots, p_{m(n)}^n$, join p_1^n with ∂W_{n-1} by γ_n^1, p_2^n with $\partial W_{n-1} \cup \gamma_n^1$ by γ_n^2 etc. Define $\gamma_n = \bigcup_{i=1}^{m(n)} \gamma_n^i$ and $W_n = W_{n-1} \setminus \gamma_n$. Finally $\Lambda = \bigcap_{n \ge 1} W_n \cap \mathbb{J}$. Now the branches f_t^{-n} are defined as continuous branches on W_n.

Observe that we can assure for every i, j that $\eta(\gamma_i^j) = 0$ because η has no atoms (otherwise it would be infinite) and we have a freedom of choosing our geodesics. It is not hard to see that this implies $\bigcup \gamma_i^j$ is nowhere dense in $\mathbb{J}$.

The key observation is that for each n due to the construction and the fact that for the set $\mathrm{Critv}(f^n)$ of all critical values for f^n

$$\mathrm{Card}(\mathrm{Critv}\, f^n) \le C_1 n, \quad \text{for a constant} \quad C_1 > 0,$$

$\bigcup_{m\le n}\gamma_m$ dissects each set of U_n (cf. Lemma 1) into at most $C_1 n + 1$ components. (As each geodesic dissects a small disc into at most 2 components.) Now intersections of the sets from U_n with W_n give a new covering $\widehat{U}_n$ of a neighbourhood $\widehat{W}_n$ of Λ.

Let us use Lemma 1. For every $B \in \widehat{U}_n$ we have (iii) satisfied, so

$$\mathrm{Card}\{f_t^{-n} \in \mathcal{T}_n : \mathrm{diam}\, f_t^{-n}(\widehat{W}_m) > \mathrm{Card}\, \widehat{U}_n \cdot \exp -n\delta_2\} \le$$
$$\le \mathrm{Card}\, \widehat{U}_n \cdot K \exp 2n\delta_2.$$

If δ_1, δ_2 are chosen so that $0 < \delta < \delta_2 - \delta_1$, $3\delta > 2\delta_2 + \delta_1$, then we obtain (2).

Step 2. (Denker, Urbański). We consider $g_n = \mathcal{P}^n_{\varphi-\log\lambda}(1)$ leaving considering $\mathcal{P}^n_{\varphi-\log\lambda}(\psi)$ for an arbitrary ψ to the reader.

By the definition of η and λ

$$\int g_n d\eta = 1, \quad \text{for every} \quad n. \tag{3}$$

Denote

$$\widehat{E}_n(z) = \exp\left(\sum_{i=0}^{n-1}\left(\varphi(f^i(z)) - \log\lambda\right)\right).$$

Because $\sup\varphi < \log\lambda$, there exists $\nu : 0 < \nu < 1$, such that $\widehat{E}_n(z) < \nu^n$ for every $z \in \mathcal{J}$ and $n \ge 0$. Finally fix an integer $n_0 > 0$ (to be specified later on) and denote for $x \in \Lambda$

$$\widehat{g}_n(x) = \sum\{\widehat{E}(f_t^{-n}(x)) : f^k \circ f_t^{-n} \notin \mathcal{T}'_{n-k} \text{ for every } 0 \le k \le n - n_0\}.$$

Suppose now that for some $K > 0$, for every $x \in \mathcal{J}, m < n$

$$|g_m(x)| \le K \tag{4}$$

Then for every $x \in \Lambda$

$$|g_n(x)| \le |\widehat{g}_n(x)| + \sum_{k=0}^{n-n_0} \sum_{f_t^{-n+k}\in\mathcal{T}'_{n-k}} \left|\widehat{E}_{n-k}(f_t^{-n+k}(x)) \cdot g_k(f_t^{-n+k}(x))\right|$$
$$\le |\widehat{g}_n(x)| + \left[\sum_{k=0}^{n-n_0} \nu^{n-k} \cdot \mathrm{Card}\, \mathcal{T}'_{n-k}\right] \cdot K. \tag{5}$$

The coefficient in the square bracket is < 1 if $3\delta + \log\nu < 0$ (by (2)) and if n_0 is large enough.

It is clear from the construction and Hölder continuity of φ that $|\widehat{g}_n(z)/\widehat{g}_n(w)| < C_1$ so by (3) $|\widehat{g}_n(z)| < C_1$ for a constant $C_1 > 0$ independent of n and every $z, w \in \Lambda$. So if K is chosen large enough (4) for all $m < n$ implies (4) for $m = n$ and every $x \in \Lambda$. By the density of Λ in $\mathcal{J}$ we conclude (4) for every $x \in \mathcal{J}$.

Let us prove now that $g_n(x) \ge \mathrm{Const} > 0$ for every x, n. Fixed n, due to (3) there exists $x_0 \in \Lambda$ such that $g_n(x_0) \ge 1$. We have

$$|g_n(x_0)| \le |\breve{g}_n(x_0)| + A_{n_1} \cdot \sup_{m,x} |g_m(x)|$$

for

$$A_{n_1} = \sum_{k=0}^{n-n_1} \nu^{n-k}\, \mathrm{Card}\, \mathcal{T}'_{n-k}$$

similarly to (5). We write $\breve{g}_n$ instead of $\widehat{g}_n$ because now it depends on n_1 maybe different from n_0. If n_1 is large enough that $A_{n_1} \cdot \sup_{m,x}|g_m(x)| = B < 1$, we deduce that $|\breve{g}_n(x_0)| \ge 1 - B > 0$ (the estimate independent of n). So the same holds with another $B < 1$, for every x by $|\breve{g}_n(z)/\breve{g}_n(w)| <$ Const. Having $0 < C^{-1} < |g_n(x)| < C$ for every n, x, a proof of the equicontinuity of $\{g_n\}$ is standard, see Section 4. □

Appendix B.

The assumption $P > \sup\varphi$ seems to have been introduced for the first time by Urbański in [U]. This assumption is essential. Indeed, take for example Blaschke product $f(z) = (\frac{3z+1}{3+z})^2$ and $\varphi = -\log|f'|$. There is a neutral fixed point $p = 1$. We have $\varphi(p) = 0$ and $P(f,\varphi) = 0$, (the latter follows from [U], Corollary 3.7). In this example neither $\{\mathcal{P}^n_\varphi(\psi)\}$ for any continuous function ψ is equicontinuous nor an equilibrium state equivalent to the $|f'|$-conformal measure (which is just the length measure on the unit circle) exist. The latter follows from [T] and implies the preceding.

The reader may ask why not to assume $\log\lambda > \sup\varphi$ from the beginning? The matter is that this assumption seems usually uncheckable. The condition $P > \sup\varphi$ is easier to check. For example it follows ([U], Remark 2) from $h_{\mathrm{top}}(f) > \sup\varphi - \inf\varphi$, (the condition introduced in [**H-K**]). The latter, as

$h_{top}(f) = \deg f$, has plenty of examples. (I owe most of the above remarks to M. Urbański.)

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