Differentiable maps of S^3 into S^2 with given inverse images

by

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1. *Introduction* – The purpose of this paper is to study the following general question. Let M^m and N^n be differentiable manifolds of dimensions m and n, respectivelly. Let $M_i^{m-n} (i = 1, \ldots, k)$ be disjoint submanifolds of M^m and let $a_i \, (i = 1, \ldots, k)$ be distinct points of N^n. We ask: under what conditions is there a differentiable map $f : M^m \to N^n$ such that $f^{-1}(a_i) = M_i$, with a_i a regular value of $(i = 1, \ldots, k)$?

In what follows, by a "differentiable map" we understand a C^∞-differentiable map; we notice that if a is a regular value of $f : M^m \to N^n$, $f^{-1}(a)$ is either empty or is a differentiable submanifold of M^m, having dimension $m-n$.

We shall only consider the case in which $M = S^m$ and $N = S^n$, particularly, $M = S^3$ and $N = S^2$.

Main Theorem – Let γ_1 and γ_2 be two disjoint and linked knots, imbedded in S^3 and let a_1, a_2 be two distinct points of S^2. There exists a differentiable map $f : S^3 \to S^2$ such that $f^{-1}(a_i) = \gamma_i$, with a_i a regular value of $f (i = 1, 2)$.

Observe that if we do not assume that the points a_1 and a_2 are regular values of f, then the problem might be trivial. As an example, suppose that M_1 is a compact manifold imbedded in R^m and let

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*I am grateful to Professor Gilberto Francisco Loibel for his encouragement, his guidance and many helpful suggestions in the preparation of this thesis.

$\Omega = T_\varepsilon(M_1)$ be an open tubular neighborhood of M_1 with radius ε; let $\rho(P)$ be the distance from $P \in \bar\Omega$ to M_1. Define the map $\phi: R^m \to R$ by

$$\phi(P) = \begin{cases} \rho^2(P), & \text{if } P \in \bar\Omega \\ \varepsilon^2, & \text{if } P \notin \bar\Omega \end{cases};$$

clearly, ϕ is differentiable in Ω and is the zero function on M_1. It follows from Lemma 1.1 below that there exists a differentiable map $f: R^m \to R$ which is a δ-aproximation to ϕ and which coincides with ϕ on a closed tubular neighborhood $\bar\Omega_1$ with radius $\varepsilon_1 < \varepsilon$. Here $f^{-1}(0) = M_1$ but $0 \in R$ is not a regular value of f.

1.1 – *Lemma* – Let $g: M_1 \to M_2$ be a continuous function of differentiable manifolds, differentiable on a closed subset A of M_1 (this means that g is differentiable in some open set U of M_1 containing A). Let δ be a positive continuous function on M_1; give to M_2 the metric determined by some imbedding $M_2 \subset R^p$. Then there exists $f: M_1 \to M_2$ such that:

1) f is differentiable;
2) f is a δ-approximation to g, and
3) $f \mid A = g \mid A$

(for the proof, see |12|, 3.11 or |14| p. 25).

Remark – In the special case in which M_1^{m-n} is a compact submanifold of a compact Riemannian manifold M^m, it is possible to show that, given a point $a \in N = S^n = R^n \cup \{\omega\}$, there exists a differentiable map $f: M \to S^n$ for which a is a regular value and such that $f^{-1}(a) = M_1^{m-n}$ if there are n independent differentiable vector fields normal to M_1^{m-n}.

In fact, suppose that such fields exist. Then there is a C^∞-diffeomorphism $\theta: \Omega = T_\varepsilon(M_1) \sim M_1^{m-n} \times B_\varepsilon^n$, where Ω is a closed tubular neighborhood of M_1^{m-n} in M^m and B_ε^n is the closed ball of radius ε centered at $0 \in R^n$.

Let $\phi: \Omega \to S^n$ be the composition

$$\Omega \xrightarrow{\theta} M_1^{m-n} \times B_\varepsilon^n \xrightarrow{\pi_2} B_\varepsilon^n \xrightarrow{j_1} R^n \xrightarrow{j_2} S^n.$$

Notice that $\phi(\Omega)$ is the closed ball D_ε^n in S^n centered at $a = j_2(0)$; furthermore, every $x \in D_\varepsilon^n$ is a regular value of ϕ and $\phi^{-1}(a) = M_1^{m-n}$.

Denote by $\dot\Omega$ the boundary of Ω and write $S_1^{n-1} = \phi(\dot\Omega) = \dot D_\varepsilon^n$. Next, consider the triangulable pair $(M^m\text{-int}\,\Omega, \dot\Omega)$. Since the homotopy groups of $S^n\text{-int}\,D_\varepsilon^n$ are all trivial, there are no obstructions to extend the restriction

$$\psi = \phi \mid \dot\Omega \quad \dot\Omega \to S_1^{n-1} \subset S^n\text{-int}\,D_\varepsilon^n$$

to a continuous function $h: M^m\text{-int}\,\Omega \to S^n\text{-int}\,D_\varepsilon^n$.
In this way we obtain a continuous function $g: M^m \to S^n$ by setting

$$g(P) = \begin{cases} \phi(P), & \text{if } P \in \Omega \\ h(P), & \text{if } P \in M^m\text{-int}\,\Omega \end{cases}$$

Since g is differentiable in a closed tubular neighborhood $\Omega_1 = T_\eta(M_1)$ with $\eta < \varepsilon$, Lemma 1.1 above shows that there is a differentiable map $f: M^m \to S^n$ which is a δ-approximation to g such that $f \mid \Omega_1 = g \mid \Omega_1$. One should observe that taking $\delta = \eta/2$, $f(M^m - \Omega) \cap \text{int}\,D_{\eta/2}^n = \phi$ and hence, $f^{-1}(a) = M_1^{m-n}$; moreover, a is a regular value of f.

2. *Differentiable Maps of S^3 into S^2*

2.1 – *Theorem* – *Let γ_1 be a knot imbedded in S^3 and let a be a point of S^2. There exists a differentiable map $f: S^3 \to S^2$ having a as a regular value and such that $f^{-1}(a) = \gamma_1$.*

We simply observe that the proof of this Theorem is a consequence of the Remark written in the Introduction: in fact, there are always two independent vector fields normal to γ_1,

In the next result we shall deal with two *separated knots* of S^3, that is to say, knots contained in the interior of two disjoint closed balls.

2.2 – *Theorem* – *Let γ_1 and γ_2 be two separated and plane circles in S^3. Let a_1 and a_2 be two distinct points of S^2. There exists a differentiable map $f: S^3 \to S^2$ such that $f^{-1}(a_1) = \gamma_1$, $f^{-1}(a_2) = \gamma_2$ and f has a_1 and a_2 as regular values.*

Proof – Let V_1 and V_2 be closed tubular neighborhoods of γ_1 and γ_2 respectively, constructed as rotation tori. Let $\phi_i: V_i, V_i \to S^2 (i = 1, 2)$ be the differentiable map obtained by projecting $V_i \sim \gamma_i x B_i^2$ into closed disks $D_i^2 \subset S^2$, with center a_i. The maps ϕ_1 and ϕ_2 define a differentiable map $\phi: V_1 \cup V_2 \to S^2$ with a_1 and a_2 as regular values, and $\phi^{-1}(a_i) = \gamma_i (i = 1, 2)$. Next we extend ϕ to a continuous map of S^3 into S^2.

Let us consider the points of $S_i^1 = \phi(\dot{V}_i)$ as images of regular arcs $C_i : I \to S^2$, where I is the unit interval $[0, 1]$. For each $t \in I$ the fibres $\phi^{-1}(C_i(t)) = \alpha_t^i$ is a circle contained in $\dot{V}_i$; also, if $t \neq t'$, α_t^i and $\alpha_{t'}^i$ are disjoint and $\bigcup_{t \in I} \alpha_t^i = \dot{V}_i$.

Write d^i for the plane disk having boundary γ_i; we can assume that $\alpha_0^i = d^i \cap \dot{V}_i \, (i = 1, 2)$.Let d_0^i be the plane disk $d^i - V_i$; notice that $\dot{d}_0^i = \alpha_0^i$. Consider, for each $t \in I$, the superimposed rotation surfaces d_t^i with boundary α_t^i. The bottom surface is the plane disk d_0^i. The union of these surfaces with with the tubular neighborhood V_i is a geometric solid W_i with boundary $d_0^i \cup d_1^i$. Figure 1 shows a transversal section of W_1.

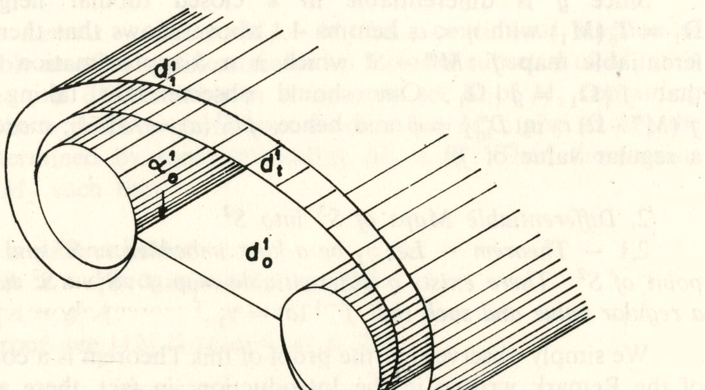


FIG. I

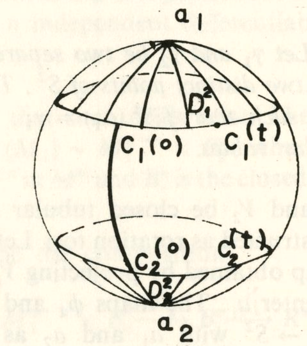


FIG. 2

Of course, we can suppose that W_1 and W_2 are contained in the interior of two disjoint closed balls B_1^3 and B_2^3. On the other hand, there is a homeomorphism θ of $S^3 - (B_1^3 \cup B_2^3)$ onto $S^2 \times I$ such that $\theta(\dot{B}_1^3) = S^2 \times 0$ and $\theta(\dot{B}_2^3) = S^2 \times 1$. Next, let us consider a regular arc

$$\lambda : I \to S^2 - (D_1^2 \cup D_2^2)$$

having end-points $C_1(0)$ and $C_2(0)$. Thus, define a continuous function

$$g : S^3 \to S^2$$

by the conditions:

$$g(x) = \begin{cases} \phi(x) & \text{if} \quad x \in V_1 \cup V_2 \\ C_i(t) & \text{if} \quad x \in d_t^i, \quad i = 1, 2, \quad 0 \leq t \leq 1 \\ C_i(0) & \text{if} \quad x \in B_i^3 - W_i, \quad i = 1, 2 \\ \lambda \pi_2 \, \theta(x), & \text{if} \quad x \in S^3 - \operatorname{int}(B_1^3 \cup B_2^3) \end{cases}$$

obviously, $g(x)$ is differentiable in int $(V_1 \cup V_2)$. Let V_i' be a closed tubular neighborhood with radius ε of γ_i, $i = 1, 2$, where ε is smaller that the radius of V_i. Fom Lemma 1.1, we get a differentiable function

$$f : S^3 \to S^2$$

which is a δ-approximation to g, and which coincides with g in $V_1' \cup V_2'$. Taking $\delta < \varepsilon/2$, we have $f^{-1}(a_i) = \gamma_i$, a_i a regular value of f.

Remark – It is possible to prove the Theorem above in the case where the manifolds $M_i = f^{-1}(a_i)$, $i = 1, \dots, k$ are obtained as unions of a finite number of mutually separated circles γ_{ij} $(j = 1, \dots, \alpha_i)$ of S^3.

In fact, using the same notations as in Theorem 2.2, we observe that the continuous function

$$g : S^3 \to S^2$$

is modified only on $S^3 - (B_1^3 \cup \dots \cup B_k^3)$. Notice that $g(\dot{B}_i^3) = P_i \in D_i^2$. For each $i = 1, \dots, k$, let X_i be the union of all disjoint segments $H_i K_i$, which are normal to B_i^3 (we let H_i run over $\dot{B}_i^3$).

We now extend g so that each segment $H_i K_i$ is taken homeomorphically onto the regular arc $P_i E$ in $S^2 - (D_1^2 \cup \dots \cup D_k^2)$.

Write $U_i = B_i^3 \cup X_i$; since g take $\dot{U}_i$ into E, we obtain an extension of g to the whole S^3 by setting $g(S^3 - (U_1 \cup \dots \cup U_k)) = E$. From here, we get a differentiable map $f : S^3 \to S^2$ as a δ-approximation to g and with the required properties.

3. *A fundamental operation*

Let a_1, a_2 be two distinct points of S^2 and let γ_1', γ_2' be two disjoint linked knots, imbedded in S^3. Assume that $f' : S^3 \to S^2$ is a differentiable map such that $f'^{-1}(a_i) = \gamma'_i$ and a_i is a regular value of f' $(i = 1, 2)$; suppose that f' is obtained as an extension of $\phi : V'_1 \cup V'_2 \to D_1^2 \cup D_2^2 \subset S^2$, where ϕ is the projection of the tubular neighborhoods V'_1 and V'_2 of γ'_1 and γ'_2 respectively over the disks D_1^2, D_2^2 centered at a_1 and a_2.

Identify a_1 with the north pole and a_2 with the south pole of S^2; consider a meridian through a_1 and a_2 and let x_1, x_2 and y_0 its intersection with $S_1^1 = \dot{D}_1^2$, $S_2^1 = \dot{D}_2^2$ and the equator S_0^1, respectively.

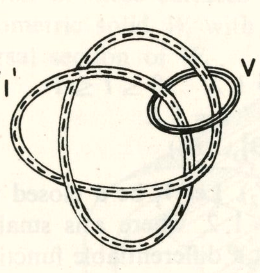


FIG. 3

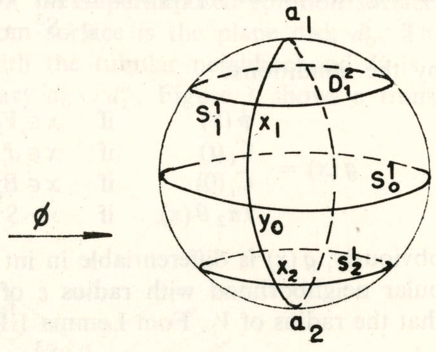


FIG. 4

Let C_1 be a small cube so that $A_1 = V'_1 \cap C_1$ and $B_1 = V'_2 \cap C_1$ are parallel circular straight cylinders and furthermore, with $L_1 = \phi^{-1}(x_1) \cap C_1$ and $L_2 = \phi^{-1}(x_2) \cap C_1$ parallel segments (see figure 5); we can always get down to that situation by a diffeomorphism of S^3.

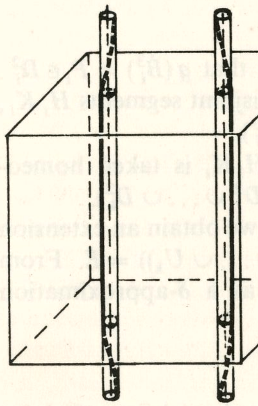


FIG. 5

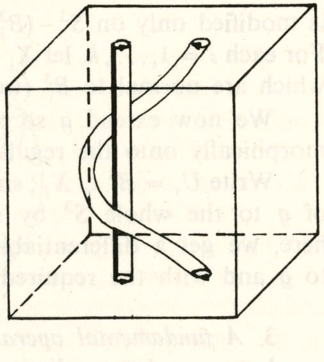


FIG. 6

Consider next the map $\theta : S^2 \to S^2$ which takes D_1^2 and D_2^2 homeomorphically onto the north and south hemispheres, stretching the arc $a_1 x_1$ into $a_1 y_0$ and $a_2 x_2$ into $a_2 y_0$. The composition $f_0 = \theta f : S^3 \to S^2$ is differentiable in the interior of $V'_1 \cup V'_2$ and also, $f_0^{-1}(a_i) = \gamma'_i$ with a_i still a regular value for f_0 $(i = 1, 2)$.

Let us substitute a twisted cylinder B' for B_1 in the configuration presented in figure 5 in such a way that $B' \cap \dot{C}_1 = B_1 \cap \dot{C}_1$ and the knot γ_2 relative to B' has a linking number $\varepsilon(\gamma_1, \gamma_2)$ with $\gamma_1 = \gamma'_1$ equal to $\varepsilon(\gamma'_1, \gamma'_2) \pm 1$ (see figure 6).

We ask now the following question: does there exist a continuous function

$$g : C_1 \to S^2$$

such that:

a) $g \,|\, \dot{C}_1 = f_0 \,|\, \dot{C}_1$;

b) $g \,|\, A_1 \cup B'$ is differentiable;

c) $\bar{f} : S^3 \to S^2$ defined by $\bar{f} \,|\, C_1 = g$ and $\bar{f} \,|\, S^3 - \operatorname{int} C_1 = f_0$ has a differentiable δ-approximation f such that $f^{-1}(a_i) = \gamma_i$, with a_i regular value for f?

We shall see that this question has an affirmative answer.

The construction of f out of f_0, which essentially is performed in the interior of C_1, will be called "the operation detour". We shall show that the "operation detour" alters the Hopf invariant of any pair of fibres obtained as inverse images of regular values of f, by the same constant. This is the crux of the matter.

In what follows all our constructions will be performed in $R^3 = S^3 - \{\omega\}$.

Let $W_1 = \overline{C_1 - (A_1 \cup B_1)}$ be the double solid torus with surface $\tilde{B}_1$ (see figure 7) and let

$$f'_0 = f_0 \,|\, \tilde{B}_1 : \tilde{B}_1,\ L_1 \cup L_2 \to S_0^1,\ y_0.$$

Homotopies of loops in $\tilde{B}_1$ *and* S^2.

Consider the lateral surfaces of the cylinders A_1 and B_1, which will be denoted by T_1 and T'_1; let c_1, c_2, c_3 and c_4 be the intersections of T_1 and T'_1 with the top and bottom of C_1, having the orientations indicated in figure 7. We construct next the loops $\alpha, \beta : S^1, p_0 \to \tilde{B}_1, P_1$ with base-point P_1 where:

a) $\alpha(S^1) = \overline{P_1 Q_1}\, \Gamma_1\, \overline{Q_1 P_1}$ is the composition of the segment $\overline{P_1 Q_1}$, followed by a closed curve $\Gamma_1 = Q_1 E_1 F_1 Q_1$ which turns around the curves c_1 and c_2 (having the same orientation as c_1) and finally, of the segment $\overline{Q_1 P_1}$;

b) $\beta(S^1) = P_1 c_1 \Sigma_1 c_2^{-1} \Sigma_1^{-1}$, where Σ_1 is an arc connecting P_1 to R_1 (see figure 7).

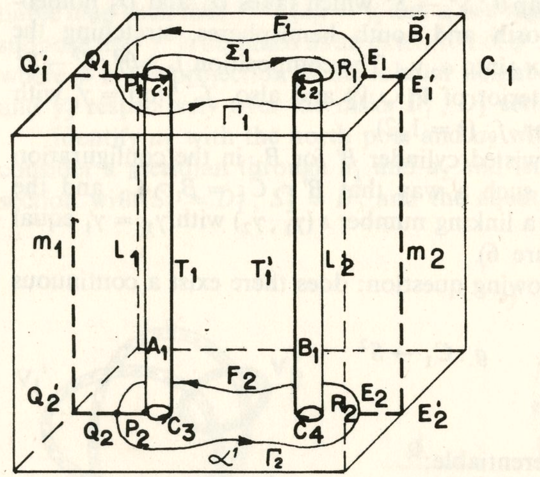


FIG. 7

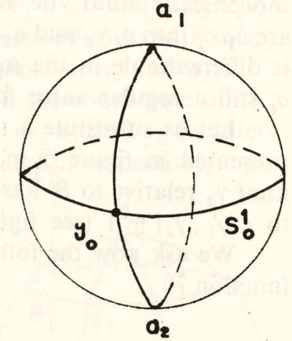


FIG. 8

One can show that there exists a relative (to p_0) homotopy k_t connecting α to β.

The function

$$k_t' = f_0' \, k_t : S^1 , p_0 \to S_0^1 , \; y_0$$

is a homotopy, with base point $y_0 \in S_0^1$, connecting the loops $k_0' = f_0' \, \alpha$ and $k_1' = f_0' \, \beta$ of S^2.

Let $\alpha' : S^1 , p_0 \to \tilde{B}_1 , P_2$ be a loop based on P_2, contained in the bottom of C_1, and defined in the same fashion as α (see figure 7).

3.1 – *Lemma* – *There exists a homotopy*

$$h_t : (\alpha (S^1) \cup \alpha' (S^1) , \; \{P_1\} \cup \{P_2\}) \to S_0^1 , \; y_0 \;\; \text{rel} \;\; \{P_1\} \cup \{P_2\}$$

connecting

$$h_0 = f_0' \,|\, (\alpha (S^1) \cup \alpha' (S^1))$$

to the constant map $h_1 (\alpha (S^1) \cup \alpha' (S^1)) = y_0$.

Proof – We might assume that the cylinders A_1 and B_1 are such that whenever a point moves on the circles c_1, c_2, c_3 and c_4 with the directions as indicated in Fig. 7, its image under f_0' moves on the equator S_0^1 in the same direction. (This is coherent with the definition of f_0, because this function projects $V_1' \cup V_2'$ over the north and south hemispheres of S^2). Let a, b and c be the restriction of f_0' to the arcs $P_1 Q_1$, Σ_1 and Γ_1 respectively; let us recall that $f'_0(c_1) = f'_0(c_2) = S_0^1$. Then,

$$k_0' = f_0' \, \alpha = aca^{-1}$$

and

$$k_1' = f_0' \, \beta = S_0^1 \, b \, (S_0^1)^{-1} \, b^{-1} .$$

Since the fundamental group of S_0^1 is abelian, the homotopy class of k_1' is trivial and thus, $f_0' \, \alpha$ is homotopic to the constant map

$$k' : S^1 , \; p_0 \to S_0^1 , \; y_0 .$$

In the same way one shows that $f_0' \,|\, \alpha' (S^1)$ is homotopic to the constant map to y_0; this completes the proof of 3.1.

Homotopies of maps from $\tilde{B}_1$ *into* $S_0^1 \subset S^2$

The closed arcs $\Gamma_1 = Q_1 E_1 F_1 Q_1$ and $\Gamma_2 = Q_2 E_2 F_2 Q_2$ divide $\tilde{B}_1$ into two regions, one of them containing T_1 and T_1'; the other region, which is exterior to the arcs Γ_1 and Γ_2, will be denoted by D. Notice that D is homeomorphic to $S_1 \, xI$.

3.2 – *Theorem* – *There exists a map*

$$f_1 : B_1 , \;\; L_1 \cup L_2 \to S_0^1 , \; y_0$$

such that $f_1 \simeq f'_0$ rel $T_1 \cup T_1'$ *and* $f_1 (D) = y_0$.

For the proof of this theorem we shall need several additional results.

3.3 – *Lemma* – *There exists a map*

$$g' : \tilde{B}_1 , \;\; L_1 \cup L_2 \to S_0^1 , \; y_0$$

such that $g' \simeq f_0'$ rel $T_1 \cup T_1'$ *and* $g' (\alpha (S^1) \cup \alpha' (S^1)) = y_0$.

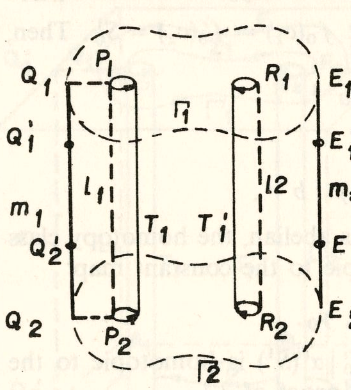
Proof – Consider the pair $(\tilde{B}_1 , \; L')$ where L' is the subcomplex of $\tilde{B}_1$ defined as $L' = (T_1 \cup T_1') \cup (\alpha (S^1) \cup \alpha' (S^1))$.

Take the retraction

$$r : \tilde{B}_1 \, xI \to M' = (\tilde{B}_1 \, x0) \cup (L' \, xI)$$

and define a map $H' : M' \to S_0^1$ by the conditions:

$$H' (x, t) = \begin{cases} f_0' (x) & \text{if} \quad x \in \tilde{B}_1 , \; t = 0 \\ f_0' (x) & \text{if} \quad x \in T_1 \cup T_1' , \; t \in I \\ h_t (x) & \text{if} \quad x \in \alpha (S^1) \cup \alpha' (S^1), \; t \in I \end{cases}$$



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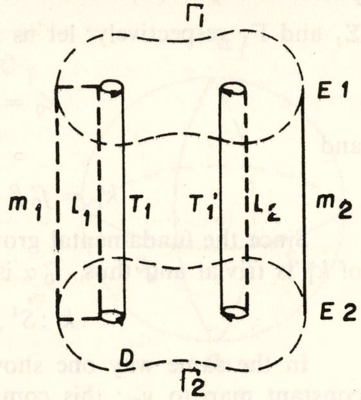

FIG. 10

The homotopy $G' : \hat{B}_1\, xI \to S_0^1$ defined by $G'(x, t) = H' r(x, t)$ for every $x \in \hat{B}_1$ and $t \in I$ gives the deformation of f_0' into g'.

We modify figure 7 into a new one (figure 9) to show the parts of $\hat{B}_1$ which are taken into y_0 by g' : there are indicated by dotted lines.

Let m_1 and m_2 be the polygonal lines $m_1 = Q_1\, Q_1'\, Q_2'\, Q_2$ and $m_2 = E_1\, E_1'\, E_2'\, E_2$.

3.4 – *Lemma – There exists a map*

$$g'' : \hat{B}_1 ,\ L_1 \cup L_2 \to S_0^1 ,\ y_0$$

such that $g'' \simeq g'$ rel $L' = T_1 \cup T_1' \cup \alpha(S^1) \cup \alpha'(S^1)$ *and* $g''(m_1) = y_0$.

Proof – We begin by observing that g' can be extended to the double solid torus W_1 because it is homotopic to f_0', restriction of f_0 to $B_1 = \dot{W}_1$.

There is an obvious homeomorphism between the rectangle $P_1\, Q_1\, Q_1'\, Q_2'\, Q_2\, P_2$ (figure 9) and $m_1\, xI$. The extension of g' to the interior of the former rectangle gives a homotopy

$$h_t' \text{ rel } \{Q_1\} \cup \{Q_2\}$$

so that $h_0' = g' \,|\, m_1$ and $h_1' = g' \,|\, L_1 = y_0$.

We now use an argument similar to that of 3,3: let $r' : \hat{B}_1\, xI \to M'' = (\hat{B}_1\, x0) \cup (L''\, xI)$ (where $L'' = L' \cup m_1$) be a retraction and define $H'' : M'' \to S_0^1$ by

$$H''(x, t) = \begin{cases} g'(x) & \text{if } x \in \hat{B}_1 ,\ t = 0 \\ g'(x) & \text{if } x \in L' ,\ t \in I \\ h_t'(x) & \text{if } x \in m_1 ,\ t \in I \end{cases}$$

The homotopy $G'' : \hat{B}_1\, xI \to S_0'$ given by $G''(x, t) = H'' r'(x, t)$ solves the problem.

Note – The dotted lines of figure 10 indicate the parts of $\hat{B}_1$ taken into y_0 by g''.

3.5 – *Lemma – There exists a map*

$$g''' : \hat{B}_1 ,\ L_1 \cup L_2 \to S_0^1 ,\ y_0$$

such that $g''' \simeq g''$ rel $L'' = L' \cup m_1$ *and* $g'''(m_2) = y_0$.

Proof – Consider the cylinder D we have spoken of before Lemma 3.2 (limited by Γ_1 and Γ_2) and divide it into two parts R_1 and R_2 by m_1 and m_2 ; notice that R_1 and R_2 are homeomorphic to a rectangle (see figure 10).

Since g'' is defined on $\hat{B}_1$ and $g''(m_1) = y_0$, it follows that

$$g'' \,|\, m_2 \simeq y_0 \text{ rel. } \{E_1\} \cup \{E_2\}.$$

This partial homotopy can be extended to a homotopy

$$G''' : \hat{B}_1\, xI \to S_0^1$$

connecting g'' and the required g''', as one can see with arguments similar to those of the previous Lemmas.

3.6 – *Lemma – There exists a map*

$$f_1 : \hat{B}_1 ,\ L_1 \cup L_2 \to S_0^1 ,\ y_0$$

such that $f_1 \simeq g'''$ rel. $L''' = L'' \cup m_2$ *and* $f_1(D) = y_0$.

Proof – The map g''' is defined on the "rectangles" R_1 and R_2 ; furthermore $g'''(\dot{R}_1) = g'''(\dot{R}_2) = y_0$ and $[g''' \,|\, R_i] \in \pi_2(S_0^1) = 0$. Hence, $g''' \,|\, R_i$ is homotopic to the constant map y_0, relatively to the boundary of R_i, $i = 1, 2$.

Taking next the triangulable pair $(\hat{B}_1 ,\ L''' \cup D)$ we can extend the homotopy $g''' \,|\, D \simeq y_0$ to a homotopy which proves the Lemma.

We are now ready for the proof of Theorem 3,2: Lemmas 3.3 to 3.6 show that

$$f_1 \simeq g''' \simeq g'' \simeq g' \simeq f_0' \text{ rel. } T_1 \cup T_1'$$

and $f_1(D) = y_0$.

Homeomorphisms of W_1 onto another solid double torus $W_2 \subset W_1$ and of $\hat{B}_1 x I$ onto $X = \overline{W_1 - W_2}$

Let us construct a cube C_2 concentric to C_1 and contained in the interior of C_1. Let A_2 and B_2 be solid cylinders contained in C_2, as indicated by figure 11; the lateral surfaces of these cylinders will be denoted by T_2 and T_2', respectively. Finally, let $W_2 = \overline{C_2 - (A_2 \cup B_2)}$ be the solid double torus of surface $\hat{B}_2$.

One can see that there is a homeomorphism

$$H : W_1 \to W_2$$

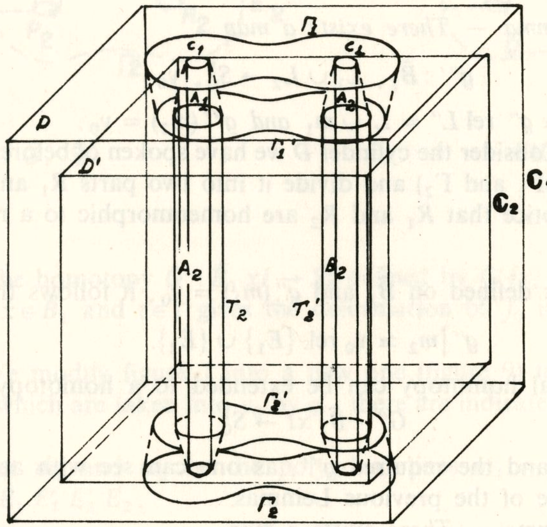

FIG. 11

of W_1 onto W_2 which is homotopic to the identity map on W_1 and which contracts W_1 into W_2, taking $\hat{B}_1$ onto $\hat{B}_2$. Let

$$F : W_1 \times \to W_1$$

be such that $(\forall x \in W_1)\, F(x, 0) = x$ and $F(x, 1) = H(x)$. This function F can be viewed as a continuous family of homeomorphisms

$$h_t : W_1 \to W_1 \qquad (0 \leq t \leq 1)$$

satisfying the conditions $h_0 = 1$ and $h_1 = H$.

The map

$$\nu = F \,|\, \hat{B}_1 \times I : \hat{B}_1 \times I \to W_1$$

is then a homeomorphism of $\hat{B}_1 \times I$ onto $X = \overline{W_1 - W_2}$.

Extension of f_0' to $X = \overline{W_1 - W_2}$

The homotopy $G : \hat{B}_1 \times I \to S_0^1$ of Theorem 3.2 which coincides with f_0' for $t = 0$ and with f_1 at the stage $t = 1$, can be viewed as an extension to $\hat{B}_1 \times I$ of the map

$$f_0' : \hat{B}_1,\ L_1 \cup L_2 \to S_0^1,\ y_0\,.$$

If we call $h = H \,|\, \hat{B}_1$, $p_1 \nu^{-1} \,|\, \hat{B}_2 = h^{-1}$ where $p_1 : \hat{B}_1 \times I \to \hat{B}_1$ is the first projection; furthermore, we have the following commutative diagram:

$$\begin{array}{ccc} \hat{B}_1 \xleftarrow{p_1} \hat{B}_1 \times I,\ \hat{B}_1 \times 0,\ \hat{B}_1 \times 1 & \xrightarrow{\nu} & X = \overline{W_1 - W_2},\ \hat{B}_1,\ \hat{B}_2 \\ f_1 \simeq f_0' \qquad \downarrow G & & \\ S_0^1 \leftarrow & - - - - - - & F_x,\ f_0',\ \bar{f}_1 \end{array}$$

Notice that $\bar{f}_1(D') = y_0$ where $D' = H(D)$.

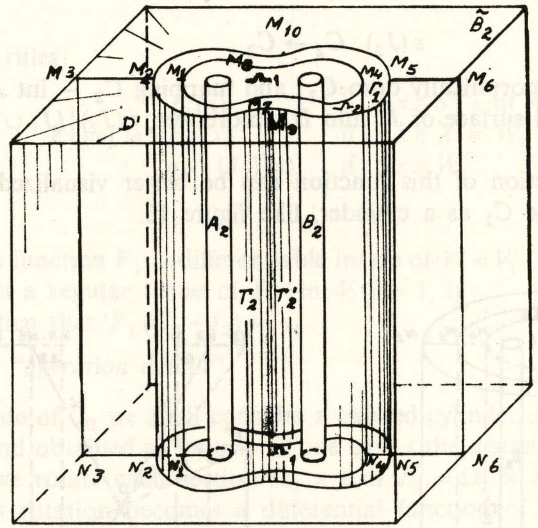

FIG. 12

Let Ω_1 and Ω_2 be the regions of the top of cube C_2, limited by the closed curves $M_1 M_7 M_4 M_8 M_1$ and $\Gamma' = H(\Gamma_1) = M_2 M_9 M_5 M_{10} M_2$, respectively; also, let Ω_1' and Ω_2' be the regions corresponding to Ω_1 and Ω_2 on the bottom of C_2 (see figure 12). Then,

$$D' = \dot{C}_2 - (\text{int}\, \Omega_2 \cup \text{int}\, \Omega_2').$$

Consider in figure 12 the cylinders $J_1 = \Omega'_1 \times I$ and $J_2 = \Omega'_2 \times I$, where Ω'_1 is identified to $\Omega'_1 \times 0$, Ω'_2 to $\Omega'_2 \times 0$, Ω_1 to $\Omega_1 \times 1$, Ω_2 to $\Omega_2 \times 1$ and the height of the cube C_2 is identified to the unit interval I.

Extension of $\bar{f}_1$ *to* W_2

Consider the following commutative diagram

$$\begin{array}{ccc} W_1, \tilde{B}_1 & \xrightarrow{f_0,\, f'_0} & S_0^1 \\ \uparrow {\scriptstyle H^{-1}, h^{-1}} & \nearrow {\scriptstyle \bar{f}_0, \bar{f}'_0} & \\ W_2, \tilde{B}_2 & & \end{array}$$

On the other hand, Theorem 3.2 gives a homotopy $f_1 \simeq f'_0$; thus, let $\bar{f}_1$ be the composition $f_1 h^{-1}$. Since $\bar{f}'_0$ has the extension $\bar{f}_0$ over W_2 and since $\bar{f}_1$ is homotopic to $\bar{f}'_0$, it follows that one can extend $\bar{f}_1$ to W_2. Let $g : W_2 \to S'_0$ be such extension.

Expansion of the cylinder J_2

We shall show that there exists a continuous function (expansion of J_2)

$$\varepsilon(J_2) : C_2 \to C_2$$

taking J_2 homeomorphically onto C_2, and mapping $C_2 - \operatorname{int} J_2$ into $\dot{C}_2$ and the lateral surface of J_2 into D'; moreover, $\varepsilon(J_2) \mid (J_1 \cup \dot{C}_2) =$ = identity.

The construction of this function can be better visualized if we represent the cube C_2 as a cylinder, like figure 13.

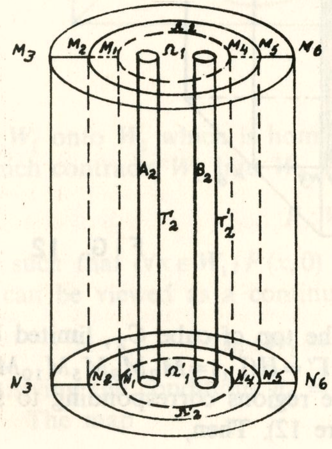

FIG. 13

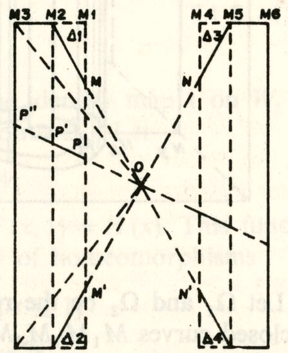

FIG. 14

Let 0 be the center of the square $M_3 N_3 N_6 M_6$, vertical section of the representing cylinder. Let M and M' (N and N') be the projections from 0, of M_2 and N_2 (M_5 and N_5) over the side $M_1 N_1$ ($M_4 N_4$). To each point $P \in MM'$ ($P \in NN'$) we associate the intersections P' and P'' of the line segment OP with $M_2 N_2$ and the line $M_2 M_3 N_3 N_2$ ($M_5 N_5$ and $M_5 M_6 N_6 N_5$). Let us denote the square $M_1 N_1 N_4 M_4$ with the letter Q; we also indicate the triangles $MM_1 M_2$, $M' N_1 N_2$, $NM_4 M_5$ and $N' N_4 N_5$ by Δ_1, Δ_2, Δ_3 and Δ_4, respectively. With this, one can see that

$$\varepsilon(J_2) \mid (Q \cup \Delta_i) = \text{identity} \qquad (i = 1, 2, 3, 4)$$

and also, we see that $\varepsilon(J_2)$ takes the segment PP' linearly over PP'' and the segment $P'P''$ onto the point P''.

Construction of $F_1 : S^3 \to S^2$

We define now the continuous function

$$F_1 : S^3 \to S^2$$

by the rules:

$$F_1(x) = \begin{cases} f_0(x) & \text{if } x \in S^3 - \operatorname{int} W_1; \\ F_X(x) & \text{if } x \in X = \overline{W_1 - W_2}; \\ g \circ \varepsilon(J_2)(x), & \text{if } x \in W_2 \end{cases}$$

The function F_1 is differentiable inside of $V' = V'_1 \cup V'_2$, $F_1^{-1}(a_i) = \gamma'_i$ and a_i is a regular value of $F_1 \mid \operatorname{int} V_i$ $(i = 1, 2)$.

Notice that $F_1(C_2 - J_2) = y_0$.

The "operation detour"

Inside of C_2 we shall consider a twisted cylinder J'_2 diffeomorphic to J_2, and obtained as follows: if one calls t the distance to the botton of J_2, we rotate each section $\Omega'_2 \times t$ of $J_2 = \Omega'_2 \times I$ in such a way that this rotation becomes a differential function of t. This rotation $\theta(t)$ must vary from 0 to 2π as t increases from 0 to 1; moreover it must be extended to a C^∞-function which is zero for ≤ 0 and 2π for every $t \geq 1$. Observe that we should allow room inside of C_2 to perform this rotation; in others words, we should assume either C_2 as sufficiently large or J_2 sufficiently small.

Let $\mu : J'_2 \to J_2$ be the diffeomorphism just described. Notice that

$$\mu^{-1}(\gamma'_1 \cap A_2) = \gamma'_1 \cap A_2, \; \mu^{-1}(A_2) = A_2$$

and $\mu^{-1}(B_2)$ is a tubular neighborhood of $\mu^{-1}(\gamma'_2 \cap B_2)$. Then, attaching this last curve to the curve $\gamma'_2 \cap (S^3 - \text{int } C_2)$, we obtain a curve γ_2 which has with γ'_1 the linking number

$$\varepsilon(\gamma_2, \gamma'_1) = \varepsilon(\gamma'_2, \gamma'_1) \pm 1.$$

The continuous function

$$F_2 : S^3 \to S^2$$

defined by

$$F_2(x) = \begin{cases} F_1(\mu(x), & \text{if } x \in J'_2 \\ y_0, & \text{if } x \in C_2 - J'_2 \\ F_1(x), & \text{if } x \in S^3 - C_2 \end{cases}$$

is differentiable in the interior of the tubular neighborhoods $V_1 = V'_1$ and V_2 of $\gamma_1 = \gamma'_1$ and γ_2 respectively.

Let $f : S^3 \to S^2$ be a differentiable δ-approximation of F_2, which coincides with F_2 on a closed tubular neighborhood $\tilde{V}_i$ of γ_i contained in V_i; it is clear that $f^{-1}(a_i) = \gamma_i$ and a_i is a regular value of f, $i = 1, 2$.

This completes the construction of the "operation detour"; the reader is asked to observe that the Hopf invariant for any pair of anti-images of regular values of f, is altered by the same constant (either 1 or -1).

The previous constructions show that the "operation detour" can also be applied whenever the cylinders A and B belong to the same tubular neighborhood, that is to say, this operation can be applied to transform a knot into a trivial one. In this case the linking number of every pair of curves, which are anti-images of regular values of f, is not altered by the operation.

4. *The Main Theorem*

In this section we shall prove the Main Theorem stated at the beginning of this paper.

A diffeomorphism of R^3 into itself which takes a differentiable knot into one with straight segments.

Crowell and Fox ([1], Appendix) have shown that a C^1-knot K K parametrized by arc length is ε-equivalent to a polygonal knot, that is to say, for every $\varepsilon > 0$ there exists a homeomorphism h of R^3 onto itself so that $h(K)$ is polygonal and $\| h(p) - p \| < \varepsilon$, for every $p \in R^3$.

Here, we shall take advantage of some of their ideais to construct a diffeomorphism H of R^3 onto itself which takes a differentiable knot K into a differentiable knot with straight segments.

We shall assume that the rectifiable knot K is expressed by a vector valued function of the arc length

$$p(s) = (x(s), y(s), z(s)).$$

Let l be the length of K and consider the set of n points $p(s_j) \in K$, where $s_{j+1} - s_j = l/n$, $j = 1, \dots, n$.

It is shown in [1] (Appendix) that, given $\varepsilon > 0$ there are a convenient angle $0 < \alpha_0 < \pi/4$ and a number n sufficiently large, so that for each s_j it is possible to construct a doublecone C_j (i.é., the union of two circular symmetric cones with common base) with axis equal to the segment having end-points $p(s_j)$, $p(s_{j+1})$ and with angle α_0 at the vertices, satisfying the following conditions:

1) the double cones C_j are arbitrarily small, i.e., the maximum diameter is smaller than $\varepsilon > 0$;

2) two adjacent double-cones intercept only at the common vertex;

3) if $s_j \le s \le s_{j+1}$ then $p(s_j) \in C_j$;

4) for each normal section D of C_j there is only one s, $s_j \le s \le s_{j+1}$, such that $p(s) \in D$;

5) non-adjacent double-cones are disjoint.

Let then γ_1 and γ_2 be disjoint linked knots imbedded in S^3 and let d be the minimum distance between them.

Given $0 < \varepsilon < d/2$, one can divide γ_1 and γ_2 by n_1 and n_2 points respectively, and construct the double-cones C_{ij} ($i = 1, 2$; $j = 1, \dots, n_i$) having vertices v_{ij} on those points, satisfying the preceeding conditions 1) to 5).

Let K_i be the polygonal knots with vertices v_{ij} and sides given by the axis of C_{ij} ($i = 1, 2$).

By [1] (p. 7), $K = K_1 \cup K_2$ is in regular position with respect to a certain direction u of $R^3 = S^3 - \{\omega\}$.

Let $K' = \pi(K)$ be the orthogonal projection of K to a plane normal to u. Because K is in regular position with respect to u, the multiple points of K' are all double; moreover, K' has only a finite number of double points and these are not images of vertices belonging to K.

On the plane which contains K', take disks D_{ij} centered at $\pi(V_{ij})$, having radius r_{ij} so small that the sides of K' not adjacent to $\pi(v_{ij})$ do not intercept D_{ij}.

Each cylinder V_{ij}, projected from the disk D_{ij} according to the direction u, meets only the sides of K which intercept on the vertex v_{ij}.

Consider the balls B_1^{ij} and $B_1^{i,j+1}$ of radius r_{ij}, center on v_{ij} and $v_{i,j+1}$ respectively; these balls are contained on the appropriate cylinders V_{ij}. The disks with boundary $\dot{B}_1^{ij} \cap \dot{C}_{ij}$ and $\dot{B}_1^{i,j+1} \cap \dot{C}_{ij}$ are normal to the segment $(v_{ij}, v_{i,j+1})$ at points which will be denoted by P_1^{ij} and Q_1^{ij}. On the other hand, the balls B_2^{ij} and $B_2^{i,j+1}$ of radii $r_{ij}/2$ and centered at v_{ij} and $v_{i,j+1}$, intercept $(v_{ij}, v_{i,j+1})$ at $\bar{P}_2^{ij}$ and $\bar{Q}_2^{ij}$. The planes normal to $(v_i, v_{i,j+1})$ passing through $\bar{P}_2^{ij}$. and $\bar{Q}_2^{ij}$ intercept the curve γ_i at P_2^{ij} and Q_2^{ij}, respectively.

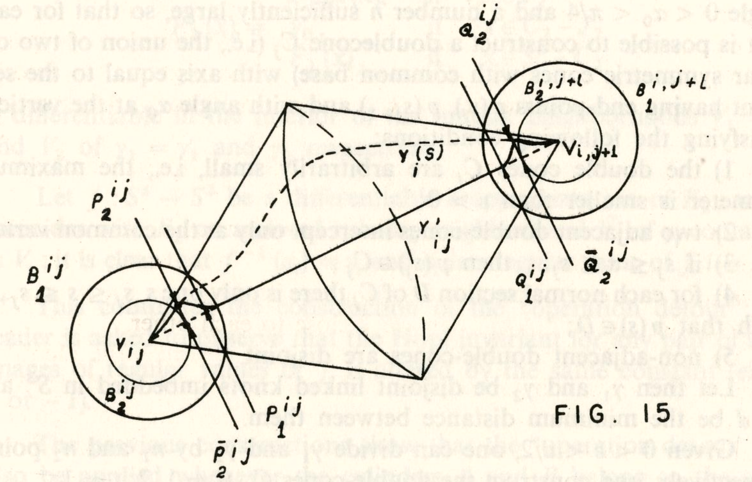

FIG 15

On each double cone C_{ij} we consider a differentiable curve γ'_{ij} obtained by taking:

a) the arcs (v_{ij}, P_2^{ij}) and $(Q_2^{ij}, v_{i,j+1})$ of the knot γ_i;

b) the segment $P_1^{ij} Q_1^{ij}$ on $(v_{ij}, v_{i,j+1})$;

c) regular arcs which connect P_2^{ij} to P_1^{ij} and Q_1^{ij} to Q_2^{ij} and are attached differentiably to the arcs of *a*) and *b*); furthermore, the arcs $P_2^{ij} P_1^{ij}$ and $Q_1^{ij} Q_2^{ij}$ can be taken so to meet a plane normal to $(v_i, v_{i,j+1})$ at a unique point.

The construction of the differentiable curves $\gamma_i = \bigcup \gamma'_{ij} \, (i = 1, 2)$ shows that there exists a C^∞-isotopy h_t, $0 \leq t \leq 1$, such that:

i) $h_0 \gamma_i = \gamma_i$, $h_1 \gamma_i = \gamma'_i$;

ii) h_t is a diffeomorphism for each t;

iii) h_t is the identity on the arcs $\gamma_i \cap B_2^{ij}$.

This partial isotopy can be extended to a global one

$$H_t : S^3 \to S^3 = \mathbb{R}^3 \cup \{\omega\}$$

in such a way that H_0 = identity (cf. [15], 157-03)

Proof of the Main Theorem

Every double point of the projection $K' = \pi(K)$ is the image of two points belonging to straight segments of $H_1(\gamma_1 \cup \gamma_2) = \gamma'_1 \cup \gamma'_2$. The point having larger Z-coordinate is called an *overcrossing*; the one with smaller z-coordinate is an *undercrossing*; the segment containing an overcrossing (undercrossing) is called an *overpass* (*underpass*).

If we keep γ'_1 fixed and move γ'_2 according to u in the direction of the increasing z-coordinates, any underpass of γ'_2 will meet an overpass of γ'_1 in just one point; after a finite number of crossings the two knots will be completely separated. Let us write γ''_2 for the knot γ'_2 when separated from γ'_1.

We shall apply global diffeomorphisms of S^3 and operations detour to the knots γ'_1 and γ''_2.

Step 1 — Let A be an infinite rectangular prism parallel to u, which contains an overpass of γ''_2, an underpass of γ'_1 and which does not contain anything else of both knots. One can assume that the overpass and the underpass in question are very close to each other.

Consider a diffeomorphism of S^3 over itself which is the identity outside A and on a tubular neighborhood of γ'_1. This diffeomorphism can be chosen so to transform γ''_2 into a knot $\bar{\gamma}_2$ having a straight segment L_2 parallel to the underpass of γ'_1. Let $C_1 \subset A$ be a cube containing L_2 and a corresponding segment L_1 on the underpass of γ'_1.

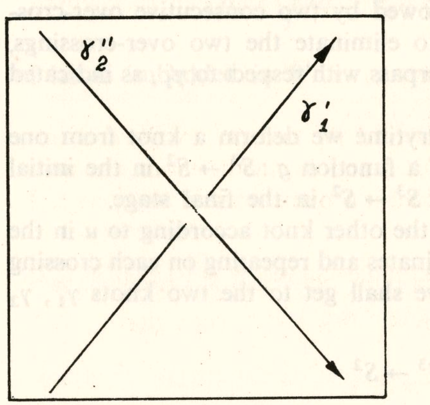

FIG. 16

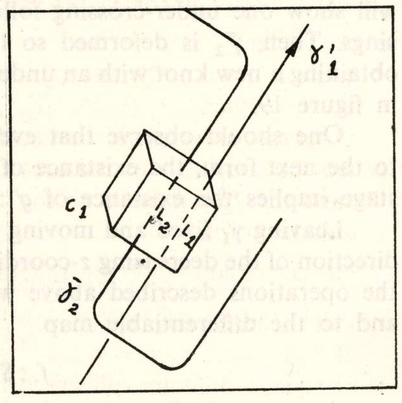

FIG. 17

Let us assume that there exists a differentiable map

$$g : S^3 \to S^2$$

such that $g^{-1}(a_1) = \gamma'_1$, $g^{-1}(a_2) = \bar{\gamma}_2$ and having a_1 and a_2 as regular values. Since the operation detour can be applied to γ'_1 and $\bar{\gamma}_2$ on the cube C_1, we can construct a map

$$g' : S^3 \to S^2$$

with $g'^{-1}(a_1) = \gamma'_1$ and $g'^{-1}(a_2) = \bar{\bar{\gamma}}_2$, where $\bar{\bar{\gamma}}_2$ and γ'_1 are linked.

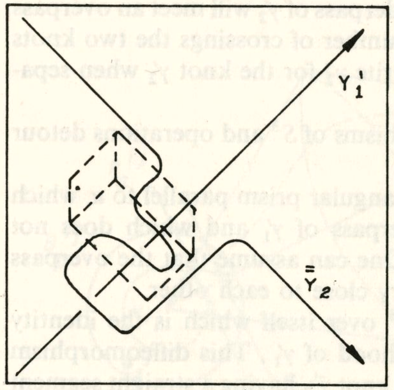

FIG. 18

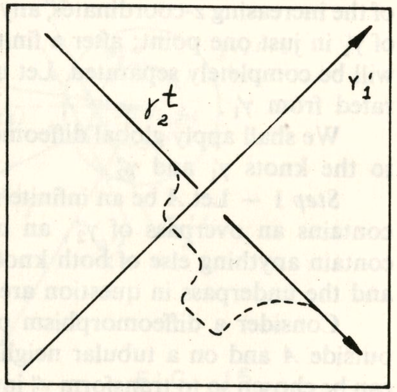

FIG. 19

Step 2 — As soon as the operation detour is effected the knot $\bar{\bar{\gamma}}_2$ will show one under-crossing followed by two consecutive over-crossings. Then, $\bar{\bar{\gamma}}_2$ is deformed so to eliminate the two over-crossings, obtaining a new knot with an underpass with respect to γ'_1, as indicated in figure 19.

One should observe that everytime we deform a knot from one to the next form, the existance of a function $g : S^3 \to S^2$ in the initial stage implies the existance of $g' : S^3 \to S^2$ in the final stage.

Leaving γ'_1 fixed and moving the other knot according to u in the direction of the decreasing z-coordinates and repeating on each crossing the operations described above we shall get to the two knots γ_1, γ_2 and to the differentiable map

$$f : S^3 \to S^2$$

with $f^{-1}(a_1) = \gamma_1$, $f^{-1}(a_2) = \gamma_2$, a_1, a_2 regular values of f.

The main theorem is then proved except for the existance of the differentiable map $g : S^3 \to S^2$ mentioned before.

If we consider an overpass and an underpass of the same knot, γ'_1 or γ''_2 (they are separated), after a finite number of operations similar to those described before, we shall transform them into trivial knots $\bar{\gamma}_1$ and $\bar{\gamma}_2$.

Consider now a diffeomorphism

$$\psi : S^3 \to S^3$$

isotopic to the identity and taking the trivial knots $\bar{\gamma}_1$ and $\bar{\gamma}_2$ into separated plane circles $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. For these last two theorem 2.2 shows that there exists a differentiable map $\tilde{f} : S^3 \to S^2$ such that $\tilde{f}^{-1}(a_i) = \tilde{\gamma}_i$, a_i regular value of $\tilde{f}$. The composite map

$$\bar{f} = \tilde{f}\psi : S^3 \to S^2$$

is a differentiable map such that $(\bar{f})^{-1}(a_i) = \bar{\gamma}_i$, a_i regular value of $\bar{f}$. Then from $\bar{f}$ and with the operations described before, now performed in the opposite sense, we shall arrive after a finite number of steps to the map

$$g : S^3 \to S^2$$

and then to the differentiable map

$$f : S^3 \to S^2$$

with $f^{-1}(a_i) = \gamma_i$, a_i regular value of f, where γ_i are two knots trivial or not, separated or not, imbedded in S^3. This shows the Main Theorem.

We observe that this Theorem is still true if each γ_i is a union

$$\gamma_i = \bigcup_j \gamma_{ij} \qquad (i = 1, 2;\ j = 1, \ldots, \alpha_i)$$

of knots imbedded in S^3.

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