Differentiable Conjugacy Near Compact Invariant Manifolds

by

CLARK ROBINSON

0. *Introduction*

In this paper we show how the differentiable linearization of a diffeomorphism near a hyperbolic fixed point (a la Sternberg) can be adapted to a neighborhood of an invariant compact manifold. There are two parts of the standard proof. The first part says that if two diffeomorphisms have all their derivatives equal at a hyperbolic fixed point then they are C^∞ conjugate to one another in a neighborhood. This result is true in a neighborhood of a compact manifold with little change in the statement or proof. See Theorem 1. The second part says that if a diffeomorphism f satisfies eigenvalue conditions at a hyperbolic fixed point then there is a C^∞ diffeomorphism h such that all the derivatives of $g = h^{-1} fh$ at the fixed point are equal to the derivatives of the linear part of g. Near a manifold there is no general condition that replaces the eigenvalue condition so we get only a very much weakened result in this direction, see Theorem 2. However Theorem 2 does imply that under some conditions the strong stable manifolds of points vary differentiably. See Corollary 3.

We were aware that Theorem 1 was true before reading the recent paper of Takens [10]. However his proof is the easiest to adapt to our setting and also saves one more derivative than some other proofs. We could just say that Theorem 1 follows from the proof in [10], however for clarity we repeat the proof with the necessary modifications. The only essential changes are in the definitions of $\eta(\delta)$ and $\mathcal{O}$. All other changes are a matter of style.

To prove Theorem 2 we adapt the type of proof used for Theorem 1. At a hyperbolic fixed point this can be solved much more directly by solving for coefficients of polynomials using eigenvalue conditions. See [7], [9], or [10].

1. *Statement of the theorems*

Let V be a compact submanifold of M. Give M a Riemannian metric. Let ρ be the distance between points of M induced by the metric. Let $p : TM \to M$ be the usual projection. Let $T_x M = p^{-1}(x)$ and $T_V M = p^{-1}(V)$. A diffeomorphism $f : M \to M$ is called *hyperbolic along* V if $fV = V$, there is a splitting $T_V M = TV \oplus E^u \oplus E^s$ as the Whitney sum of subbundles, and there is an integer n such that $\| Df^n(x) \,|\, E^s_x \| < 1$ and $\| Df^{-n}(x) \,|\, E^u_x \| < 1$ for all $x \in V$ where $E^s_x = E^s \cap T_x M$.

For $h : M \to M$ and $x \in V$ let $D_1 h(x) = Dh(x) \,|\, T_x V$, $D_2 h(x) = Dh(x) \,|\, E^u_x$, and $D_3 h(x) = Dh(x) \,|\, E^s_x$. Let $j^r h(x) = (x, h(x), Dh(x), \dots, D^r h(x))$. This is called the *r-jet of h at x* in local coordinates on the domain.

Let $W^s(V,f) = \{x \in M : \rho(f^j x, V) \to 0 \text{ as } j \to \infty\}$ and $W^u(V,f) = \{x \in M : \rho(f^{-j} x, V) \to 0 \text{ as } j \to \infty\}$. These are called the *stable and unstable manifolds of V for f*. If these manifolds are differentiable then $T_V(W^s(V,f)) = TV \oplus E^s$ and $T_V(W^u(V,f)) = TV \oplus E^u$.

Now assume $\lambda_x = \| D_2 f^{-n}(x) \| < \| D_1 f^n(x) \|^{-1}$ and $\mu_x = \| D_3 f^n(x) \| < \| D_1 f^{-n}(x) \|^{-1}$ for $x \in V$. This says f is more hyperbolic normally to V than along V. Let $W^{ss}(x,f) = \{ y \in M$: there exists a constant c_y such that

$$\rho(f^{jn}(x),\ f^{jn}(y)) \le c_y \mu_x \dots \mu_{f^{(j-1)n}x}\} \text{ and } W^{uu}(x,f) = \{ y \in M :$$

there exists a constant c_y such that $\rho(f^{-jn}(x), f^{-jn}(y)) \le c_y \lambda_x \dots \lambda_{f^{(-j+1)n}x}\}$. These are called the *strong stable* and *strong unstable manifolds of x for f* [5], shows that $W^s(V,f) = \cup \{ W^{ss}(x,f) : x \in V \}$ and $W^u(V,f) = \cup \{ W^{uu}(x,f) : x \in V \}$. A more general theorem of this kind is contained in [6].

Now we define the loss of derivatives that occurs in the conjugation of Theorem 1. Given α, let $\beta = \beta(f, \alpha)$ be the largest integer such that

$$\| Df^{-n}(f^n x) \| \cdot \| Df^n(x) \|^{\beta} \cdot \| D_3 f^n(x) \|^{\alpha - \beta} < 1$$

for all $x \in V$. Next let $\gamma = \gamma(f, \beta)$ be the largest integer such that

$$\| Df^n(f^{-n} x) \| \cdot \| Df^{-n}(x) \|^{\gamma} \cdot \| D_2 f^{-n}(x) \|^{\beta - \gamma} < 1 \quad \text{for all} \quad x \in V.$$

Theorem 1. Let $V \subset M$ be a compact C^1 submanifold. Assume $f, g : M \to M$ are C^α diffeomorphisms that are hyperbolic along V and such that $j^\alpha f(x) = j^\alpha g(x)$ for all $x \in V$. $f(V) = V$. Assume $W^s(V,f)$ is a C^β submanifold near V. Then there exist a neighborhood U of V and a C^β diffeomorphism $h : U \to M$ such that $k = h^{-1} g h$ has $j^\beta k(x) = j^\beta f(x)$ for $x \in W^s(V,f) \cap U$. Also there exists a C^γ diffeomorphism $h' : U \to M$ such that $(h')^{-1} g h'(x) = f(x)$ for $x \in U$. h/V, $h'/V = id$. β and γ are as defined above.

The proof is contained in § 3.

Theorem 2. Let $f : M \to M$ be a C^α diffeomorphism, and $V \subset M$ a compact C^α submanifold. Assume f contracts along V, i.e. E^u is the zero section in the definition of g being hyperbolic along V. Assume also that

$$\| D_1 f^{-1}(fx) \| \cdot \| Df(x) \|^{\alpha - 1} \cdot \| D_3 f(x) \| < 1$$

for all $x \in V$. Then there exists a neighborhood U of V and a C^α diffeomorphism $h : U \to M$ such that $h/V = id$ and $g = h^{-1} f h$ has $D^j_2 (pr \circ g)(x) = 0$ for $1 \le j \le \alpha$ where $pr : U \to V$ is a differentiable normal bundle projection. Thus infinitesimally g preserves the fibers of $pr : U \to V$.

The proof is contained in § 4.

Corollary 3. Let f be a C^α diffeomorphism contracting along V. Assume

$$\| D_1 f^{-1}(fx) \| \cdot \| Df(x) \|^{\alpha - 1} \cdot \| D_3 f(x) \| < 1$$

for all $x \in V$. Let p and r be the integers such that

$$\| D_3 f^{-1}(fx) \| \cdot \| D_3 f(x) \|^p \le 1$$

$$\| D_1 f(x) \|^{\alpha - p} (\| D_3 f(x) \| / \| D_1 f(x) \|)^{r+1} \le 1.$$

Let $\beta = \alpha - 1 - p - r$. Then there exist a neighborhood U of V and a C^β diffeomorphism $h : U \to M$ such that $g = h^{-1} f h$ preserves the fibers of $pr : U \to V$. Actually h has all derivatives $D^j D^k_2 h(x)$ for $x \in U$, $0 \le j \le \beta$, and $0 \le j + k \le \alpha$. In particular the set of $W^{ss}(x,f)$ for $x \in V$ form a foliation of $W^s(V,f) \cap U$ such that each leaf is C^α and they vary C^β.

Proof :

By applying theorem 2 we can assume $D^j_2 (pr \circ f)(x) = 0$ for $1 \le j \le \alpha$ and $x \in V$. Define $g_1 : U \to V$ by $g_1(x) = f_1(pr\, x)$. In vector bundle charts of $pr : U \to V$ define $g_2(x) = f_2(x)$. Use bump functions to define $g = (g_1, g_2) : U \to U$. Then $g_1(x) = g_1(pr\, x) = f_1(pr\, x)$ for $x \in M$ and $j^\alpha f(x) = j^\alpha g(x)$ for $x \in V$. Theorem 1 gives the C^β conjugacy of f and g where $\beta = \alpha - 1 - p - r$ since

$$\| D_3 f^{-1} \| \cdot \| D_1 f \|^{\alpha - 1 - p - r} \| D_3 f \|^{1 + p + r} \le \| D_1 f \|^{\alpha - 1 - p - r} \| D_3 f \|^{1 + r}$$
$$\le \| D_1 f \|^{\alpha - p} (\| D_3 f \| / \| D_1 f \|)^{1 + r} < 1.$$

The extra derivatives of h exist as remarked in the proof of Theorem 3.

Q.E.D.

Using the methods of the proof of Theorem 1 differently we can get a stronger statement about the differentiability of the foliation $W^s(x,f)$.

Corollary 4. Let f be a C^α diffeomorphism contracting along V. Assume

$$|D_1 f^{-1}(fx)\| \cdot \| Df(x)\|^{\alpha-1} \| D_3 f(x)\| < 1$$

for all $x \in V$. Then the set of $W^{ss}(x,f)$ for $x \in V$ form a $C^{\alpha-1}$ foliation of $W^s(V,f) \cap U$, where U is a neighborhood of V.

The proof is contained in § 5.

Using the estimates in [7] the above proofs should go over to flows. However beware of the proof of linearization there. "By induction" does not work since the variation equation does not satisfy a global Lipschitz constant.

We would like to discuss how the above theorems relate to some of the results in [4], [5] and [8]. In [5] f is called *r-normally hyperbolic* if there is an integer n such that

$$\| D_1 f^{-n}(f^n x)\|^r \cdot \| D_3 f^n(x)\| < 1$$

and
$$\| D_1 f^n (f^{-n} x)\|^r \cdot \| D_2 f^{-n}(x)\| < 1.$$

for all $x \in V$. This condition is similar but different than the condition we require in Theorem 2 and Corollaries 3 and 4. If f is r-normally hyperbolic, then $W^s(V,f)$, $W^u(V,f)$, and V are C^r submanifolds. See [5]. Also for each $x \in V$ $W^{ss}(x,f)$ and $W^{uu}(x,f)$ are C^r and they vary continuously in the C^r topology. Corollaries 3 and 4 give that they vary differentiably.

[8] shows that if f is 1-normally hyperbolic then f is C^0 conjugate to a map g that preserves the fibers of $pr : U \subset M \to V$ and such that g is linear on fibers of $pr : U \subset M \to V$. Corollary 3 gives a differentiable conjugacy in the contracting case to a fiber preserving map g but g is not necessarily linear on fibers.

If V is replaced by an expanding attractor, then [4, 6.4] gives conditions under which the stable manifolds of points form a C^1 foliation of a neighborhood. Corollary 4 possibly could be adapted to this setting to give the same answer. The result in [4] only apply to stable manifolds of points $W^s(x,f) = \{y \in M : \rho(f^j(x), f^j(y)) \to 0 \text{ as } j \to \infty\}$ and not the strong stable manifolds of points. Thus when V is only an attractor the results are different. Also we give a condition that insures higher differentiability.

Added in proof: M. Shub pointed out to me that [4, 6.4] and the C^h section theorem prove Corollary 4.

2. *Notation and definitions*

Since we are only interested in a conjugacy of diffeomorphisms in a neighborhood of V, we can take a tubular neighborhood of V. Thus we can consider M as a vector bundle over V, $pr : M \to V$. Let $p : TM \to M$ be the projection of the tangent bundle of M to $M \cdot |\cdot|$ is a norm induced by a Riemannian metric on TM. Let ρ be the distance between points of M induced by $|\cdot|$.

Let $L_s^r(T_x M, T_y M)$ be the (linear) space of all symmetric r-linear maps from $T_x M$ to $T_y M$.* Let $J^0(M,M) = M \times M$ and $J^r(M,M) = \cup \{(x,y) L_s^1(T_x M, T_y M) \times \ldots \times L_s^r(T_x M, T_y M) : x, y \in M\}$. If $h : M \to M$ is C^r let $j^r h(x) = (x, h(x), Dh(x), \ldots, D^r h(x)) \in J^r(M,M)$. This is called the *r-jet of h at x*. Let $\pi_0 : J^0(M,M) \to M$ be the projection on the first factor and $\pi_r : J^r(M,M) \to J^{r-1}(M,M)$ be the natural projection for $r \geq 1$. Let $\psi_r = \pi_0 \circ \ldots \circ \pi_r : J^r(M,M) \to M$. All of these projections are fiber bundles. $\psi_r : J^r(M,M) \to M$ is called the *r-jet bundle*. Define a distance on $J^0(M,M)$ by

$$\rho_0((x_1, x_2), (y_1, y_2)) = \max\{\rho(x_i, y_i) : i = 1, 2\}.$$

Let the distance on each fiber of $\pi_r : J^r(M,M) \to J^{r-1}(M,M)$ be the usual one induced by $|\cdot|$ on TM,

$$\rho_r((x, y, A_0, \ldots, A_r), (x, y, A_0, \ldots, A_{r-1}, B_r)) = \| A_r - B_r \|_r =$$
$$\sup\{|(A_r - B_r)(v_1, \ldots, v_r)| : v_i \in T_x M \text{ and } |v_i| = 1 \text{ for all } i\}.$$

By using the distance on the base space there is an induced (noncanonical) distance on $J^r(M,M)$. Given a subset $U \subset M$ let $J^r((M,U),M) = \psi_r^{-1}(U)$. Let $\Gamma J^r((M,U)M)$ be the space of continuous sections of $\psi_r : J^r((M,U),M) \to U$.

3. *Proof of Theorem* 1

I. First we prove that the conjugacy exists along $W^s(V,f)$. We use the notation given in § 1 and § 2. By the assumptions of Theorem 1 there exists an integer n and a $0 < \mu < 1$ such that

$$\| Df^{-n}(f^n x)\| \cdot \| Df^n(x)\|^\beta \cdot \| Df^n(x) \,|\, E_x^s\|^{\alpha-\beta} < \mu$$

*Note: Higher derivatives are only defined in terms of local coordinates. Therefore cover a neigborhood of V with a finite number of coordinate charts and define the jets and norms in terms of these coordinate charts.

for all $x \in V$. Below we construct a conjugacy h between f^n and g^n. Because of its special form, $h \sim \lim_{j\to\infty} g^{-nj} f^{nj}$, h is also a conjugacy between f and g. Thus for convenience we take $n = 1$. The reader can check the details for $n > 1$. The constant μ is fixed during the proof.

We define the following numbers,

$$a_x = \| Dg^{-1}(x) \| \qquad x \in M$$

$$A_x = \begin{cases} \rho(fx, V)\,\rho(x, V)^{-1} & x \in W^s(V,f) - V \\ \lim \{A_y : y \in W^s(V,f) - V \text{ and } y \to x\} & x \in V \end{cases}$$

$$B_x = \| Df(x) \| \qquad x \in W^s(V,f).$$

Note for $x \in V$, $A_x \leq \| Df(x) \mid E^s_x \| < 1$ and

$a_{fx}^{-1} \leq A_x < 1 \leq B_x$. By our assumption $a_{fx} B^\beta_x A_x^{\alpha-\beta} < \mu$

for $x \in V$. There exist neighborhoods

$\eta(\delta) = \{x \in W^s(V,f) : \rho(x, V) < \delta\}$ and $\mathcal{O}$ of $\{(m,m) : m \in V\}$ in $M \times M$

such that (i) $f\eta(\delta) \subset \eta(\delta)$ and (ii) if $x \in \eta(\delta)$ and $(fx, y) \in \mathcal{O}$ then $a_y B^\beta_x A_x^{\alpha-\beta} < \mu$.

For simplicity of notation let

$J^r = J^r((M, \eta(\delta)), M) = \psi_r^{-1}(\eta(\delta))$ and ΓJ^r be the

continuous sections of $\psi_r : J^r \to \eta(\delta)$.

We define a second norm on the fibers of $\pi_r : J^r \to J^{r-1}$ (possibly infinite) by $\sigma_r(c^1, c^2) =$

$$\sup\{\rho_r(c^1 x, c^2 x)\,\rho(x, V)^{-(\alpha-r)} : x \in \eta(\delta) - V\} \quad \text{for} \quad \pi_r c^1 = \pi_r c^2.$$

If $c \in \Gamma J^0$ then we can identify it with the map $c_0 : \eta(\delta) \to M$ such that $c(x) = (x, c_0(x))$. Let

$\Phi_0 : \Gamma J^0 \to \Gamma J^0$ be defined by $\Phi_0(c) = d$ where

$d(x) = (x, g^{-1} c_0 f(x)) = j^0(g^{-1} c_0 f)(x)$. Let $\Phi_r : \Gamma J^r \to \Gamma J^r$ be defined by $\Phi_r(c) = d$ such that for each x $d(x) = j^r(g^{-1} h f)(x)$ where $j^r h(fx) = c(fx)$. First we prove Φ_r contracts along fibers of π_r.

Lemma 3.1: Let $c^1, c^2 \in \Gamma J^r$ with $\pi_r c^1 = \pi_r c^2$, $\sigma_r(c^1, c^2) < \infty$

and $\pi_1 \circ \ldots \circ \pi_r c^i(fx) \in \mathcal{O}$ for all $x \in \eta(\delta)$, $i = 1, 2$.

Then $\sigma_r(\Phi_r c^1, \Phi_r c^2) \leq \mu \sigma_r(c^1, c^2)$.

Proof:

Assume $r \geq 1$. $\sigma_r(\Phi_r c^1, \Phi_r c^2) =$

$$\sup\{\rho_r(\Phi_r c^1(x), \Phi_r c^2(x))\,\rho(x, V)^{-(\alpha-r)} : x \in \eta(\delta) - V\}$$

$$\leq \sup\{\rho_r(c^1(fx), c^2(fx))\, a_y B^r_x\, \rho(x, V)^{-(\alpha-r)} :$$

$$x \in \eta(\delta) - V \text{ and } \pi_1 \circ \ldots \circ \pi_r c^i(fx) = (fx, y)\}.$$

This last inequality is true using the formula for higher derivatives of a composition of functions and the fact that $\pi_r c^1 = \pi_r c^2$. Then this is

$\leq \sup\{\rho_r(c^1(fx), c^2(fx))\,\mu\rho(fx, V)^{-(\alpha-r)} : x \in \eta(\delta) - V\} \leq \mu\sigma_r(c^1, c^2)$.

When $r = 0$, $\rho(x, V)^{-\alpha} \leq \mu\rho(fx, V)^{-\alpha}$. The details are left to the reader.

Q.E.D.

Let $I_r \in \Gamma J^r$ be defined by $I_r(x) = j^r(id)(x) = (x, x, id_x, 0, \ldots, 0)$ where $id : M \to M$ is the identity map and $id_x : T_x M \to T_x M$ is the identity map. Let $C_0 = \sigma_0(\Phi_0 I_0, I_0)$. C_0 is finite because $j^\alpha f(x) = j^\alpha g(x)$ for all $x \in V$ and V is compact. Let $D_0 = C_0(1-\mu)^{-1}$. Let O_r be the zero section of $\pi_r : J^r \to J^{r-1}$. Let $\mathscr{F}_0 = \{c \in \Gamma J^0 : \sigma_0(c, I_0) \leq D_0\}$, and $\mathscr{F}_r = \{c \in \Gamma J^r : \pi_r c \in \mathscr{F}_{r-1}$ and $\sigma_r(c, O_r \pi_r c) < \infty\}$ for $r \geq 1$.

Since $\sigma_0(c, I_0) \leq D_0$ for $c \in \mathscr{F}_0$, there exists a $\delta > 0$ smaller than above if necessary, such that for $c \in \mathscr{F}_0$ and $x \in \eta(\delta)$, *then* $c(fx) \in \mathcal{O}$.

Lemma 3.2: $\Phi_r : \Gamma J^r \to \Gamma J^r$ maps $\mathscr{F}_r$ into itself.

Proof:

We prove the lemma by induction. $\mathscr{F}_{-1} = \phi$ is invariant by Φ_{-1}. Assume $\mathscr{F}_{r-1}$ is invariant by Φ_{r-1}. Let $c \in \mathscr{F}_r$. Then $\sigma_r(\Phi_r c, O_r \pi_r \Phi_r c) \leq \sigma_r(\Phi_r c, \Phi_r O_r \pi_r c) + \sigma_r(\Phi_r O_r \pi_r c, O_r \pi_r \Phi_r c) \leq \mu\sigma_r(c, O_r \pi_r c) + \sigma_r(\Phi_r O_r \pi_r c, O_r \Phi_{r-1} \pi_r c)$. For $r = 0$ this last term is $\leq \mu D_0 + C_0 \leq D_0$. For $r > 0$ it is $< \infty$.

Lemma 3.3: $\Phi_r : \mathscr{F}_r \to \mathscr{F}_r$ is continuous in terms of σ_r.

Proof:

We use the chain rule for higher derivatives of a composition.

$\sigma_r(\Phi_r c^1, \Phi_r c^2) = \sup\{\rho_r(\Phi_r c^1, \Phi_r c^2)\,\rho(x, V)^{-(\alpha-r)} : x \in \eta(\delta) - V\}$

$\leq$ (constant) $\sup\{\|D^i g^{-1}(y_2)\|\,\rho_{j_1}(c^1(fx), c^2(fx))\ldots$

$\rho_{j_i}(c^1(fx), c^2(fx))\,\|D^{k_1} f(x)\|\ldots\|D^{k_j} f(x)\|\,\rho(x, V)^{-(\alpha-r)}$:

$x \in \eta(\delta) - V,\ \pi_1 \circ \ldots \circ \pi_r c^2(fx) = (fx, y_2),\quad 1 \leq i \leq r,$

$j = j_1 + \ldots j_r,\ k_1 + \ldots + k_j = r\} +$

$+$ (constant) $\sup\{\rho_i(g^{-1}(y_1), g^{-1}(y_r)\,\|D^{j_1}(fx)\|\ldots$

$\|D^{j_i} c^1(fx)\|\cdot\|D^{k_1} f(x)\|\ldots\|D^{k_j} f(x)\|\,\rho(x, V)^{-(\alpha-r)}$:

$\pi_1 \circ \ldots \circ \pi_r c^1(fx) = (fx, y_1)\}.$

Here the constants depend only on the binomial coefficients. We look at the first summation and leave the second to the reader. It is $\leq$(constant) $\sup\{\sigma_{j_1}(c^1, c^2)\ldots\sigma_{j_i}(c^1, c^2)\rho(fx, V)^{(\alpha-j)}\rho(x, V)^{-(\alpha-r)} : 1 \leq i \leq r,$ $1 \leq j \leq r\} \leq$ (constant) $\sigma_r(c^1, c^2)^r$. These last two constants include the supremum of derivatives of f and g^{-1}. From this it follows that σ_r is continuous.

Q.E.D.

By lemma 3.1, $\Phi_0 : \mathscr{F}_0 \to \mathscr{F}_0$ is a contraction in terms of σ_0. Thus there is a unique attractive fixed point, c^0. Attractive means that for each $c \in \mathscr{F}_0$, $\sigma_0(c^0, \Phi_0^j c) \to 0$ as $j \to \infty$. Assume that $\mathscr{F}_{r-1}$ has an attractive fixed point. By Lemma 3.3, Φ_r is continuous. By Lemma 3.2, $\Phi_r : \mathscr{F}_r \to \mathscr{F}_r$ contracts along fibers of $\pi_r : \mathscr{F}_r \to \mathscr{F}_{r-1}$ by a factor of μ. By the fiber contraction theorem, [4, 1.2], Φ_r has a unique fixed point in $\mathscr{F}_r$ and it is attractive.

Let $id : M \to M$ be the identify diffeomorphism and $I_r(x) = j^r(id)(x)$. Then $\Phi_\beta(I_\beta)$ converges (in the uniform topology of sections of $\psi_r : J^r \to \eta(\delta)$) to a section $c \in \Gamma J^r$. Let $c(x) = (x, c_0(x), \ldots, c_\beta(x))$ with $c_i(x) \in L^i_s(T_x M, T_{c_0 x} M)$ for $i \geq 1$. By the uniform convergence it follows that $c_i : \eta(\delta) \to$ $\to UL^i_s(T_x M, T_y M) : x \in \eta(\delta),\ y \in M\}$ is $c^{\beta-i}$. Thus the conditions of the Whitney Extension Theorem are satisfied. See [1, p. 120] for a statement of the theorem. There exists a C^β function $h : M \to M$ such that for $x \in \eta(\delta)\, j^\beta h(x) = c(x)$. $Dh(x) = id_x$ for $x \in V$ so h is a local diffeomorphism in a neighborhood of V. Thus $h^{-1} gh$ is defined in a neighborhood of V in M and $j^\beta(h^{-1}gh)(x) = j^\beta f(x)$ for $x \in \eta(\delta) \subset W^s(V, f)$. This completes the conjugacy of f and g along $W^s(V, f)$.

Remark: In Corollary 3 we noted that more derivatives of the conjugacy existed along the fibers. In that setting we have C^α diffeomorphisms such that $D_2^j(pr \circ g)(x) = 0$ for $1 \leq j \leq \alpha$ and $x \in V$ and $f(pr\, x) = pr \circ f(x)$. Here $pr : M \to V$ is a normal bundle. For $\alpha \geq r \geq \beta$ let $J^{\beta r}$ be the jet bundle of maps with all derivatives $D^j D_2^k h(x)$ for $0 \leq j \leq \beta$ and $0 \leq j + k \leq r$. Let $\rho_{\beta r}$ be the associated norm. Let $\pi_r : J^{\beta r} \to J^{\beta r - 1}$ be as before. For $\pi_r c^1 = \pi_r c^2$ let $\sigma_{\beta r}(c^1, c^2) =$ $\sup\{\rho_{\beta r}(c^1 x, c^2 x)\,\rho(x, V)^{-(\alpha-r)} : x \in \eta(\delta) - V\}$. If $c^1, c^2 \in J^{\beta r}$ and $\pi_r c^1 = \pi_r c^2$ then

$$\rho_{\beta r}(\Phi_{\beta r} c^1(x), \Phi_{\beta r} c^2(x))\,\rho(x, V)^{-(\alpha-r)} \leq$$

$\leq \rho_{\beta r}(c^1(fx), c^2(fx))\, a_y B_x^\alpha A_x^{r-\beta} \rho(x, V)^{-(\alpha-r)}$. A little check is necessary to show this depends only on $\rho_{\beta r}$ and not ρ_r. (f preserves fibers). Then this is $\leq \mu\rho_{\beta r}(c^1(fx), c^2(fx))\,\rho(fx, V)^{-(\alpha-r)}$: Lemma 3.1 follows. The other details are left to the reader.

II.

Now we can assume f and g are C^β and $j^\beta f(x) = j^\beta g(x)$ for $x \in W^s(V, f) = W^s$ and x near V. For $x \in M$ define the following numbers $a_x = \|Df^{-1}(x)\|$

$$b_x = \begin{cases} \rho(f^{-1}, W^s)\,\rho(x, W^s)^{-1} & x \notin W^s \\ \lim\{b_y : y \notin W^s \text{ and } y \to x\} & x \in W^s \end{cases}$$

$B_x = \|Dg(x)\|$.

For $x \in V$, $a_x^{-1} < 1 < b_x^{-1} \leq B_{f^{-1}x}$ and $B_{f^{-1}x} a_x^\gamma b_x^{\beta-\gamma} < \mu < 1$. By using a bump function we can make $g(x) = f(x)$ at points x such that $\rho(x, V) \geq \delta$. (g is then defined on all of M). Also g can be left unchanged at points x with $\rho(x, V) \leqq \delta/2$. Let $\eta(\delta) = \{x \in M : \rho(x, W^s) < \delta\}$ and $\eta'(\delta) = \{x \in M : \rho(x, V) < \delta\}$. By taking δ smaller if necessary and taking $\mathscr{O}$ to be a small neighborhood of $\{(m, m) : m \in W^s\}$ in $M \times M$, we can insure that for $x \in \eta'(\delta)$ and $(f^{-1} x, y) \in \mathscr{O}$ it follows that $B_y a_x^\gamma b_x^{\beta-\gamma} < \mu$.

Let Φ_r be induced by $h \to g\, h\, f^{-1}$. That is, in the earlier definition replace f by f^{-1} and g^{-1} by g. Continue as before taking sections c of $\psi_r : J^r(\eta(\delta), M) \to \eta(\delta)$ such that $c(x) = j^r id\,(x)$ for $x \in \eta(\delta) - \eta'(\delta)$. Lemmas 3.1, 3.2, and 3.3, apply to these sections. The limit $\Phi_\gamma(I_\gamma)$ gives the γ-jet of the conjugacy h on $\eta(\delta)$.

Q.E.D.

4. *Proof of Theorem* 2

In this section we assume $T_V M = TV \oplus E^s . F^1 = TV$ is differentiable. Since we do not assume the bundles are invariant we can approximate E^s by F^3 that is differentiable. Write $D_i h(z) = D h(z) \mid F^i$. We assume in the theorem that $\|D_1 f^{-1}(fz)\| \cdot \|Df(z)\|^{\alpha-1} \|D_3 f(z)\| < \mu < 1$ for all $z \in V$. $pr : M \to V$ is the normal bundle projection. For $c \in J^r((M, V), V)$ we write $c(z) = (z, c_0(z), \ldots, c_r(z))$ with $c_k(z) \in L^k_s(T_z M, F^1_{co(z)})$.

Let $\mathscr{F}_r$ be the set of sections c of $J^r((M, V), V)$ such that for each $z \in V$, there is a C^r function $h : M \to V$ such that $h/V = id$ and $c(z) = j^r h(z)$. This is equivalent to assuming for each $z \in V$ (*i*) $\pi_1 \circ \cdots \circ \pi_r c(z) = (z, z)$ and (*ii*) $c^k(z)/F^1 x \cdots x F^1 = D^k(id)(z)$ where $id : V \to V$ is the identify function.

Let $g_1 : M \to V$ be defined by $g_1(z) = f \circ pr(z)$. When we write g_1^{-1} we mean $g_1^{-1} : V \to V$.

Define $\Phi_r : \mathscr{F}_r \to \mathscr{F}_r$ by $\Phi_r c = s$ such that for each z $s(z) = j^r(g_1^{-1} h f)(z)$ where $j^r h(fz) = c(fz)$. By abuse of notation

$$\Phi_r c(z) = j^r(g_1^{-1} c f)(z).$$

Lemma 4: Let $c^1, c^2 \in \mathscr{F}_r$ be such that $\pi_r c^1 = \pi_r c^2$. Then $\rho_r(\Phi_r c^1, \Phi_r c^2) \leqq \mu \rho_r(c^1, c^2)$.

Proof:

$\rho_r(\Phi_r c^1, \Phi_r c^2) \leqq \sup\{\|D_1 g_1^{-1}(fz)\| \cdot \|(c_r^1(fz) - c_r^2(fz))(Df(z)^r\| : z \in V\}$ $\leq \sup\{\|D_1 f_1^{-1}(fz)\| \rho_r(c^1, c^2) \| D_3 f(z)\| \cdot \| D f(z)\|^{r-1} : z \in V\}$ since $c_r^1(z) \mid F^1 x \cdots x F^1 = c_r^2(z) \mid F^1 \cdots F^1$.

Then

$$\rho_r(\Phi_r c^1, \Phi_r c^2) \leqq \rho_r(c^1, c^2)\mu$$

Q.E.D.

As in the proof of Theorem 1, we can apply the fiber contraction principle to find a $c \in \mathscr{F}_\alpha$ such that $\Phi_\alpha(c) = c$. Let $s \in \Gamma J^\alpha((M, V), M)$ be given by $s(z) = (c(z), j^\alpha(id_3(z))$, i.e. the component of s in F^3 in the range is like the jet of the identify function on fibers. (This has meaning at the jet level but not as maps). By the uniform convergence of $\Phi^k_\alpha(j^\alpha pr)$ to c, it follows that s satisfies the conditions of the Whitney Extension Theorem. There exists a C^α h such that $j^\alpha h(z) = s(z)$ for $z \in V$. h is a diffeomorphism on a neighborhood of V because of the form of the derivatives at points of V. Then $g = h\ f\ h^{-1}$ has $D^j{}_3(pr \circ g)(z) = D^j{}_3 g_1(z) = 0$ for $z \in V$ where g_1 is as above.

Q.E.D.

5. *Proof of Corollary* 4.

By applying Theorem 2 we can assume $D^j_2(pr \circ f)(x) = 0$ for $1 \leqq j \leqq \alpha$ and $x \in V$. Define $g_1 : U \to V$ by $g_1(x) = f_1 \circ pr(x)$. In the proof of Theorem 1, replace $a_x = \|Dg^{-1}(x)\|$ by $a_x = \|Dg_1^{-1}(x)\|$ where $g_1^{-1} : V \to V$. Next consider jets in $J^r = J^r(\eta(\delta), V)$ instead of $J^r(\eta(\delta), M)$. Define $\Phi_r : \Gamma J^r \to \Gamma J^r$ by $\Phi_r(c) = d$ such that for each x $d(x) = j^r(g_1^{-1} h f)(x)$ where $j^r h(fx) = c(fx)$.

As in the earlier proof we can find a c such that $\Phi_{\alpha-1} c = c$ and c satisfies the conditions of the Whitney Extension Theorem. There exists a $C^{\alpha-1}$ function $h : M \to V$ such that $g_1^{-1} h f = h$. h is a projection onto V and defines a $C^{\alpha-1}$ foliation · Since $h f = g_1 h$ it follows that f preserves this foliation · Since the foliation is tangent to E^s it follows it is $W^{ss}(x, f)$.

Q.E.D.

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Instituto de Matemática Pura e Aplicada

Rio de Janeiro, Brasil