The riemannian geometry of holomorphic curves

by

H. BLAINE LAWSON, JR.

The classical theory of curves in euclidean space $\mathbb{R}^n$ centers on the concept of certain curvature functions $K_1, \ldots, K_{n-1}$ which appear in the Frenêt formulas. Since the intrinsic geometry of a real curve is trivial, these functions are decidedly extrinsic invariants. Moreover, by the fundamental theorem for space curves, they are mutually independent and completely determine the path of the curve in $\mathbb{R}^n$. This situation, of course, holds in any manifold of constant sectional curvature.

Our purpose here is to explore the analogous situation in the complex case. Most of the results we discuss are due to E. Calabi and S. S. Chern. Calabi showed [2] that associated to a complex analytic curve in complex euclidean space $\mathbb{C}^n$ there are also $n-1$ real-valued curvature functions, defined in a manner entirely analogous to the real case. In this situation, however, the curvature functions are *intrinsic.* That is, they can be computed in terms of the induced riemannian metric.

Calabi actually gives explicit formulas for these curvatures in terms of the metric. However, in principle their intrinsic nature can be deduced from his general theorem [1] which states that: *any complex submanifold of a (simply-connected, complete) space of constant holomorphic curvature is completely determined, up to holomorphic isometries of the ambient space, by its induced metric.*

This theorem has the following corollary. Let C be a non-singular, algebraic curve in complex projective n-space $\mathbb{C}P^n$, and assume that $\mathbb{C}P^n$ is equipped with the standard Fubini-Study metric. (See below.) Then all the extrinsic (projective) invariants of C can be computed in terms of the induced metric. In what follows we shall show how to compute certain of these invariants explicitly. For example, it will be shown that the degree v_k of the k^{th} order osculating curve can be realized as

$$v_k = \frac{1}{2\pi} \iint_C K_k \, dA$$

where K_k is the k^{th} curvature function. (See Chern [3].) From this one can establish the classical Plücker formulas.

We will then formulate and prove some results concerning the differential geometry of algebraic curves, which include a theorem of Nomizu and Smyth [5]. We shall also establish a sequence of "Plücker formulas" for compact, complex curves in a complex torus.

Finally we shall define a sequence of higher order integral invariants for complete minimal surfaces in euclidean space, and establish some formulas relating them.

The main purpose of this exposition is to establish some ground work for the study of complex submanifolds from the point of view of riemannian geometry. In general, such a study should consider the higher order osculating spaces of the submanifold. We shall do this for the one-dimensional case. The viewpoint developed here should be useful in studying, for example, non-compact varieties and varieties with singularities.

1. *A Brief Geometrical Description of the Curvature Functions.*

Let M be a complex n-manifold with a hermitian metric of constant holomorphic curvature. Let ∇ denote the riemannian connection and J the almost complex structure of M. Since M is Kahlerian we have

$$\nabla_X (JY) = J(\nabla_X Y) \tag{1.1}$$

for any tangent vector fields X, Y on M.

Let $C \subset M$ be a non-singular holomorphic curve, that is, a complex submanifold of complex dimension one. At any point $p \in C$, the *second fundamental form* of C is defined as follows. Let X, $Y \in T_p(C)$ (the tangent space of C at p) and extend these to local tangent vector fields $\tilde{X}$, $\tilde{Y}$ on C. Then the normal vector

$$B^1_{X,Y} = (\nabla_{\tilde{X}} \tilde{Y})_p^{N_1} \tag{1.2}$$

where N_1 denotes projection onto $N_p(C) = T_p(C)^\perp$, is independent of the choice of the extensions $\tilde{X}$, $\tilde{Y}$. Furthermore, since $\nabla_{\tilde{X}}\tilde{Y} - \nabla_{\tilde{Y}}\tilde{X} = [\tilde{X}, \tilde{Y}]$, we have $B^1_{X,Y} = B^1_{Y,X}$. Thus, the map $B^1 : T_p(C) \times T_p(C) \to N_p(C)$ is a well defined, symmetric, bilinear mapping, called the second fundamental form of C at p.

Since C is a complex submanifold, both $T_p(C)$ and $N_p(C)$ are complex, i.e., J-invariant subspaces of $T_p(M)$. We now observe that *the second fundamental form B is, in fact, complex bilinear.* To see this, let X, Y, $\tilde{X}$, $\tilde{Y}$ be as above. Then $(\nabla_{\tilde{X}} J\tilde{Y})^N = (J\nabla_{\tilde{X}}\tilde{Y})^N = J(\nabla_{\tilde{X}}\tilde{Y})^N$, and so from (1.2) and the symmetry of B^1 we have

$$B^1_{X,JY} = JB^1_{X,Y} = B^1_{JX,Y}. \tag{1.3}$$

(This is exactly the statement that B^1 is complex bilinear.)

It follows from (1.3) that

$$|B^1_{e_1,e_2}|^2 = K_1(p)$$

constant for all unit vectors e_1, $e_2 \in T_p(C)$. The function K_1 is called the *first curvature function* of the surface. From the complex analyticity of B it follows that the function K_1 vanishes only at isolated points of C. Furthermore, taking the images of B^1, we get a holomorphic complex line bundle over that part of where $K_1 \neq 0$. It can be shown that this bundle, denoted $N_1(C)$ extends to all of C.

Suppose $K_1(p) \neq 0$ and choose vectors X, Y, $Z \in T_p(C)$. Extend these vectors to local fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ on C as above. We now consider the normal vector

$$B^2_{X,Y,Z} = (\nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z})_p^{N_2} \tag{1.4}$$

where N_2 denotes projection onto $[T_p(C) \oplus N_1(C)]^\perp$. Using (1.1) and the fact that M has constant holomorphic sectional curvature (See the formula in [6] for example.), one can easily show that B^2 is a complex trilinear function from $T_p(C)$ into $N_p(C)$. We then define

$$K_2(p) = \frac{|B^2_{e_1,e_2,e_3}|^2}{K_1(p)}$$

for any unit vectors e_1, e_2, $e_3 \in T_p(C)$. This again vanishes only at isolated points, and we obtain a second holomorphic complex line bundle N_2 defined over C.

Proceeding in the same way, we obtain for each k, $1 \leqq k \leqq n-1$, a $(k+1)^{st}$ fundamental form

$$B^k_{X_1, \ldots, X_{k+1}} = (\nabla_{\tilde{X}_1} \nabla_{\tilde{X}_2} \cdots \nabla_{\tilde{X}_k} \tilde{X}_{k+1})^{N_k} \tag{1.5}$$

which is complex multilinear, and we set

$$K_k = \frac{|B^k_{e_1, \ldots, e_{k+1}}|^2}{K_1 \ldots K_{k-1}} \tag{1.6}$$

where $e_1, \ldots, e_{k+1} \in T_p(C)$ are unit vectors.

We have shown that at each point $p \in C$, the normal space $N_p(C)$ decomposes into one-(complex)-dimensional subspaces $N_1 \oplus \ldots \oplus N_{n-1}$ such that for each k, $T_p(C) \oplus N_1 \oplus \ldots \oplus N_k$ is the k^{th} order osculating space of the curve. A similar situation occurs at non-degenerate points of a real curve in $\mathbb{R}^n$ It is straightforward to see that the definition of the curvature functions above is in complete analogy with the definition in the real case.

2. *An Intrinsic Computation of the Curvature Functions.*

Let $M^n(c)$ denote the complete, simply-connected complex manifold of constant holomorphic curvature c. If $c > 0$, $M^n(c)$ is the complex projective n-space with the Fubini-Study metric; if $c = 0$, $M^n(c) = \mathbb{C}^n$ with the flat metric; if $c < 0$, $M^n(c)$ is the unit complex n-disk with the Poincare-Bergmann metric. Let R be a Riemann surface and let $\psi: R \to M^n(c)$ be a holomorphic immersion. In terms of a local coordinate $z = x + iy$ on R we can express the metric induced by ψ as

$$ds^2 = 2F\,|dz|^2. \tag{2.1}$$

By means of the operators $d/dz = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$ and $(d/d\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, the Laplace-Beltrami operator of this metric can be written

$$\Delta = \frac{2}{F}\frac{d}{dz}\frac{d}{d\bar{z}}. \tag{2.2}$$

Therefore, the Gauss curvature is given by the formula

$$K = -\frac{1}{F}\frac{d}{dz}\frac{d}{d\bar{z}} \log F. \tag{2.3}$$

It is now possible to define inductively a sequence of non-negative functions on M as follows.

Lemma 1. (E. Calabi [2]) *Suppose that the image $\psi(R) \subset M^n(c)$ lies in no proper, totally geodesic submanifold, of $M^n(c)$. Then we can define a sequence of functions $\{F_k\}_{k=0}^{n+1}$ by setting*

$$\begin{aligned} F_0 &= 1 \\ F_1 &= F \\ \textit{and'}\ F_{k+1} &= \frac{F_k^2}{F_{k-1}}\left(\frac{d}{dz}\frac{d}{d\bar{z}} \log F_k + \frac{(k+1)}{2} cF\right) \end{aligned} \tag{2.4}$$

for $k = 1, \ldots, n$. For $0 \leqq k \leqq n$, F_k is non-negative and vanishes only at isolated points. The succeeding function F_{k+1} is defined by (2.4) away from those points but extends to a real analytic function on all of R. The function $F_{n+1} \equiv 0$.

Conversely, we also have

Theorem 1. (E. Calabi [1], [2]). *Let $ds^2 = 2F\,|dz|^2$ be a real analytic metric on R, and suppose that the sequence of functions $\{F_k\}_{k=1}^{n+1}$ given by (2.4) can be defined with the same properties as in Lemma 1. Then there exists a unique, holomorphic, isometric immersion of R into $M^n(c)$.*

These intrinsically defined functions F_k carry information about the higher order jets of the (uniquely determined) immersion. To exploit this information we look for combinations of the F_k's which are independent of local coordinates. The most important of these is the following. For each k, $1 \leqq k \leqq n$, we define the function

$$K_k = \frac{F_{k+1} F_{k-1}}{F F_k}, \tag{2.5}$$

and for convenience we set

$$K_0 = \frac{c}{2}.$$

Clearly $K_k \geqq 0$, and for $k < n$, $K_k = 0$ only at isolated points. ($K_n \equiv 0$.) Furthermore, from (2.2) and (2.4) we see that

$$K_k = \tfrac{1}{2}(\Delta \log F_k + (k+1)c). \tag{2.6}$$

Since these curvature functions are defined recursively, there exist certain non-trivial relationships between them. The ones of interest to us here are the following.

Lemma 2. Let F_k and K_k be defined as above. Then

$$\tfrac{1}{2}\Delta \log K_k = K_{k+1} + K_{k-1} - 2K_k + K \text{ for } k = 1, \ldots, n-1 \tag{2.7}$$

$$\tfrac{1}{2}\Delta \log (K_1 \ldots K_{n-1}) = (2n-1)K_0 - nK_1 - K_{n-1} \tag{2.8}$$

$$K_1 = 2K_0 - K \tag{2.9}$$

where K is the Gauss curvature of the surface.

Proof. (2.3), (2.5) and (2.6) ⇒ (2.7). (2.3) and (2.6) ⇒ (2.9). (2.7) and (2.9) ⇒ (2.8).

We leave it to the reader to verify that the functions K_k defined here coincide with the curvature functions in 1 up to a multiplicative factor of 2. This is most easily done by using the formulas (3.4) or (4.1), derived later. We point out, however, that equation (2.9) is simply the Gauss curvature equation of the immersed surface.

3. *Curves in Complex Projective Space.*

We turn our attention now to the case where the holomorphic curvature $c = 1$. Let $\mathbb{C}^{n+1}$ be the $(n+1)$-dimensional complex number space and define an equivalence relation $\sim$ on $\mathbb{C}^{n+1}$ {0} by setting $Z \sim Z'$ iff $\exists \alpha \in \mathbb{C}$ such that $Z = \alpha Z'$. The quotient space $\mathbb{C}P^n = (\mathbb{C}^{n+1} \sim \{0\})/\sim$ is a complex manifold called *complex projective n-space*. The $(n+1)$-tuples $Z = (Z_0, \ldots, Z_n) \in \mathbb{C}^{n+1} \sim \{0\}$ are called *homogeneous coordinates* for $\mathbb{C}P^n$.

Consider the symmetric 2-form given in $\mathbb{C}^{n+1} \sim \{0\}$ by

$$ds_0^2 = 4\frac{|Z \Lambda dZ|^2}{|Z|^4} \tag{3.1}$$

where $|Z \Lambda dZ|^2 = |Z|^2 |dZ|^2 - |< Z, dZ >|^2$. This form is invariant under complex scalar multiplication in $\mathbb{C}^{n+1}$ and projects to a riemannian metric on $\mathbb{C}P^n$, of constant holomorphic curvature 1, called the Fubini-Study metric.

The projective lines $\mathbb{C}P^1 \subset \mathbb{C}P^n$, given by complex planes in homogeneous coordinates, are all isometric to the unit 2-sphere in $\mathbb{R}^3$ and, therefore, have area 4π.

We define a closed exterior 2-form in $C^{n+1} \sim \{0\}$ by setting

$$\omega = 4\partial\bar{\partial} \log |Z|^2$$

where $\partial = \sum_{k=0}^{n} dZ_k \Lambda \frac{\partial}{\partial Z_k}$ is the usual operator on forms. ω projects to a closed non-degenerate 2-form in $\mathbb{C}P^n$, called the *Kähler form*. It has the property that restricted to any complex line in any tangent space of $\mathbb{C}P^n$, it is the positively oriented unit 2-form. Since $H_2(\mathbb{C}P^n;\ Z) = Z$, where the generator is any projective line, we have the following. *Let ψ: $R \to \mathbb{C}P^n$ be a holomorphic mapping where R is a compact Riemann surface. Then the volume of R induced by this mapping is $4\pi n$ where n is the homology degree of ψ.*

We now consider a non-singular algebraic curve in $\mathbb{C}P^n$, that is, a holomorphic immersion ψ: $R \to \mathbb{C}P^n$ where R is a compact Riemann surface. Given a local parameter z on R, we may express this immersion in homogeneous coordinates by an $(n+1)$-tuple of holomorphic functions.

$$\psi(z) = (\psi_0(z), \ldots, \psi_n(z)). \tag{3.2}$$

Note that the expression (3.2) is well-defined up to scalar multiplication by a holomorphic function. We may assume that the ψ_k are never simultaneously zero, for if, say, they had a common zero of order m at a point z_0, then we could replace $\psi(z)$ by the functions $(1/(z-z_0)^m)\psi(z)$.

The metric induced on R by ψ has the form

$$ds^2 = 2F\,|dz|^2$$

where

$$F = 2\frac{|\psi \Lambda \psi'|^2}{|\psi|^4} = 2\frac{d}{dz}\frac{d}{d\bar{z}} \log |\psi|^2. \tag{3.3}$$

Lemma 3. Let $ds^2 = 2F\,|dz|^2$ be the metric given by (3.3) and let F_k be the sequence of associated functions given in Lemma 1. Then

$$F_k = 2\frac{|\psi \Lambda \psi' \Lambda \ldots \Lambda \psi^{(k)}|^2}{|\psi|^{2k+2}} \tag{3.4}$$

for each k.

Proof. By equation (3.3), the lemma holds for $k = 1$. We shall proceed by induction.

To begin we observe that formulas (3.3) and (3.4) remain invariant under unitary transformations of the homogeneous coordinates of ψ and under scalar multiplication of ψ by a non-zero holomorphic function. Consequently, if we fix a point z_0 in the coordinate neighborhood, then we may assume, without loss of generality, that the functions $d^k\psi/dz^k$ have the form:

$$\begin{aligned} \psi(z_0) &= (a_0, 0, 0, 0, \ldots, 0) \\ \psi'(z_0) &= (b_{1,0}, a_1, 0, 0, \ldots, 0) \\ \psi''(z_0) &= (b_{2,0}, b_{2,1}, a_2, 0, \ldots, 0) \\ \psi^{(n)}(z_0) &= (b_{n,0}, b_{n,1}, \ldots, b_{n,n}, a_n). \end{aligned} \tag{3.5}$$

For each $k = 0, \ldots, n$, consider the holomorphic vector

$$\psi_k = \psi \wedge \psi' \wedge \ldots \wedge \psi^{(k)} \tag{3.6}$$

(having $\binom{n+1}{k+1}$ components). Then at $z = z_0$ we have

$$\psi_k(z_0) = (a_0 a_1 \ldots a_k, 0, 0, \ldots, 0).$$

We may, of course, assume that $\psi(R)$ lies in no proper linear subspace of $\mathbb{C}P^n$. Hence, the function ψ_n vanishes only at isolated points, and we may assume that z_0 is not one of those points, so that $a_0 \ldots a_n \neq 0$.

Suppose now (3.4) holds for $F_1, \ldots, F_k$. Then

$$\begin{aligned} &\frac{d}{dz}\frac{d}{d\bar{z}} \log F_k - \binom{k+1}{2} F \\ &= \frac{d}{dz}\frac{d}{d\bar{z}} \log |\psi_k|^2 = \frac{|\psi_k \wedge \psi_k'|^2}{|\psi_k|^4} \\ &= \frac{|(a_0 \ldots a_{k-1})^2 a_k a_{k+1}|^2}{|a_0 \ldots a_k|^4} \end{aligned}$$

at $z = z_0$. Thus, at $z = z_0$

$$\begin{aligned} F_{k+1} &= \frac{F_k^2}{F_{k-1}} \left(\frac{d}{dz}\frac{d}{d\bar{z}} \log F_k + \binom{k+1}{2} F \right) \\ &= 2|a_0 \ldots a_{k+1}|^2 \\ &= 2|\psi_{k+1}|^2 . \end{aligned}$$

The formula now extends to the points where $\psi_n = 0$ by continuity, and the lemma is proved.

Observe now that the local maps $\psi_k(z)$ of equation (3.6) give a well defined, holomorphic mapping

$$\psi_k : R \to \mathbb{C}P^{\binom{n+1}{k+1}-1}$$

called the k^{th} *osculating curve* of our original curve. By equation (3.3) the metric $d\sigma_k^2$ induced on R by ψ_k has the form

$$d\sigma_k^2 = 2G_k |dz|^2$$

where

$$G_k = 2 \frac{d}{dz}\frac{d}{d\bar{z}} \log |\psi_k|^2 .$$

It now follows directly from Lemma 3 and equation (2.6) that the k^{th} curvature can be expressed as

$$2K_k = \frac{G_k}{F} = \frac{d\sigma_k^2}{ds^2} \tag{3.7}$$

in analogy with the classical formula of Gauss.

As remarked above, the volume of the k^{th} osculating curve is $4\pi\nu_k$ where ν_k is an integer (the homology degree of ψ_k) called the *order* of the k^{th} osculating curve. From (3.7) we immediately obtain the following.

Proposition 1. *Let ds^2 be a metric on a compact Riemann surface R, which satisfies the conditions of Theorem 1. Let K_k be the k^{th} curvature function, defined by formula (2.5). Then the integrated k^{th} curvature*

$$\nu_k = \frac{1}{2\pi} \iint_R K_k \, dA \tag{3.8}$$

is an integer equal to the order of the kth osculating curve of the (uniquely determined) isometric immersion of R into $\mathbb{C}P^n$.

We now examine a second invariant. Let $p \in R$ and choose a local coordinate z for R such that p corresponds to $z = 0$. By making a suitable unitary change of homogeneous coordinates we may assume that the functions $\psi(z) = (\psi_0(z), \ldots, \psi_n(z))$ have the form:

$$\begin{aligned}
\psi_0(z) &= c_{0,0} + c_{0,1} z + \cdots \\
\psi_1(z) &= z^{\delta_0}(c_{1,1} z + c_{1,2} z^2 + \ldots) \\
&\vdots \\
\psi_k(z) &= z^{\delta_0 + \cdots + \delta_{k-1}}(c_{k,k} z^k + c_{k,k+1} z^{k+1} + \cdots) \\
&\vdots
\end{aligned}$$

where each $c_{k,k}$ is non-zero and each $\delta_k = \delta_k(p)$ is a non-negative integer which is zero except at a finite number of points. We define

$$\sigma_k = \sum_{p \in R} \delta_k(p).$$

(Note that since ψ: $R \to \mathbb{C}P^n$ is non-singular, $\sigma_0 = 0$.)

A straightforward computation from (3.4) and (2.5) shows that if ψ has the form (3.9) above, then locally

(3.10) $$K_k(z) = c\,|z|^{2\delta_k} + 0(|z|^{2\delta_k+1}).$$

It now follows from Green's Identity that

(3.11) $$\sigma_k = -\frac{1}{4\pi} \iint_R \Delta \log K_k \, dA.$$

Hence, from equations (2.7),(3.8), (3.11) and the classical Gauss-Bonnet formula, we get an intrinsic proof of the following. (See [7].)

Theorem 2 (The Plücker formulas).

(3.12) $$\sigma_k = 2v_k - v_{k+1} - v_{k-1} + 2(g-1)$$

where $v_{-1} = 0$ *and* g = *the genus of* R.

Recall now that each of the functions $K_1, \ldots, K_{n-1}$ is non-negative and vanishes only at isolated points. We now have .

Corollary 1. *Let* ψ: $R \to \mathbb{C}P^n$ *be a compact, non-singular curve of genus* g *for which* $K_1 > 0, \ldots, K_{n-1} > 0$. *Then* $g = 0$ *and*

$$v_k = (k+1)(n-k); \; k = 0, \ldots, n.$$

Proof. Since $K_k > 0$, it follows from (3.11) that $\sigma_k = 0$; $k = 0, \ldots, n-1$. Applying the Plücker formulas inductively we get

$$v_k = (k+1)(v_0 + k(g-1))$$

for $k = 0, \ldots, n$. However, $v_n = 0$ since $K_n \equiv 0$. Thus, we have $v_0 = n(1-g)$; and since $v_0 = (1/4\pi) \times$ (area of the curve) > 0, we must have $g = 0$, $v_0 = n$. This completes the proof.

We now introduce the canonical example of a curve with the properties assumed in Corollary 1. It has been shown by Calabi [1] that, modulo holomorphic congruences, there is only one curve C_n of constant Gauss curvature in $\mathbb{C}P^n$ which does not lie in any linear subspace. This curve has curvature $1/n$ and is given by the following embedding of $\mathbb{C}P^1 \to \mathbb{C}P^n$:

$(3.13)_n$ $$(z_0, z_1) \to \left(z_0^n, \sqrt{n}\, z_0^{n-1} z_1, \ldots, \sqrt{\tbinom{n}{k}}\, z_0^{n-k} z_1^k, \ldots, z_1^n\right).$$

The curvatures for this curve are

(3.14) $$K_k = \frac{(k+1)}{2n}(n-k),$$

Therefore, by equation (3.7) the k^{th} osculating curve of C_n also has constant Gauss curvature, and is exactly $C_{(k+1)(n-k)}$. Concerning these curves we have the following interesting fact.

Proposition 2. (The Quantization Lemma.) *Let* ψ: $R \to \mathbb{C}P^n$ *be a non-singular curve whose Gauss curvature* K *satisfies*

$$\frac{1}{k} \leqq K < \frac{1}{k-1}$$

for some k, $1 < k \leqq n$. *Then* $K \equiv \frac{1}{k}$ *and* $\psi(R) = C_k$.

Proof. Integrating the inequality over R we have

$$\frac{1}{k}\iint_R dA \leqq \iint_R K dA < \frac{1}{k-1}\iint_R dA.$$

Therefore, by the Gauss-Bonnet theorem

$$4\pi \frac{v_0}{k} \leqq 4\pi < 4\pi \frac{v_0}{k-1},$$

and it follows that $k = v_0$. We now have $K - 1/k \geqq 0, \iint (K - 1/k) dA = 0$.

We conclude that $K \equiv 1/k$, and that the curve must be congruent to C_k by the uniqueness statement of Theorem 1.

There is, of course, a completely similar statement if we require

$$\frac{1}{k} < K \leqq \frac{1}{k-1}$$

Observe that the pinching must be between numbers of the sort $1/k$ and $1/(k-1)$. In fact, by perturbing the coefficients of the monomials in equation $(3.13)_n$, we obtain curves whose Gauss curvature functions take values in arbitrarily small, but non-trivial, intervals about $1/n$.

As a natural complement to Proposition 2 one might conjecture that if $K \leqq \frac{1}{n}$ (for a curve in $\mathbb{C}P^n$), then $K \equiv 1/n$. For $n = 2$ this is indeed the case. Observe that from equations (2.7) and (2.9) we have that for a curve in $\mathbb{C}P^2$

(3.15) $$\Delta \log K_1 = 6\left(K - \frac{1}{2}\right)$$

where

(3.16) $$K_1 = 1 - K.$$

From these equations we obtain

Proposition 3 (Nomizu and Smyth [5]). *Let C be a compact, non-singular curve in* $\mathbb{C}P^2$ *whose Gauss curvature K satisfies*

$$K \leqq \frac{1}{2}$$

Then $K \equiv \frac{1}{2}$ *and* $C = C_2$.

Proof. By (3.16) we have $K_1 > 0$, and therefore, by (3.15),

$$\iint_R \left(K - \frac{1}{2}\right) dA = 0.$$

It follows immediately that $K \equiv \frac{1}{2}$, and so $C = C_2$.

Using the equations of 2 we have an immediate generalization of this proposition.

Theorem 3. *Let C be a compact, non-singular curve in* $\mathbb{C}P^n$ *for which* $K_1 > 0, \ldots, K_{n-1} > 0$. *Suppose that*

$$K_{n-1} \geqq \frac{1}{2},$$

and the Gauss curvature

$$K \leqq \frac{1}{n}.$$

Then $K \equiv \frac{1}{n}$ *and* $C = C_n$.

Proof. By (3.16) our conditions imply that

$$nK_1 + K_{n-1} \geqq n - \frac{1}{2}.$$

Hence, by equation (2.8) we have

$$\frac{1}{2}\Delta \log (K_1 \ldots K_{n-1}) = \left(n - \frac{1}{2}\right) - (nK_1 + K_{n-1}) \leqq 0.$$

Since $K_1 \ldots K_{n-1} > 0$, this implies $K_1 \ldots K_{n-1}$ = constant and $nK_1 + K_{n-1} \equiv n - \frac{1}{2}$. We conclude that $K \equiv 1/n$ and $C = C_n$.

Note that Theorem 3 is exactly Proposition 3 for the case $n = 2$.

Note also that the conditions of Theorem 3 can be reinterpreted by saying

$$K_1 \geq \bar{K}_1$$

$$K_{n-1} \geq \bar{K}_{n-1}$$

where $\bar{K}_1$ and $\bar{K}_{n-1}$ are the curvatures of C_n given in (3.14).

It would be interesting to know whether Theorem 3 continues to hold under slightly weaker hypotheses.

4. *Curves in a Complex Torus.*

Let R be a compact Riemann surface and $\psi: R \to T^n$ a holomorphic immersion into a complex n-torus T^n with the flat metric $d\sigma^2 = 2|dz|^2$. Such curves arise from the study of periodic minimal surfaces. For example, let $\phi: \Delta = \{|z| < 1\} \to \mathbb{R}^3$ be the classical Schwartz minimal surface and let $\tilde{\phi}: \Delta \to \mathbb{R}^3$ be its conjugate surface. We then define a holomorphic map $\psi: \Delta \to \mathbb{C}^3 = \mathbb{R}^3 \times i\ \mathbb{R}^3$ by $\psi = (\phi, \tilde{\phi})$. The image

$\psi(R) \subset \mathbb{C}^3$ is an embedded curve, invariant under a 6-dimensional lattice $\mathbb{Z}^6$ of translations. Thus, $\psi(\Delta)/\mathbb{Z}^6 \subset C^3/\mathbb{Z}^6 = T^3$ is a non-singular compact curve in T^3.

A second method of generating such curves is to take the Picard mapping $\psi \colon R \to T^g$ obtained by integrating a properly chosen basis of holomorphic differentials on R.

The map $\psi \colon R \to T^n$ induces a metric on R of the form $ds^2 = 2F|dz|^2$, and we can obtain inductively the functions $F_0, F_1, F_2, \ldots$ and the curvatures $K_0 = 0, K_1, K_2, \ldots$ as in 2. Proceeding as in the proof of Lemma 3 one can show that

$$F_k = 2|\psi' \wedge \cdots \wedge \psi^{(n)}|^2 \; ; \; k = 1, 2, \ldots \tag{4.1}$$

In analogy with 3 we now define

$$\nu_k = \frac{1}{2\pi} \iint_R K_k \, dA$$

and

$$\sigma_k = -\frac{1}{4\pi} \iint_R \Delta \log K_k \, dA.$$

It follows from Lemma 2 that we again have the "Plücker formulas"

$$\sigma_k = 2\nu_k - \nu_{k-1} - \nu_{k+1} + 2(g-1). \tag{4.2}$$

Observe that by using (4.1) and proceeding as in 3 we can show that σ_k is a non-negative integer equal to the number of singularities in the $(k+1)^{st}$ osculating bundle N_{k+1}.

Note, moreover, that $\nu_0 = 0$ and $\nu_1 = \iint K_1 \, dA = -\iint K dA = 2g - 2$. (Therefore $g \geqq 1$, and if $g = 1$, the curve is a totally geodesic torus.) From (4.1) it then follows that

$$\nu_0 = 0$$

$$\nu_{k+1} = (k+2)(k+1)(g-1) - \sigma_k - 2\sigma_{k-1} - \cdots - k\sigma_1.$$

In particular, each of the curvatures ν_k is an integer.

5. *Complete Minimal Surfaces of Finite Total Curvature.*

By a minimal surface in $\mathbb{R}^n$ we mean a conformal immersion $\phi \colon R \to \mathbb{R}^n$ of a Riemann surface R into $\mathbb{R}^n$ such that each component

function of $\phi = (\phi_1, \ldots, \phi_n)$ is harmonic. In terms of local coordinates, ϕ is harmonic if and only if

$$\frac{d}{dz}\frac{d}{d\bar{z}}\phi = 0. \tag{5.1}$$

Let $\phi \colon R \to \mathbb{R}^n$ be a minimal surface. Then for each local coordinate z on R we define the local, $\mathbb{C}^n$-valued function

$$\psi(z) = (\psi_1(z), \ldots, \psi_n(z)) \stackrel{\text{def.}}{=} \frac{d\phi}{dz}(z). \tag{5.2}$$

Under a change of coordinates $w = w(z)$ we have that $\psi(z) = \psi(w) \cdot (dw/dz)$. Thus, (5.2) defines a holomorphic map $\psi \colon R \to \mathbb{C}P^{n-1}$. The fact that ϕ is conformal is equivalent to the condition that

$$\psi^2 = \sum_k \psi_k^2 = 0, \tag{5.3}$$

so, in fact, $\psi(R)$ lies in the hyperquadric $Q_{n-2} \subset \mathbb{C}P^{n-1}$, defined in homogeneous coordinates by the equation

$$Z_1^2 + \cdots + Z_n^2 = 0.$$

Observe that the metric of the minimal surface has the form

$$ds^2 = 2F|dz|^2$$

where

$$F = |\psi|^2.$$

Locally, this metric can also be induced by the *holomorphic* map $\Phi^0 \colon R \to \mathbb{C}^n = R^n \times i\,R^n$ given by $\Phi = (1/\sqrt{2})(\phi, \tilde{\phi})$ where each $\tilde{\phi}_k$ is the harmonic conjugate of ϕ_k. It follows, therefore, from Lemma 1 that we may define functions $F_0, F_1, F_2, \ldots$ and curvatures $K_0 = 0$, $K_1, K_2, \ldots$ for this metric.

We now introduce on $\mathbb{C}P^{n-1}$ the renormalized Fubini-Study metric.

$$ds_0^2 = 2\,\frac{|Z \wedge dZ|^2}{|Z|^4}.$$

The mapping $\psi \colon R \to \mathbb{C}P^{n-1}$ then induces a metric

$$ds^2 = 2\tilde{F}\,|dz|^2$$

on R which has associated functions $\tilde{F}_0, \tilde{F}_1, \tilde{F}_2, \ldots$ and curvatures $\tilde{K}_0 = 1, \tilde{K}_1, \tilde{K}_2, \ldots$.

These two metrics are related by the equation

$$\frac{d\tilde{s}^2}{ds^2} = -K \tag{5.4}$$

where K is the Gauss curvature of the minimal surface [4]. Moreover, we have that for each k

$$F_k = |\psi \wedge \psi' \wedge \cdots \wedge \psi^{(k-1)}|^2$$

$$\tilde{F}_k = \frac{|\psi \wedge \psi' \wedge \cdots \wedge \psi^{(k)}|^2}{|\psi|^{2k+2}}. \tag{5.5}$$

Consequently, the curvatures of the metrics are related by the equations

$$K_k = -K\tilde{K}_{k-1}\;;\; k = 1, 2, \ldots . \tag{5.6}$$

It follows immediately from (5.6) and (5.4) that

$$v_k = \tilde{v}_{k-1}\;;\; k = 1, 2, \ldots .$$

In general, of course, these curvatures will be infinite. Suppose, however, that at least the first one is finite, that is, suppose that $\phi: R \to \mathbb{R}^n$ is a complete minimal surface whose total curvature

$$v_1 = \tilde{v}_0 = -\iint_R K dA$$

is finite. Then by a theorem of Chern and Osserman [4], the Riemann surface R is conformally equivalent to a compact Riemann surface R' punctured at a finite number of points, and the mapping $\psi: R \to \mathbb{C}P^{n-1}$ extends to a holomorphic mapping of R'. We can therefore conclude that each of the total curvatures v_k is an integer. Furthermore, the function $\Delta \log K_k$ is integrable over R, since by (2.7) it is a finite sum of integrable functions. Hence, we may define

$$\sigma_k = -\frac{1}{4\pi}\iint \Delta \log K_k \, dA \tag{5.7}$$

for $k = 1, 2, \ldots$ and obtain, via (2.7), a sequence of "Plücker formulas". Putting this together, we have.

Theorem 4. *Let $\phi: R \to \mathbb{R}^n$ be a complete minimal surface of finite total (Gaussian) curvature. Then each of the total curvatures*

$$v_k = \frac{1}{2\pi}\iint_R K_k \, dA$$

of the surface is a non-negative integer. Furthermore, these integers satisfy the formulas

$$\sigma_k = 2v_k - v_{k+1} - v_{k-1} + v_1$$

for $k = 1, 2, 3, \ldots$.

We shall give a geometric interpretation at least of the integer σ_1. From equation (5.4) and (5.5) we see that

$$K_1 = -K = \frac{|\psi \wedge \psi'|^2}{|\psi|^6} \tag{5.8}$$

Therefore,

$$\Delta \log K_1 \, dA = 4\frac{d}{dz}\frac{d}{d\bar{z}}\left[\log\frac{|\psi \wedge \psi'|^2}{|\psi|^4} - \log|\psi|^2\right] dxdy$$
$$= -2\tilde{K}d\tilde{A} + 2KdA$$

where $\tilde{K}$ is the Gauss curvature of the mapping ψ. It follows that

$$\sigma_1 = v_1 + \frac{1}{2\pi}\iint_R KdA. \tag{5.9}$$

In the case that $n = 3$, the quadric $\mathbb{Q}_1 \subset \mathbb{C}P^2$ is just the curve C_2, and so $\tilde{K} \equiv 1$. This shows that for $n = 3$,

$$\sigma_1 = 2v_1 .$$

In general the integral in equation (5.9) can be interpreted, via the Gauss-Bonnet Theorem, in terms of the topology of R and the number of umbilic points on the minimal surface.

Note that a non-singular holomorphic curve C in $\mathbb{C}^n$ is, in particular, a minimal surface. Therefore, if C is complete in the induced metric and has finite total curvature, Theorem 4 can be applied.

I. M. P. A., Rio de Janeiro
and University of California, Berkeley

Bibliographie

[1] E. Calabi, *Isometric imbeddings of complex manifolds*, Ann. of Math. 58 (1953), 1-23.

[2] E. Calabi, *Metric Riemann surfaces*, Contributions to the Theory of Riemann Surfaces, Princeton Univ. Press, Princeton, 1953.

[3] S. S. Chern, *Geometry of submanifolds in a complex projective space*, Symposium internacional de topologia algebraica, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 87-96.

[4] S. S. Chern, and R. Osserman, *Complete minimal surfaces in euclidean n-space*, J. d'Analyse Math. 19 (1967), 15-34.

[5] K. Nomizu and B. Smyth, *Differential geometry of complex hypersurfaces*, II, J. Math. Soc. Japan 20 (1968), 498-521.

[6] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math., 85 (1967), 246-266.

[7] H. Weyl, *Meromorphic Function and Analytic Curves*, Ann. Math. Studies No. 12, Princeton, 1943.