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Dynamics of Complexity of Intersections

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—To S. Smale on the occasion of his 60th birthday

Abstract. The topological complexity of the intersection of a submanifold, moved by a dynamical system, with a given submanifold of the phase space, can increase with time. It is proved that the Morse and Betti numbers of the transversal intersections "generically" grow at most exponentially, while for some special infinitely smooth systems the topological complexity of the intersections can become larger than any given function of time (for a growing sequence of integer time moments).

0. Introduction

Let us consider a diffeomorphism of a compact manifold to itself and two submanifolds. We move the first submanifold applying the diffeomorphism N times and we intersect the images with the second submanifold. The intersections $Z(N)$ are generically smooth manifolds and we shall majorate their topological complexity by an exponential function of N, provided that the second submanifold is generic.

I prove, among other facts, that "generically" the sum $|Z|$ of the Betti numbers of the intersections verifies the inequalities

$$|Z(N)| \leq Ce^{\lambda N} \quad \text{for all} \quad N \geq 0 \tag{1}$$

C and λ independent of N and $Z(N) = (A^N X) \cap Y$, where A: $M^m \to M^m$ is our diffeomorphism and X^k and Y^ℓ the first and second submanifolds.

The word "generically" means "for almost all values of the parameters t in any sufficiently rich family $\{Y_t\}$".

The meaning of the expression "sufficiently rich" is explained below ("very rich" in section 2). The sufficiently rich families of deformations of Y form an open dense set in the space of families, provided that the dimension of the space

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of the parameters exceeds some value (depending only on m).

Remark 1. If (M, A, X, Y) are algebraic, the exponential majoration of the Betti numbers of $Z(N)$ may be obtained by the same reasoning which Artin-Mazur [1] have used for an exponential majoration of the number of periodic points (see [2]).

Remark 2. There exist C^∞ examples, where all the intersections are transversal and still $|Z(N)|$ grows faster than *any* given function of N (at least for a growing subsequence N_i of values of N). In an example, described in [2],

$$M = \mathbf{T}^2 = \{x, y \mod 1\},$$

$$X = Y = \mathbf{S}^1(y = 0),$$

$$A(x, y) = (x, y + f(x)).$$

where f is a C^∞ function. No analytical examples of super exponential growth are known.

Remark 3. It seems that the exponential majoration holds generically for any reasonable differential-topological invariant of $Z(N)$ (for instance, for the sum of the Betti numbers, for the number of generators of the fundamental group and so on). In [2] the exponential majoration has been proved for $|Z|$ =volume of Z (of dimension $= k + \ell - m$). In the present article it is proved for $|Z| =$ volume (of dimension $k + \ell - m$) of the set of tangent planes of Z in the total space of the bundle of Grassmann manifolds of s-planes tangent to M. Other aspects of the relations between the dynamical systems and volume growth has been discussed by Y. Yomdin [6].

Remark 4. It seems probable, that the exponential majoration always holds for the transversal intersections generated by the *analytical* quadruples (A, M, X, Y), but this is not proved even for the apparently simpler local problems.

For instance, let us consider two analytical submanifolds X^k and Y^ℓ, of complementary dimensions $(k + \ell = m)$, intersecting at a fixed point of A. Then the multiplicities of the (complex isolated) intersection of $A^N X$ with Y at that point should be majorated by an exponent of N.

As far as I know, even the bounds of the corresponding oriented intersection multiplicities are unknown. The only exception is the Shub-Sullivan [3] bound for the Poincaré indices of a fixed point. These indices are the oriented intersection

multiplicities of the diagonal of $M \times M$ with its images under the iterations of the mapping $id \times A$.

Remark 5. Similar results hold perhaps for the continuous time dynamical systems (flows). In this case we consider the chords varieties $Z(N) =$ the set of starting points in X of those orbits of the flow which intersect Y at some moment at most N later.

Remark 6. Similar results should hold for the smooth mappings $A\colon M \to M$ which are not invertible, but in this paper A is supposed to be a diffeomorphism.

Remark 7. Probably, similar results hold for the sufficiently rich deformations of the quadruples, (M_t, A_t, X_t, Y_t).

Remark 8. Let us consider a homoclinical saddle fixed point 0 of a mapping A of a plane to itself. Let us fix a neighbourhood of the fixed point and let us consider an intersection point of the attracted and of the repulsed separatrices of the saddle. The images of this point under the iterations A^N of A tend to 0 for $N \to \pm\infty$. We define the *order* of the point as the number of its images outside the neighourhood we have fixed. It seems, that the number of orbits of order N grows with N at most exponentially, if A is generic (in the sense described above). In this connection see [5].

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1. The Exponential Majoration of the Volume

Let us consider a p-parameter family of deformations of Y. Such a family is a mapping $i\colon Y \times T \to M$, where the base T of the family is a p-dimensional manifold with a distinguished point 0 and where i restricted to $Y \times 0$ is the identity mapping. The submanifolds $i(Y \times t)$ will be denoted by Y_t, and the intersections $(A^N X^k) \cap Y_t$ by $Z_t(N)$.

Definition. A family is *rich*, if the derivative of i at each point of $Y \times 0$ is a mapping onto the tangent plane of M.

Rich families exist and even form an open dense set in the space of families, provided that the parameter space dimension p is sufficiently large (namely, if $p \geq m$).

Indeed, the "codimension=product of the co-ranks" formula (see, for instance, [4]) implies that generically

$$\text{codim(bad points of } Y) = p - m + \ell + 1 > \ell = \dim Y.$$

Let us fix a rich family. We replace its base T by a small ball B centered at 0. The restriction of the family to B is a smooth mapping

$$j\colon Y \times B \to M$$

which is a restriction of a smooth proper fibration over a compact base manifold to a compact submanifold with boundary, $Y \times B$.

We fix on B and on M (and hence on its submanifolds X, Y, Z, etc) Riemannian metrics, normalizing the first by the condition volume $(B) = 1$.

Let $|Z|$ be the non-oriented Riemannian volume (of dimension $k + \ell - m$).

Theorem 1. *For almost every $t \in B$ there exist $C > 0, \lambda > 0$ such that*

$$|Z_t(N)| < Ce^{\lambda N} \quad \textit{for all} \quad N > 0 \tag{2}$$

provided that the family $\{Y_t\}$ is rich.

Proof. Let us consider the integral

$$I(X) = \int_B |Z_t|\,|dt| \tag{3}$$

where $Z_t = X^k \cap Y_t^\ell$, $|dt|$ denote the Riemann volume element on B^p. It is well known that this Lebesgue integral exists. The bad values of t, where $|Z_t| = \infty$, contribute nothing to this integral, since they form a set of measure zero. Indeed, let us consider the manifold $W = j^{-1}X$. The bad values are the critical values of the restriction of the natural projection $Y \times B \to B$ to W, hence they form a zero measure set by the Sard theorem. □

Lemma 1. *There exists a constant C_1, independent on X, such that*

$$I(X) \leq C_1\,|X| \tag{4}$$

where $|X|$ is the k-dimensional Riemannian volume of the submanifold X (provided that the family $\{Y_t\}$ is rich).

Proof. It follows from (3) that

$$I(X) \leq |W|$$

where $W^p = j^{-1}X^k$ is the submanifold of $Y^\ell \times B^p$, which is sent to X^k by j, and where $|W|$ denote the Riemann volume.

On the other side, there exists a constant $C_1 > 0$, independent on X, such that

$$|W| \leq C_1\,|X|$$

since j is a restriction of a proper fibration over a compact manifold. □

Now we apply (4) to the consecutive images $X_N = A^N X$ of X under our diffeomorphism A.

Lemma 2. *The integral $I_N = I(X_N)$ grows at most exponentially:*

$$|I_N| \leq C_2 e^{\alpha N} \quad \textit{for all} \quad N \geq 0 \tag{5}$$

where $C_2(X)$ and $\alpha(A)$ are positive constants.

Proof. Let a be the norm of the derivative of A. Then from (4) we obtain

$$|I(X_N)| \leq C_1 a^{KN}\,|X|$$

and (5) is proved with $\alpha = K \ell n\, a$ and $C_2 = C_1\,|X|$.

The rest of the proof of the theorem is the standard Borel-Cantelli reasoning, as in [2]. The function $|Z_t|$ is non-negative. The Tchebyshev inequality implies that for any $C_3 > 0$

$$\text{mes}\,\{t \in B\colon |Z_t(N)| \geq C_2 C_3 N^2 e^{\alpha N}\} \leq 1/(C_3 N^2)$$

(the measure on B being normalized, mes $B = 1$). Hence we have

$$\text{mes}\{t \in B\colon \exists N > 0\colon |Z_t(N)| \geq C_2 C_3 N^2 e^{\alpha N}\} \leq \pi^2/(6C_3).$$

Intersecting those sets, corresponding to large C_3, we obtain

$$\text{mes}\{t \in B\colon\ \forall C_3 > 0\ \exists N > 0\colon\ |Z_t(N)| \geq C_2 C_3 N^2 e^{\alpha N}\} = 0.$$

Hence for almost every $t \in B$ there exists $C_3 > 0$ such that for every $N > 0$

$$|Z_t(N)| < C_2 C_3 N^2 e^{\alpha N}$$

Finally, $N^2 \leq C_4(\beta)e^{\beta N}$ for any $\beta > 0$, hence we obtain the required majoration (2) with $C = C_2C_3C_4$, $\lambda = \alpha + \beta$ for any $\lambda > \alpha$. □

2. The Exponential Majoration of the Total Curvature

Now let us denote the dimension $k + \ell - m$ of Z by s and let us consider the manifold $\tilde{M}$ of the s-dimensional linear subspaces of the tangent spaces of M. $\tilde{M}$ is a compact Riemann manifold of dimension $\tilde{m} = m + s(m - s)$ – the total space of the bundle of Grassmann manifolds over M. Let us denote $\tilde{X}$ the manifold of the s-dimensional subspaces of the tangent spaces of X^k. $\tilde{X}$ is a compact submanifold of $\tilde{M}$ of dimension $\tilde{k} = k + s(k - s)$. Similarly, the manifold $\tilde{Y}$ of the s-dimensional subspaces of the tangent spaces of Y^ℓ is a compact submanifold of $\tilde{M}$ of dimension $\tilde{\ell} = \ell + s(\ell - s)$.

Finally, $\tilde{X} \cap \tilde{Y} = \tilde{Z}$ is the manifold of tangent planes of $Z = X \cap Y$.

The manifold $\tilde{Z}$ is diffeomorphic to Z and

$$\tilde{k} + \tilde{\ell} - \tilde{m} = k + \ell - m = s.$$

Let us consider a family i: $T \times Y \to M$ of deformations Y_t of Y. It induces a family $\tilde{i}$: $T \times \tilde{Y} \to \tilde{M}$ of deformations $\tilde{Y}_t$ of $\tilde{Y}$ ($\tilde{Y}_t$ is the manifold of tangent s-dimensional subspaces of Y_t).

Definition. The family $\{Y_t\}$ is *very rich*, if the induced family $\{\tilde{Y}_t\}$is rich.

It is easy to see that very rich families exist (and that they are generic among the families of deformations of Y with sufficiently many parameters). Indeed, any s-dimensional subspace of a tangent space of M at a point neighbouring to Y is tangent to an ℓ-submanifold, which is close to Y.

We fix a small ball B in T centered at 0, as in section 1.

Theorem 2. *For almost every $t \in B$ there exist $C > 0, \lambda > 0$ such that*

$$\left|\tilde{Z}_t(N)\right| < Ce^{\lambda N} \quad \textit{for all} \quad N > 0, \qquad (\tilde{5})$$

provided that the family $\{Y_t\}$ is very rich.

Proof. We apply theorem 1 to the submanifolds $\tilde{X}, \tilde{Y}$ of the manifold $\tilde{M}$ and to the family $\{\tilde{Y}_t\}$. The diffeomorphism A: $M \to M$ induces a diffeomorphism

$\tilde{A}$: $\tilde{M} \to \tilde{M}$, and we have

$$\tilde{Z}_t(N) = (A^N X \cap Y_t) = (\tilde{A}^N \tilde{X}) \cap \tilde{Y}_t.$$

Since the family $\{\tilde{Y}_t\}$ is rich, theorem 1 implies the majoration $(\tilde{5})$. □

Remark. The volume $\left|\tilde{Z}\right|$ of $\tilde{Z}$ depends on the derivatives of the tangent planes of Z along Z, hence it measures some "total curvature" of Z.

Corollary. *Any characteristic number of the intersection $Z_t(N)$ grows for almost every t at most exponentially with N.*

Proof. We imbed M into an euclidean space diffeomorphically, u: $M \to \mathbb{R}^q$. Since M is compact, the total curvatures of the images of the submanifolds of M will be majorated by the product of their total curvature in M and of a constant, independent on the submanifold.

Any characteristic number of Z is equal to the integral of the corresponding universal differential form, defined on the bundle of the Grassmann manifold of s-planes in $\mathbb{R}^q$, along the manifold of tangent planes of uZ. Hence it is majorated by the total curvature of uZ and hence by the total curvature $\left|\tilde{Z}\right|$ multiplied by a constant, independent on Z.

Thus the exponential majoration of the total curvatures $(\tilde{5})$ implies a similar exponential bound for any characteristic number (for instance, for the Euler characteristics of the intersections). □

3. The Exponential Majorations of the Total Morse and Betti Numbers

The exponential majoration of the total curvature implies a similar majoration for the total Morse number (the minimal number of critical points of a Morse function) and for the total Betti number (the sum of Betti numbers) of the intersections $Z_t(N)$ for almost all t (provided that the family $\{Y_t\}$ is very rich).

The following Lemma is perhaps well known(1)

Lemma 3. *The total Morse number μ and hence the total Betti number of*

(1) L. Jorge has provided me with reference [7] for this lemma.

a compact smooth submanifold of the Euclidean space $\mathbb{R}^q$ does not exceed the product of its total curvature with a positive constant, independent on the submanifold.

Proof. Let us consider the contact elements bundle $E^{2q-1} \to \mathbb{R}^q$ (whose fiber over a point consists of the contact elements, that is of $q-1$- dimensional subspaces of the tangent space at that point). We equip E with its natural Riemannian metrics.

Those contact elements, which are tangent to the given compact submanifold Z (of *any* dimension) of $\mathbb{R}^q$ form a $q-1$-dimensional smooth (Legendre) compact submanifold Z' of E. Let us consider the natural projection $E \to \mathbb{R}P^{q-1}$ (which sends each contact element to a parallel element at the origin). The projective space $\mathbb{R}P^{q-1}$ is also equipped with its natural Riemann metrics (induced from the Euclidean metrics of $\mathbb{R}^q$).

We define now the *Gauss mapping* of Z as the restriction $g: Z' \to \mathbb{R}P^{q-1}$ of the above projection. Let us consider the integral of the multiplicity function of g

$$k(Z) = \int_{\mathbb{R}P^{q-1}} \left|g^{-1}(p)\right| |dp| = \int_{Z'} |\partial g/\partial z| \, |dz| \,. \qquad (6)$$

Since $\|\partial g/\partial z\| \leq 1$, the last integral is majorated by the volume of Z' and hence by a product of the total curvature of Z with a constant independent on Z. Indeed, let us consider the bundle over the set of s-planes tangent to $\mathbb{R}^q$, whose fiber over an s-plane consists of those contact elements, which contain the s-plane. The manifold Z' is the restriction of this bundle to the submanifold $\tilde{Z}$ of the base ($\tilde{Z}$ is the manifold of tangent planes of Z). Hence

$$k(Z) \leq |Z'| \leq C_5(q) \left|\tilde{Z}\right| \qquad (7)$$

where $\left|\tilde{Z}\right| = \text{vol } \tilde{Z}$ is the total curvature at the s- dimensional submanifold Z of the Euclidean space $\mathbb{R}^q$.

But the integrand $\left|g^{-1}(p)\right|$ in (6) is the number of critical points of the restriction to Z of a linear function, whose level planes are parallel to p. Hence

$$\left|g^{-1}(p)\right| \geq \mu(Z), \quad k(Z) \geq C_6(q)\mu(Z).$$

Comparing with (7), we obtain the majorations

$$\mu(Z) \leq C_7(q) \left|\tilde{Z}\right|, \quad \Sigma b_i(Z) \leq C_7(q) \left|\tilde{Z}\right| \qquad (8)$$

of the total Morse and Betti numbers by the total curvature, as required in Lemma 3. □

Remark. One can guess that the total curvature of a real algebraic complete intersection manifold is bounded in terms of the number of monomials, entering in the equations, that is, in terms of the complexity of the system of defining equations.

It would be interesting to know whether the number of manifolds (or of homotopy type of manifolds) with a bounded total curvature is bounded, and whether it is majorated by a polynomial function of the total curvature if it is bounded.

Lemma 4. *The total Morse number (and hence the total Betti number) of a compact smooth submanifold of a compact Riemann manifold is majorated by a product of the total curvature of the submanifold with a positive constant, independent on the submanifold.*

Proof. We fix an embedding of the given compact manifold in an Euclidean space (the metric may be non-preserved). This embedding induces a smooth mapping of the manifold of tangent s-planes of our manifold (where s is the dimension of the submanifolds that we shall consider) into the manifold of tangent s-planes of the Euclidean space. Since the manifold of tangent s- planes of our initial manifold is compact, the derivative of the induced mapping is bounded. Hence the total curvature of the image of an s- submanifold in the Euclidean space does not exceed the product of the total curvature of the s-submanifold in the initial Riemannian space with a constant, independent on the submanifold.

Now Lemma 4 follows from Lemma 3. □

Combining the Theorem 2 and the Lemma 4 we finally obtain the main result of this article.

Theorem 3. *For almost every $t \in B$ there exist $C > 0$, $\lambda > 0$, such that the total Morse and Betti numbers of the intersections $Z_t(N) = (A^N X) \cap Y_t$ verify the exponential inequalities*

$$|\mu(Z_t(N))| < Ce^{\lambda N}, \quad |\Sigma b_i(Z_t(N))| < Ce^{\lambda N}$$

provided that the family $\{Y_t\}$ is very rich.

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