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On the cohomology of one dimensional foliated manifolds

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Abstract. We show that the cohomology group $H^1(M, \mathcal{F})$ is an infinite dimensional vector space, for a dense set of one dimensional foliations on a closed manifold. In particular we compute this cohomology, for some foliations on the torus $\mathbf{T}^2$.

1. Introduction

In this paper we study the cohomology of a closed foliated manifold $(M, \mathcal{F})$ for a foliation given by the orbits of a C^∞ flow without fixed points. Our main result is the following

Theorem 2.1. *If there exist infinitely many distinct leaf closures of* $\mathcal{F}$, *then* $H^1(M, \mathcal{F})$ *is an infinite dimensional vector space.*

A statement of denseness and openness for foliations satisfying this sufficient condition is given in section 3. The result above adds the following information on the torus $\mathbf{T}^2$.

Theorem 2.3. *If* $\mathcal{F}$ *is not a minimal foliation on* $\mathbf{T}^2$, *then* $\dim H^1(M, \mathcal{F}) = \infty$.

Notice that for a linear foliation $\mathcal{L}$, on the torus $\mathbf{T}^n$ the cohomology of $(\mathbf{T}^n, \mathcal{L})$ was completely calculated [**1**],[**3**],[**4**], and [**8**]. Thus, it remains to compute the cohomology group $H^1(M, \mathcal{F})$, for foliations C^r conjugate to linear ones, $0 \leq r \leq 1$. Theorem 4.3 gives a partial answer to this question.

Theorem 4.3. *The following are equivalent on the torus* $\mathbf{T}^2$

a) $\dim H^1(M, \mathcal{F}) = 1$;

b) $\mathcal{F}$ *is* C^∞ *conjugate to a diophantine linear foliation.*

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1. Preliminaries

The cohomology of a foliated manifold $(M, \mathcal{F})$, introduced by Reinhart [7], will be denoted by $H^*(M, \mathcal{F})$. It is also called foliated cohomology, or cohomology of type $(0, q)$. Details may be found in [4]. Throughout this paper M denotes a closed manifold, and $\mathcal{F}$ a foliation given by the orbits of ϕ_t, the C^∞ flow without fixed points generated by the vector field X. For a one dimensional foliation, the complex of the $(0, q)$-forms reduces to

$$0 \to \Lambda^0(M, \mathcal{F}) \xrightarrow{d_{\mathcal{F}}} \Lambda^1(M, \mathcal{F}) \to 0.$$

Given a Riemannian metric $\langle\ ,\rangle$ on M, and a fixed 1-form $\theta = \langle X,\ \rangle$, then the complex of the $(0, q)$-forms can be described as

$$\Lambda^0(M, \mathcal{F}) = C^\infty(M),\ \Lambda^1(M, \mathcal{F}) = \{g\theta; g \in C^\infty(M)\},$$

and $d_{\mathcal{F}}(f) = X(f)\theta$. Here, $C^\infty(M)$ consists of all C^∞ real functions on M, and $X(f)$ denotes the X-directional derivative of f. Then the cohomology group $H^*(M, \mathcal{F}) = H^0(M, \mathcal{F}) \oplus H^1(M, \mathcal{F})$ is given by

$$H^0(M, \mathcal{F}) = \{f \in C^\infty(M); X(f) = 0\},$$

and

$$(1) \qquad H^1(M, \mathcal{F}) = \frac{C^\infty(M)}{\operatorname{Im}\{X: C^\infty(M) \to C^\infty(M)\}}.$$

This means that if the foliation does not have non-constant first integral then $H^0(M, \mathcal{F}) = \mathbb{R}$; otherwise $H^0(M, \mathcal{F})$ is an infinite dimensional vector space over $\mathbb{R}$.

2. The main theorem

In order to determine the dimension of $H^1(M, \mathcal{F})$, by (1) we must try to solve the partial differential equation $X(g) = f$, for a given function $f \in C^\infty(M)$. Of course, if there exists a solution it can be given on each ϕ_t-orbit because fixed a point $p \in M$, we have

$$(2) \qquad g(\phi_t(p)) = g(p) + \int_0^t f(\phi_s(p))ds$$

where the initial condition is $g(p)$. We will make use of (2) to prove our main result.

Theorem 2.1. *Let $\mathcal{F}$ be a foliation on a closed manifold M given by the orbits of a C^∞ flow, ϕ_t, without fixed points. If there exist infinitely many distinct leaf closures of $\mathcal{F}$, then $H^1(M, \mathcal{F})$ is an infinite dimensional vector space over $\mathbb{R}$.*

Proof. By (1), to prove that $\dim H^1(M, \mathcal{F}) = \infty$ it suffices to show that given $n_0 \in \mathbb{N}$, there is a linearly independent set $\{[f_i]\}_{i=1}^{n_0}$ in $H^1(M, \mathcal{F})$. Here, $[f]$ denotes the cohomology class of a C^∞ function $f: M \to \mathbb{R}$. To systematize the proof let us divide it in three cases.

Case I. *If there exist an infinite number of minimal sets of the flow, then the Theorem holds.*

Proof of Case I. Denote by μ_p a minimal set containing a point $p \in M$. Let $\{\mu_{p_i}\}_{i=1}^{\infty}$ be a collection of distinct minimal sets of the flow. Recall that two distinct minimal sets are disjoint.

Given $n_0 \in \mathbb{N}$. Take the minimal sets $\mu_{p_1}, \ldots, \mu_{p_{n_0}}$. Since they are compact and disjoint, we can choose open disjoint neighbourhoods, say $V_1, \ldots, V_{n_0}$, satisfying $\mu_{p_i} \in V_i$. By standard methods, we construct a C^∞ function $f_i: M \to [0, 1]$ with compact support contained in V_i such that $f_i^{-1}(1) = \mu_{p_i}$, for $i = 1, \ldots, n_0$. Let us show that $\{[f_i]\}_{i=1}^{n_0}$ is a linearly independent set in $H^1(M, \mathcal{F})$. Suppose that there is a zero linear combination with real coefficients

$$(3) \qquad \sum_{j=1}^{n_0} r_j[f_j] = 0.$$

By (1), there exists a C^∞ function $g: M \to \mathbb{R}$ such that

$$(4). \qquad X(g) = \sum_{j=1}^{n_0} r_j f_j.$$

Take the point $p_{i_0} \in \mu_{p_{i_0}}$. By (2) and (4), we have

$$(5) \qquad g(\phi_t(p_{i_0})) = g(p_{i_0}) + \sum_{j=1}^{n_0} r_j \int_0^t f_j(\phi_s(p_{i_0}))ds.$$

Since each minimal set μ_{p_i} is ϕ-invariant, the supports of $\{f_i\}_{i=1}^{n_0}$ are disjoint,

$f_i(\mu_{p_j}) = 0$ if $i \neq j$ and $f_i(\mu_{p_i}) = 1$, it follows that (5) reduces to

$$g(\phi_t(p_{i_0})) = g(p_{i_0}) + r_{i_0} \int_0^t f_{i_0}(\phi_s(p_{i_0}))ds.$$

If $r_{i_0} \neq 0$, this gives a continuous unbounded function on M which is impossible. Therefore $r_{i_0} = 0$, and we have shown that $\{[f_i]\}_{i=1}^{n_0}$ is a linearly independent set in $H^1(M, \mathcal{F})$. Then the theorem holds. This completes the proof of the case I.

We observe that each compact invariant set of the flow ϕ_t contains a minimal set. Since the manifold M is compact then there is at least one minimal set. In the remaining cases we will need the following lemma. Notice that $\alpha(p)$ (resp. $\omega(p)$) denotes the α-limit set (resp. ω-limit set) of a point $p \in M$ under the flow ϕ_t.

Lemma 2.2. *Suppose that there are only finitely many minimal sets of the flow ϕ_t. Then given an infinite set $S = \{p_0, p_1, \ldots, p_n, \ldots\} \subset M$ there exists an infinite subset $S' = \{p_{i_0}, p_{i_1}, \ldots, p_{i_n}, \ldots\} \subset S$ such that*

$$A_1 = \bigcap_{j=0}^{\infty} \alpha(p_{i_j}) \neq \{\ \} \text{ and } A_2 = \bigcap_{j=0}^{\infty} \omega(p_{i_j}) \neq \{\ \}.$$

Proof of the Lemma. Let $\{\mu_k\}_{k=1}^{m_0}$ be the collection of all minimal sets of the flow ϕ_t. Define a finite index set by

$$\mathfrak{I} = \{(i_1, \ldots, i_p);\ 1 \leq i_1 < \cdots < i_p \leq m_0 \text{ and } 1 \leq p \leq m_0\}$$

For $I = (i_1, \ldots, i_p) \in \mathfrak{I}$, let $\mu_I = \mu_{i_1} \cup \cdots \cup \mu_{i_p}$.

Now, given an infinite set $S = \{p_0, \ldots, p_n, \ldots\}$, for $I, J \in \mathfrak{I}$ let S_{IJ} consist of all points $p_k \in S$ such that the union of the minimal sets contained in the α-limit set $\alpha(p_k)$ is μ_I and the union of the minimal sets contained in the ω-limit set $\omega(p_k)$ is μ_J. One may show that $\{S_{IJ}\}_{I,J \in \mathfrak{I}}$ is a finite partition of S. Since S is an infinite set then there are indexes, say I_0, J_0, such that $S_{I_0 J_0} = \{p_{i_0}, \ldots, p_{i_n}, \ldots\}$ is an infinite set. So, by construction we have $\bigcap_{j=0}^{\infty} \alpha(p_{i_j}) \supseteq \mu_{I_0}$ and $\bigcap_{j=0}^{\infty} \omega(p_{i_j}) \supset \mu_{J_0}$. This completes the proof of the Lemma.

Let us return to the proof of the theorem. From now on we suppose that there are infinitely many distinct orbit closures of ϕ_t, but that there are only finitely many minimal sets. Let $T = \{\overline{\sigma}(p_i)\}_{i=0}^{\infty}$, a countable family of distinct

orbit closures. By Lemma 2.2, we may assume that $\bigcap_{j=0}^{\infty} \alpha(p_j) \neq \{\ \}$ and $\bigcap_{j=0}^{\infty} (\omega_{p_j}) \neq \{\ \}$.

Now, inclusion defines an ordering on T, so there are two possibilities: a) there exists a totally ordered infinite subset $T' \subset T$; b) any totally ordered subset $T' \subset T$ is finite.

Case II. *If there exists a totally ordered infinite subset $T' \subset T$, then the theorem holds.*

Proof of Case II. Given $n_0 \in \mathbb{N}$. Take $n_0 + 1$ leaf closures in T', say $\overline{\sigma}(p_{i_0}), \overline{\sigma}(p_{i_1}), \ldots, \overline{\sigma}(p_{i_{n_0}})$. We may assume, without loss of generality, that $i_j = j$, and that $\overline{\sigma}(p_0) \subset \overline{\sigma}(p_1) \subset \cdots \subset \overline{\sigma}(p_{n_0})$. We choose open flow boxes $B_0, B_1, \ldots, B_{n_0}$, such that $p_i \in B_i$ and $B_i \cap B_j = \{\ \}$ for $0 \leq i < j \leq n_0$. By assumption, since $\overline{\sigma}(p_i) = \overline{\sigma}(p_{i+1})$, we can choose the flow boxes small enough to ensure that if $B_i \cap \overline{\sigma}(p_j) \neq \{\ \}$ then $i \leq j$. (Figure 1).

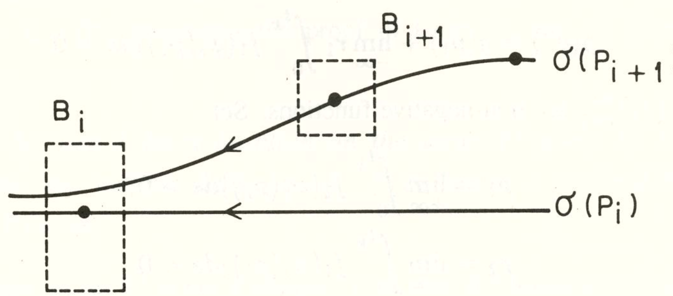

Fig. 1

Now we construct a C^∞ function $f_i\colon M \to [0,1]$ with compact support contained on B_i and $f_i^{-1}(1) = p_i$, for $i \neq 0$, $i = 1, \ldots, n_0$. Let us show that $\{[f_i]\}_{i=1}^{n_0}$ is a linearly independent set in $H^1(M, \mathcal{F})$. Suppose that there is a function $g\colon M \to \mathbb{R}$ satisfying (4). Observe that the leaf closure $\overline{\sigma}(p_0)$ does not meet the support of any f_i. Then $g(\overline{\sigma}(p_0))$ is constant because $X(g)_p = 0$, for $p \in \overline{\sigma}(p_0)$. Assume that $g(\overline{\sigma}(p_0)) = 0$. Take the point p_1. By (2) and (4), we have

$$g(\phi_t(p_1)) = g(p_1) + \sum_{i=1}^{n_0} r_i \int_0^t f_i(\phi_s(p_1))ds. \tag{6}$$

By construction, the p_1-orbit does not meet the flow boxes $B_2, \dots, B_{n_0}$. Hence it does not meet the supports of $f_2, \dots, f_{n_0}$. Therefore (6) reduces to

$$g(\phi_t(p_1)) = g(p_1) + r_1 \int_0^t f_1(\phi_s(p_1))ds. \tag{7}$$

Recall that the intersection

$$A_1 = \bigcap_{j=1}^{\infty} \alpha(p_j) \quad \text{and} \quad A_2 = \bigcap_{j=1}^{\infty} \omega(p_j)$$

are non-empty sets. Since $g(\overline{\sigma}(p_0)) = 0$ then $g(A_k) = 0$, for $k = 1, 2$. From this fact we will show that $r_1 = 0$. Take $q' \in A_1$ and $q'' \in A_2$. By definition of limit sets we have

$$q' = \lim_{t_k \to +\infty} \phi_{t_k}(p_1)$$
$$q'' = \lim_{t_k \to -\infty} \phi_{t_k}(p_1)$$

Recall that g is continuous. From (7), it follows that

$$g(q') = g(p_1) + \lim_{+\infty} r_1 \int_0^{t_k} f_1(\phi_s(p_1))ds = 0;$$
$$g(q'') = g(p_1) + \lim_{-\infty} r_1 \int_0^{t_k} f_1(\phi_s(p_1))ds = 0.$$

Note that $\{f_i\}_{i=1}^{n_0}$ are non-negative functions. Set

$$\rho_1 = \lim_{+\infty} \int_0^{t_k} f_1(\phi_s(p_1))ds > 0;$$
$$\rho_2 = \lim_{-\infty} \int_0^{t_k} f_1(\phi_s(p_1))ds < 0.$$

If $\rho_1 = \infty$ or $\rho_2 = -\infty$ (the p_1 orbit might pass through the support of f_1 infinitely many times), then r_1 must be zero because g is a bounded function; otherwise we have

$$g(q') = g(p_1) + r_1\rho_1 = 0;$$
$$g(q'') = g(p_1) + r_1\rho_2 = 0.$$

Then $r_1(\rho_2 - \rho_1) = 0$, thus $r_1 = 0$. Now repeating the process above we show that $r_i = 0$ for $i = 2, \dots, n_0$. So, we have shown that $\{[f_i]\}_{i=1}^{n_0}$ is a linearly independent set in $H^1(M, \mathcal{F})$. From this the theorem follows and the proof of Case II is complete.

Case III. *If every totally ordered subset $T' \subset T$ is finite, then the theorem holds.*

Proof of Case III. Since $T = \{\overline{\sigma}(p_i)\}_{i=1}^{\infty}$ is an infinite set, and any totally ordered subset is finite, we can choose $T' \subset T$ with infinitely many elements such that if two distinct orbit closures, $\overline{\sigma}(p_i), \overline{\sigma}(p_j)$, belong to T' then neither of them contains the other. We may assume without loss of generality that $T' = T$.

Given $n_0 \in \mathbb{N}$, take $n_0 + 1$ distinct elements in T, $\overline{\sigma}(p_0), \overline{\sigma}(p_1), \dots, \overline{\sigma}(p_{n_0})$. Since $\sigma(p_i) \cap \overline{\sigma}(p_j) = \{\ \}$ for $i \neq j$ (otherwise $\overline{\sigma}(p_i) \subset \overline{\sigma}(p_j)$ which is a contradiction) one can find disjoint flow boxes, $B_0, B_1, \dots, B_{n_0}$, such that $p_i \in B_i$ and $B_i \cap \overline{\sigma}(p_j) = \{\ \}$ for $i \neq j$, $i, j = 0, \dots, n_0$. Now take C^∞ functions $f_i \colon M \to [0,1]$ such that $f_i^{-1}(1) = p_i$ and $\operatorname{supp} f_i \subset B_i$, for $i \neq 0$, $1 \leq i \leq n_0$. As before it may be proved that $\{[f_i]\}_{i=1}^{n_0}$ is a linearly independent set in $H^1(M, \mathcal{F})$. Suppose that the function $g \colon M \to \mathbb{R}$ satisfies (4), and $g(p_0) = 0$. We use the facts that $g(\sigma(p_0)) = 0$, that the intersections $A_1 = \bigcap \alpha(p_j)$ and $A_2 = \bigcap \omega(p_j)$ are non-empty sets, and that

$$g(\phi_t(p)) = g(p) + \sum_{j=1}^{n_0} r_j \int_0^t f_j(\phi_s(p))ds.$$

By an argument similar to that used in the Case II, one can conclude that $r_1 = \dots = r_{n_0} = 0$. This completes the proof of Case III, and the proof of the theorem.

Corollary 2.3. *Let $\mathcal{F}$ be a foliation on the torus $\mathbf{T}^2$ given by the orbits of a C^∞ flow, ϕ_t, without fixed points. If ϕ_t is not a minimal flow then* $\dim H^1(\mathbf{T}^2, \mathcal{F}) = \infty$.

Proof. If a flow ϕ_t on $\mathbf{T}^2$ is not a minimal flow then the foliation has an annular surface, A, foliated by lines asymptotic to the boundary [2]. Therefore we can choose infinitely many leaves inside A whose closures are distinct sets. By 2.1, the Corollary follows.

3. Denseness and openness

Denote by $NSX(M)$ the set consisting of all C^∞ non-singular vector fields on M endowed with the usual C^1 uniform topology for vector fields. Let U consist of those vector fields whose flows have infinitely many distinct leaf closures. Denote by $\mathring{U}$ the interior of U.

Proposition 3.1. *$\mathring{U}$ is dense in $NSX(M)$.*

Proof. Let $X \in NSX(M)$. By the Closing Lemma [6], we may find a vector field $Y \in NSX(M)$ C^1 close to X whose flow has a closed orbit. By ([5], lemma 2.5 pg. 103), there exists a vector field $Z \in NSX(M)$ C^1 close to Y whose flow, ϕ_t, has a hyperbolic closed orbit, say γ. We may assume that the weak stable manifold of γ, $W^s(\gamma)$, is non-empty; otherwise γ is a source orbit, and we take the weak unstable manifold of γ. We know that $W^s(\gamma)$ is a ϕ_t-invariant immersed manifold on M whose dimension k is bigger than one. It is clear that distinct orbits on $W^s(\gamma)$ have distinct closures. Let's show that $Z \in \mathring{U}$. By [5], each vector field $Z' \in NSX(M)$ C^1 close to Z must have a hyperbolic closed orbit γ' near γ and the weak stable manifold $W^s(\gamma')$ has the same dimension as $W^s(\gamma)$. We conclude that $Z' \in U$. Then $\mathring{U}$ is dense in $NSX(M)$. The proof of the Proposition is complete.

4. Applications

Remark 4.1. By straightforward application of the method used in the proof of 2.1, one can prove a slight generalization of that theorem, namely: *If there exist n distinct orbit closures, then* $\dim H^1(M, \mathcal{F}) \geq n$.

Therefore we conclude that if $\dim H^1(M, \mathcal{F}) = k < \infty$ then there are at most k distinct orbit closures. However we do not know of any example for $2 \leq k < \infty$.

Proposition 4.2. *Let $\mathcal{F}$ be a one dimensional foliation on M given by the orbits of a smooth flow, ϕ_t, without fixed points. If* $\dim H^1(M, \mathcal{F}) = 1$ *then ϕ_t is a minimal uniquely ergodic flow.*

Proof. Assume that $\dim H^1(M, \mathcal{F}) = 1$. By remark 4.1 it follows that ϕ_t is a minimal flow.

Let us show that ϕ_t is uniquely ergodic, i.e., there is a unique probability measure, μ, on the Borel field of M satisfying $\mu(A) = \mu(\phi_t(A))$ for every Borel set A in M, and $t \in \mathbb{R}$. Indeed, by (1), a function $f: M \to \mathbb{R}$ represents the zero element in the cohomology group $H^1(M, \mathcal{F})$ if and only if there exists a function $g: M \to \mathbb{R}$ such that $X(g) = f$. Hence, given a ϕ_t-invariant probability measure μ, we have

$$\int_M (g \circ \phi_t - g) d\mu = 0.$$

Since M is compact and X is C^2 then $\dfrac{g \circ \phi_t - g}{t}$ converges uniformly to $X(g)$. Thus

$$\int_M X(g) d\mu = \lim_{t \to 0} \int_M \frac{g \circ \phi_t - g}{t} d\mu = 0. \tag{8}$$

This means that the image of $X : C^\infty(M) \to C^\infty(M)$ is contained in the kernel of $\mu: C^\infty(M) \to \mathbf{R}$. Since there exists at least one ϕ_t-invariant probability measure μ_0, the kernel of any measure is a codimension one vector space of $C^\infty(M)$, $\mu_0(X(f)) = 0$, and

$$\dim \frac{C^\infty(M)}{X(C^\infty(M))} = 1,$$

from which the proposition follows.

Let $\pi: \mathbf{R}^n \to \mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$ be the canonical projection. A symbol with a bar over it denotes an object on the torus and one without a bar its lift.

Theorem 4.3. *Let $\overline{\mathcal{F}}$ be a one-dimensional foliation on the torus $\mathbf{T}^2$ given by the orbits of a smooth flow $\overline{\phi}_t$ without fixed points. The following are equivalent*

a) $\dim H^1(\mathbf{T}^2, \overline{\mathcal{F}}) = 1$;

b) *$\overline{\mathcal{F}}$ is C^∞ conjugate to a diophantine linear foliation.*

Proof. (a $\Rightarrow$ b) Assume that $\dim H^1(M, \mathcal{F}) = 1$. Notice that a diffeomorphism of foliated manifolds $F: (M, \mathcal{F}) \to (M', \mathcal{F}')$ induces an isomorphism

$$F^*: H^*(M', \mathcal{F}') \to H^*(M, \mathcal{F}).$$

By proposition 4.2, $\overline{\phi}_t$ is a minimal flow. It is well known that in this case, up to a diffeomorphism, the foliation $\overline{\mathcal{F}}$ is transverse to a canonical circle bundle. Here we may assume that the lifting is transversal to the y-axis ant that the infinitesimal generator of ϕ_t is the vector field

$$X = \frac{\partial}{\partial x} + a \frac{\partial}{\partial y},$$

where $a: \mathbf{R}^2 \to \mathbf{R}$ is a $\mathbf{Z}^2$-periodic C^∞ function. From $\dim H^1(\mathbf{T}^2, \overline{\mathcal{F}}) = 1$ and (1), there exist $\overline{g} \in C^\infty(\mathbf{T}^2)$ and $\alpha_0 \in \mathbf{R}$ such that $\overline{X}(\overline{g}) = \alpha_0 - \overline{a}$, or equivalently $X(g) = \alpha_0 - a$. Let us show that the map $\overline{G}: \mathbf{T}^2 \to \mathbf{T}^2$ defined by its lifting $G(x, y) = (x, y + g(x, y))$ is a diffeomorphism. We only need to show that the derivative $D\overline{G}_p: \mathbf{T}^2_p \to \mathbf{T}^2_p$ is $1-1$ for every $p \in \mathbf{T}^2$ because $\overline{G}$ is homotopic to the identity. Let JG be the Jacobian matrix of G, namely

$$JG = \begin{pmatrix} 1 & 0 \\ g_x & 1+g_y \end{pmatrix}.$$

Here, f_x and f_y denote the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, respectively.

For a contradiction, suppose that the Jacobian vanishes at (x_0, y_0). This means that $1+g_y(x_0,y_0)=0$. Recall that $X(g)=\alpha_0 - a$. Then from $X(g)_y = (\alpha_0 - a)_y$, we obtain $X(1+g_y) = -a_y(1+g_y)$. The last equation can be solved

$$1+g_y(\phi_t(x,y)) = (1+g_y(x,y))\exp\{-\int_0^t a_y(\phi_s(x,y))ds\}.$$

Hence we conclude that $1+g_y(\phi_t(x_0,y_0)) = 0$ for $t \in \mathbf{R}$ because $1+g_y(x_0,y_0) = 0$. Let $p_0 = \pi(x_0,y_0) \in \mathbf{T}^2$. Now, the minimality of $\overline{\phi}_t$ and $1+\overline{g}_y(\overline{\phi}_t(p_0)) = 0$ imply that $\overline{g}_y(\mathbf{T}^2) = -1$. However, $\overline{g}_y$ must vanish at an extreme point of $\overline{g}$. This contradiction shows that the map $DG: \mathbf{T}^2_p \to \mathbf{T}^2_p$ is $1-1$ for every $p \in \mathbf{T}^2$.

Let $\mathcal{F}_{\alpha_0}$ be the foliation given by the linear vector field

$$L_{\alpha_0} = \frac{\partial}{\partial x} + \alpha_0 \frac{\partial}{\partial y}.$$

One sees that $\overline{G}_*(\overline{\mathcal{F}}) = \overline{\mathcal{F}}_{\alpha_0}$ because $G_*(X) = L_{\alpha_0}$. From the remark at the beginning of this proof, we know that $\dim H^1(\mathbf{T}^2, \overline{\mathcal{F}}_{\alpha_0}) = 1$. By [3], [8], it follows that α_0 must be a diophantine number. The proof of a) $\Rightarrow$ b) is complete. The proof of b) $\Rightarrow$ a) is immediate.

References

1. J. L. Arraut, N. M. Santos, *Linear foliations of* $\mathbf{T}^n$. to appear
2. G. Hector, U. Hirsch, "Introduction to the Geometry of foliations," Part A, 2a. ed., Vieweg & Sohn.
3. James L. Heitsch, *A cohomology for foliated manifolds*, Comment. Math. Helvetici **50** (1975), 197-218.
4. A. El Kacimi-Alaoui, *Sur la cohomology feuilletée*, Compositio Mathematica **49** (1983), 195-215.
5. J. Palis Jr., W. de Melo, "Geometric theory of dynamical systems, An introduction," Springer-Verlag. 1982
6. C. C. Pugh, *An improved closing Lemma and a general density theorem*, Am. J. Math. **89** (1967).
7. B. L. Reinhart, *Foliated Manifolds with bundle-like metrics*, Annals of Math. **69** (1959).

8. C. Roger, *Méthodes homotopiques et cohomologiques en théorie des feuilletages.*, Theses,. Orsay (1976)

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