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Limiting-type theorem for conditional distributions of products of independent unimodular 2×2 matrices

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Abstract. We consider a random process which is some version of the Brownian bridge in the space $SL(2, R)$. Under simplifying assumptions we show that the increments of this process increase as $\sqrt{t}$ as in the case of the usual Brownian motion in the Euclidean space. The main results describe the limiting distribution for properly normed increments.

0. Introduction

The analysis of the asymptotic behavior of distributions of independent unimodular two-dimensional matrices is a well-developed part of probability theory. The classical paper by H. Fürstenberg [1] states, in particular, that, under some natural conditions, typical products increase exponentially. In this paper we study a problem which is closely connected with the one concerning the asymptotical behavior of conditional distributions, under the condition that the products belong to a compact subset of the group $SL(2, R)$. To be more precise, assume that a probability distribution P on the group $SL(2, R)$ is given and has the properties

a) P is concentrated on a compact subset $K \subset SL(2, R)$,

b) P has the density $p(g)$, i.e.

$$P(C) = \int_C p(g)dg, \quad C \subset SL(2, R).$$

We consider the products $g_1^n = g_n \cdot \dots \cdot g_1$ where all $g_i \in SL(2, R)$ are independent and distributed according to the distribution P. We follow the technique by Fürstenberg and Tutubalin (see [1]-[3]) and use special coordinates on

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$SL(2,R)$. Namely, each matrix

$$g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

can be written as

$$g=o_\alpha d_\lambda o_\beta \tag{1}$$

where $o_\alpha, o_\beta \in SO(2)$ and $SO(2)$ is the abelian subgroup of matrices

$$o_\varphi=\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\operatorname{sen}\varphi & \cos\varphi\end{pmatrix}; \quad -\pi\le\varphi\le\pi;$$

d_λ is a diagonal matrix,

$$D_\lambda=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}$$

with $\lambda\ge 1$. The representation (1) is non-unique since also

$$g=(-o_\alpha)d_\lambda(-o_\beta)=o_{\alpha+\pi}d_\lambda o_{\beta+\pi}.$$

It is convenient to pass to the unit tangent bundle over the Lobachevsky plane $H=SL(2,R)/(e,-e)$. The elements $h\in H$ are pairs $(g,-g)$. Since $-g$ has the representation (1) with $\alpha+\pi,\lambda,\beta$ then each h corresponds to four triples

$$\alpha,\lambda,\beta;$$
$$\alpha+\pi,\lambda,\beta;$$
$$\alpha,\lambda,\beta+\pi;$$
$$\alpha+\beta,\lambda,\beta+\pi.$$

In order to have a one-to-one correspondence we shall assume that points of H correspond to triples α,λ,β with

$$-\frac{\pi}{2}\le\alpha<\frac{\pi}{2},\quad -\frac{\pi}{2}\le\beta<\frac{\pi}{2}.$$

It is easy to write down the explicit expressions for λ,α,β through the matrix elements of g:

$$\lambda^4-(a^2+b^2+c^2+d^2)\lambda^2+1=0 \tag{2}$$

$$\operatorname{tg}\alpha=\frac{b+\lambda^2 c}{d-\lambda^2 a} \tag{3}$$

$$\operatorname{tg}\beta=-\frac{a-d\lambda^2}{b+c\lambda^2} \tag{4}$$

The equation (2) defines λ uniquely since $\lambda\ge 1$. The equations (2), (3), (4) define uniquely an element of H because of our assumption

$$-\frac{\pi}{2}\le\alpha<\frac{\pi}{2},\quad -\frac{\pi}{2}\le\beta<\frac{\pi}{2}.$$

Returning to the original situation, suppose that for g_1^m we have a representation

$$g_1^m=o_{\varphi^{(m)}}d_{\lambda^{(m)}}o_{\psi^{(m)}}.$$

Write

$$g_{m+1}=o_{\varphi_{m+1}}d_{\lambda_{m+1}}o_{\psi_{m+1}}.$$

Then

$$g_1^{m+1}=g_{m+1}\cdot g_1^m=o_{\varphi_{m+1}}(d_{\lambda_{m+1}}o_{\psi_{m+1}+\varphi^{(m)}}d_{\lambda^{(m)}})o_{\psi^{(m)}}$$

Put

$$\gamma^{(m)}=\varphi^{(m)}+\psi_{m+1}$$

and write

$$d_{\lambda_{m+1}}o_{\gamma^{(m)}}d_{\lambda^{(m)}}=o_{\alpha^{(m)}}d_{\lambda^{(m+1)}}o_{\beta^{(m)}}.$$

Then

$$\varphi^{(m+1)}=\varphi_{m+1}+\alpha^{(m)} \pmod\pi, \tag{5}$$

$$\psi^{(m+1)}=\psi^{(m)}+\beta^{(m)} \pmod\pi. \tag{6}$$

It is essential that $\lambda^{(m+1)},\varphi^{(m+1)}$ do not depend on $\psi^{(m)}$. Therefore their evolution with m may be considered independently. The exact equation for $\lambda^{(m+1)}$ takes the form:

$$\begin{aligned}(\lambda^{(m+1)})^4-(\lambda^{(m+1)})^2\Bigg(&\lambda_{m+1}^2(\lambda^{(m)})^2\cos^2\gamma^{(m)}+\\ &+\frac{(\lambda^{(m)})^2}{\lambda_{m+1}^2}\sin^2\gamma^{(m)}+\frac{\lambda_{m+1}^2}{(\lambda^{(m)})^2}\sin^2\gamma^{(m)}+\\ &+\frac{1}{(\lambda^{(m)})^2\lambda_{m+1}^2}\cos^2\gamma^{(m)}\Bigg)+1=0\end{aligned} \tag{7}$$

Simplifying assumption. Suppose that $\lambda^{(m)}\gg 1$ for $m>1$. In this approximation (7) is replaced by

$$(\lambda^{(m+1)})^4 - (\lambda^{(m+1)})^2 \left(\lambda_{m+1}^2 (\lambda^{(m)})^2 \cos^2 \gamma^{(m)} + \right. \\ \left. + \frac{(\lambda^{(m)})^2}{\lambda_{m+1}^2} \sin^2 \gamma^{(m)} \right) = 0 \tag{7'}$$

which gives the solution

$$\lambda^{(m+1)} = \lambda^{(m)} \sqrt{\lambda_{m+1}^2 \cos^2 \gamma^{(m)} + \frac{1}{\lambda_{m+1}^2} \sin^2 \gamma^{(m)}} \tag{8}$$

In the same approximation

$$tg\alpha^{(m)} = \frac{1}{\lambda_{m+1}^2} tg\gamma^{(m)} \tag{9}$$

Now we can formulate precisely our problem. Starting with a probability distribution P take the induced distribution on H which we shall denote by the same letter. Thus P may be considered as a probability distribution on the space of triples (φ, λ, ψ). Suppose that we have n independent identically distributed triples $(\varphi_m, \lambda_m, \psi_m), 1 \le m \le n$, each having the distribution P. Consider the sequences of triples

$$\omega_m = (\varphi^{(m)}, \lambda_{m+1}, \lambda_{m+1}), \quad 0 \le m \le n,$$

where $\varphi^{(m)}$ are connected by (5) and (9) with the initial condition $\varphi^{(0)} = 0$ and λ_1, φ_1 being arbitrary. Denote by P_n the induced probability distribution on the sequences of triples $\{\omega_m\}, 0 \le m \le n$.

Lemma 1. *The probability distribution P_n is a Markov chain.*

Proof. Suppose that we are given ω_{m-1}, ω_m. The equality (5) gives a possibility to write $\alpha^{(m-1)}$ as a function of $\varphi_m, \varphi^{(m)}$, namely

$$\alpha^{(m-1)} = \varphi^{(m)} - \varphi_m \pmod{\pi}.$$

Knowing $\alpha^{(m-1)}$ we can find $\gamma^{(m-1)}$ from (9):

$$tg\gamma^{(m-1)} = \lambda_{m+1}^2 \cdot tg\alpha^{(m-1)}$$

and ψ_m from the equality

$$\psi_m = \gamma^{(m-1)} - \varphi^{(m-1)} \pmod{\pi}.$$

This shows that ψ_m is a single-valued invertible function of

$$\varphi^{(m-1)}, \lambda_m, \varphi_m, \varphi^{(m)}.$$

If the density p corresponding to P is written in the form

$$p = p(\varphi, \lambda) \cdot p(\psi \mid \varphi, \lambda)$$

where $p(\psi \mid \varphi, \lambda)$ is the conditional density of ψ, provided that φ, λ are fixed, then

$$p(\omega_m \mid \omega_{m-1}) = p(\varphi_{m+1}, \lambda_{m+1}) \cdot p(\psi_m \mid \varphi_m, \lambda_m) \cdot \left| \frac{d\psi_m}{d\varphi^{(m)}} \right| \tag{10}$$

In the last expression ψ_m is the above-mentioned function of

$$\varphi^{(m-1)}, \varphi_m, \lambda_m, \psi^{(m)}.$$

It is obvious that the conditional density $p(\omega_m \mid \omega_{m-1}, \dots, \omega_{m-k})$ depends only on ω_m, ω_{m-1}. Thus we proved that P_n is a Markov chain and found its conditional transition density. □

Formula (8) shows that

$$\lambda^{(m+1)} = \lambda^{(m)} \exp\{F(\omega_{m+1}, \omega_m)\}$$

where

$$F(\omega_{m+1}, \omega_m) = \frac{1}{2} \ln(\lambda_{m+1}^2 \cos^2 \gamma^{(m)} + \lambda_{m+1}^{-2} \sin^2 \gamma^{(m)}).$$

We shall use it in a more convenient way:

$$\ln \lambda^{(m)} = \sum_{1 \le k \le m} F(\omega_{k+1}, \omega_k) \tag{11}$$

Take two numbers a, b and denote by Q_n the conditional distribution induced by P_n under the condition $a \le \ln \lambda^{(n)} \le b$. Remark that now we have to remove our simplifying assumption and so $\lambda^{(n)}$ are no longer bigger than 1.

Fix a number $\varkappa$, $0 < \varkappa < 1$, and put $n_1 = [\varkappa n]$. Our main problem in this paper is to study the limiting probability distribution as $n \to \infty$ for $\frac{1}{\sqrt{n_1}} \ln \lambda^{(n_1)}$ where the distribution of $\lambda^{(n_1)}$ is determined by Q_n.

We shall use the Cramer's method in the theory of probabilities of large derivations for sequences of independent random variables. Write the density corresponding to P_n in the form

$$\pi_0(\omega_0,\omega_1,\dots,\omega_n) = \pi_0(\omega_0)\prod_{m=1}^{n}\pi(\omega_m \mid \omega_{m-1}) \tag{12}$$

Here $\pi(\omega_m \mid \omega_{m-1})$ is the transition density found in the Lemma 1. For any $\beta, -\infty < \beta < \infty$, introduce the new probability distribution $P_n(\beta)$ whose density equals to

$$\pi_\beta(\omega_0,\omega_1,\dots,\omega_n) = \frac{e^{\beta\sum_{m=1}^{n}F(\omega_m,\omega_{m-1})}}{\Xi_n(\beta)}\pi_0(\omega_0)\prod_{m=1}^{n}\pi(\omega_m \mid \omega_{m-1}) \tag{13}$$

and $\Xi_n(\beta)$ is the normalizing factor which is analogous to partition function in statistical mechanics. Then (13) is a non-homogeneous Markov chain.

The joint probability density corresponding to Q_n equals

$$\bar{\pi}(\omega_0,\omega_1,\dots,\omega_n) = \frac{\pi_0(\omega_0)\prod_{m=1}^{n}\pi(\omega_m \mid \omega_{m-1})}{\sum_n}$$

where $\sum_n$ is the probability that $a \le \ln\lambda^{(n)} \le b$, i.e.

$$\sum_n = \int_{\{\omega_0,\dots,\omega_n\}: a\le \ln\lambda^{(n)}\le b} \pi_0(\omega_0)\prod_{m=1}^{n}\pi(\omega_m \mid \omega_{m-1})\prod_{m=0}^{n}d\omega_m$$

Since for any β (see (13))

$$\pi(\omega_0,\omega_1,\dots,\omega_n) = \pi_\beta(\omega_0,\dots,\omega_n)\,\Xi_n(\beta)e^{-\beta\sum_{m=1}^{n}F(\omega_m,\omega_{m+1})}$$

we have

$$\sum_n \le \Xi_n(\beta)e^{-\beta a}\int_{\{\omega_0,\dots,\omega_n\}: a\le \ln\lambda^{(n)}\le b}\pi_\beta(\omega_0,\dots,\omega_n)\prod_{m=0}^{n}d\omega_m, \tag{14}$$

$$\sum_n \ge \Xi_n(\beta)e^{-\beta b}\int_{\{\omega_0,\dots,\omega_n\}: a\le \ln\lambda^{(n)}\le b}\pi_\beta(\omega_0,\dots,\omega_n)\prod_{m=0}^{n}d\omega_m. \tag{15}$$

Denote

$$\omega' = (\Phi',\lambda',\varphi'),\quad \omega'' = (\Phi'',\lambda'',\varphi''),$$

consider the positive kernel

$$K_\beta(\omega'' \mid \omega') = \pi(\omega'' \mid \omega')\exp\{\beta F(\omega'' \mid \omega')\}$$

and the corresponding integral operator

$$(K_\beta f)(\omega'') = \int K_\beta(\omega'' \mid \omega')f(\omega')d\omega'.$$

The adjoint operator has the form

$$(K_\beta^* f)(\omega') = \int K_\beta(\omega'' \mid \omega')f(\omega'')d\omega''.$$

Lemma 2. *The operator K_β has a positive eigenfunction*

$$h_\beta^*(\omega') = h_\beta(\phi',\lambda',\varphi') = p(\varphi',\lambda')g_\beta(\phi',\lambda',\varphi')$$

where g_β satisfies the following conditions:

1) $g_\beta(\phi',\lambda',\varphi') = 0$ *if* $p(\varphi',\lambda') = 0$;

2) *for some positive constants* c_1, c_2

$$c_1 < g_\beta(\Phi',\lambda',\varphi') < c_2$$

if $p(\varphi',\lambda') > 0$. The adjoint operator K_β^* has a positive eigenfunction

$$h_\beta^*(\omega'' = h_\beta^*(\Phi'',\lambda'',\varphi'')$$

such that

$$c_1 < g_\beta^*(\Phi'',\lambda'',\varphi'') < c_2.$$

The corresponding eigenvalues for K_β and K_β^* coincide. We denote this common eigenvalue by $\Lambda(\beta)$.

Proof of the lemma is given in Appendix.

Now rewrite $\pi_\beta(\omega_0,\omega_1,\dots,\omega_n)$ (see (13)) as follows:

$$\pi_\beta(\omega_0,\omega_1,\dots,\omega_n) = \Lambda^n(\beta)\frac{\pi_0(\omega_0)g_\beta^*(\omega_0)}{\Xi_n(\beta)g^*(\omega_n)}\prod_{m=1}^{n}\frac{e^{\beta F(\omega_m,\omega_{m-1})}g_\beta^*(\omega_m)}{\Lambda(\beta)g_\beta^*(\omega_{m-1})}$$

The function

$$p_\beta(\omega'',\omega') = \frac{e^{\beta F(\omega'',\omega')h_\beta^*(\omega'')}}{\Lambda(\beta)h_\beta^*(\omega')}$$

can be considered as the kernel of a stochastic operator P_β. The corresponding invariant measure for P_β has the density $\nu_\beta(\omega') = h_\beta(\omega')h_\beta^*(\omega')$. The functions g_β, g_β^* are normed in such a way that $h_\beta(\omega')$ is the density of a probability measure. Thus we have

$$\sum_n \leq \Lambda^n(\beta)e^{-\beta a}\int_{\Omega} g^*_\beta(\omega_0)\pi_0(\omega_0)\prod_{m=1}^{n} p_\beta(\omega_m,\omega_{m-1})\frac{1}{h^*_\beta(\omega_n)}\prod_{k=0}^{n} d\omega_k \quad (14')$$

$$\sum_n \geq \Lambda^n(\beta)e^{-\beta b}\int_{\Omega} g^*_\beta(\omega_0)\pi_0(\omega_0)\prod_{m=1}^{n} p_\beta(\omega_m,\omega_{m-1})\frac{1}{h^*_\beta(\omega_n)}\prod_{k=0}^{n} d\omega_k \quad (15')$$

where Ω is given by $\{\omega_0,\dots,\omega_n\}: a \leq \ln\lambda^{(n)} \leq b$.

Lemma 3. *There exists one and only one β_0 for which*

$$\int F(\omega'',\omega')p_{\beta_0}(\omega'',\omega')\nu_{\beta_0}(\omega')d\omega' d\omega'' = 0$$

Proof of this lemma is given in Appendix.

Since

$$\ln\lambda^{(n)} = \sum_{m=1}^{n} F(\omega_m,\omega_{m-1})$$

we can use the local central limit theorem for Markov chains with a compact phase space which gives

$$\int_{a\leq\ln\lambda^{(n)}\leq b} g^*_{\beta_0}(\omega_0)\pi_0(\omega_0)\prod_{m=1}^{n} p_{\beta_0}(\omega_m,\omega_{m-1})\frac{1}{h^*_{\beta_0}(\omega_n)}\prod_{k=0}^{n} d\omega_k \sim$$

$$\sim \frac{1}{\sqrt{2\pi n\sigma}}\int g^*_{\beta_0}(\omega_0)\pi_0(\omega_0)d\omega_0 h^*_{\beta_0}(\omega_n)d\omega_n \quad (16)$$

as $n\to\infty$. The constant $\sigma = \sigma(\beta) > 0$ enters into the asymptotics of the variance:

$$\mathcal{D}_{\beta_0}\left(\sum_{m=1}^{n} F(\omega_m,\omega_{m-1})\right) \sim n\sigma$$

as $n\to\infty$. Here $\mathcal{D}_{\beta_0}$ is the variance of the sum

$$\sum_{m=1}^{n} F(\omega_m,\omega_{m-1})$$

with respect to the probability distribution P_{β_0}.

Take two numbers $u_1, u_2, u_1 < u_2$ and consider the probability

$$q_n = Q_n\left\{u_1 \leq \frac{1}{\sqrt{n}}\ln\lambda^{(n_1)} \leq u_2\right\}.$$

We have

$$\begin{aligned} q_n = \frac{1}{\sum_n}\int_{\{\omega_0,\dots,\omega_n\}: u_1\leq\frac{1}{\sqrt{n_1}}\ln\lambda^{(n_1)}\leq u_2, a\leq\ln\lambda^{(n)}\leq b} &\pi_0(\omega_0)\times \\ \times\prod_{m=1}^{n}\pi(\omega_m\mid\omega_{m-1})\prod_{m=0}^{n} d\omega_m& \\ \leq \frac{\Lambda_1^n(\beta)}{\sum_n}e^{-\beta a}\int_{u_1\leq\frac{1}{\sqrt{n_1}}\ln\lambda^{(n_1)}\leq u_2, a\leq\ln\lambda^{(n)}\leq b} &\pi_0(\omega_0)g^*_{\beta_0}(\omega_0)\times \\ \times\prod_{m=1}^{n} p_{\beta_0}(\omega_m\mid\omega_{m-1})\frac{1}{g^*_{\beta_0}(\omega_n)}\prod_{k=0}^{m} d\omega_k,& \end{aligned} \quad (17)$$

and the analogous inequality from the other side. It follows from the local central limit theorem for Markov chains that for the Markov chain with the transition density p_{β_0} and stationary distribution $h_{\beta_0}\cdot h^*_{\beta_0}$ the probability density

$$p_{\beta_0}\left\{\ln\lambda^{(n_1)} = u\right\} \sim \frac{1}{\sqrt{2\pi\sigma n_1}}e^{-\dfrac{u^2}{2\sigma n_1}}$$

For $u = O(\sqrt{n}), v = O = (\sqrt{n})$ the conditional probability density

$$p_{\beta_0}\left\{\ln\lambda^{(n)} = v \mid \ln\lambda^{(n_1)} = u\right\} \sim \frac{1}{\sqrt{2\pi\sigma(1-\varkappa)n}}e^{-\frac{(v-u)^2}{2\sigma n(1-\varkappa)}}.$$

This yields

$$\int_{\substack{\sqrt{n}u_1\leq\ln\lambda^{(n_1)}\leq\sqrt{n}u_2\\ a\leq\ln\lambda^{(n)}\leq b}} \pi_0(\omega_0)h^*_{\beta_0}(\omega_0)\prod_{m=1}^{n} p_{\beta_0}(\omega_m\mid\omega_{m-1})\cdot\frac{1}{h^*_{\beta_0}(\omega_n)}\prod_{k=0}^{n} d\omega_k \sim$$

$$\sim \frac{1}{\sqrt{2\pi\sigma(1-\varkappa)}}\int_{u_1}^{u_2} e^{-\frac{u^2}{2\sigma(1-\varkappa)}}du\cdot\frac{1}{\sqrt{2\pi\sigma n}}\cdot\int g^*_{\beta_0}(\omega_0)\pi_0(\omega_0)d\omega_0$$

$$\cdot\int g_{\beta_0}(\omega_n)d\omega_n$$

Returning to (15′), (16), (17), we have

$$q_n \leq e^{-\beta_0(a-b)}\frac{1}{\sqrt{2\pi\sigma(1-\varkappa)}}\int_{u_1}^{u_2} e^{-\frac{u^2}{2\sigma(1-\varkappa)}}du. \quad (18)$$

In the same way we get the inequality from below

$$q_n \geq e^{-\beta_0(b-a)} \frac{1}{\sqrt{2\pi\sigma(1-\varkappa)}} \int_{u_1}^{u_2} e^{-\frac{u^2}{2\sigma(1-\varkappa)}} du. \tag{19}$$

Now we can formulate and complete the proof of our main theorem.

Theorem. *The limiting probability distribution of* $\frac{1}{\sqrt{n_1}} \ln \lambda^{(n_1)}$ *with respect to* Q_n *is Gaussian with the variance* $\sigma(1-\varkappa)$.

Proof. Having the interval (a,b), decompose it onto small parts $a = a_0 < a_1 < a_2 < \cdots < a_r = b$ such that $a_j - a_{j-1} \leq e$ where $e > 0$ is a given number. Then from (18) and (19)

$$\mathcal{P}_n \left\{ u_1 \leq \frac{1}{\sqrt{n_1}} \ln \lambda^{(n_1)} \leq u_2 \;\middle|\; a \leq \ln \lambda^{(n)} \leq b \right\} =$$

$$= \sum_{j=1}^{r} \frac{\mathcal{P}\{a_{j-1} \leq \ln \lambda^{(n)} \leq a_j\}}{\mathcal{P}\{a \leq \ln \lambda^{(n)} \leq b\}} \times$$

$$\times \mathcal{P} \left\{ u_1 \leq \frac{1}{\sqrt{n_1}} \ln \lambda^{(n_1)} \leq u_2 \;\middle|\; a_{j-1} \leq \ln \lambda^{(n)} \leq a_j \right\} =$$

$$= \frac{1}{\sqrt{2\pi\sigma(1-\varkappa)}} \int_{u_1}^{u_2} e^{-\frac{u_2}{2\sigma(1-\varkappa)}} du \cdot \sum_{j=1}^{r} \frac{\mathcal{P}\{a_{j-1} \leq \ln \lambda^{(n)} \leq a_j\}}{\mathcal{P}\{a \leq \ln \lambda^{(n)} \leq b\}} (1 + \delta_j^{(n)}(\epsilon))$$

where $\left|\delta_j^{(n)}(\epsilon)\right| \leq 2\epsilon$ for all sufficiently large n. This gives the desired result. □

Appendix

Proof of Lemma 2. We start with the analysis of h_β. Take an arbitrary function $u(\omega')$ of the form $u(\omega') = p(\varphi', \lambda') \cdot v(\Phi', \lambda', \varphi')$ where v is equal to zero if $p(\varphi', \lambda') = 0$ and

$$d_1 < v(\Phi', \lambda', \varphi') < d_2$$

otherwise, where d_1, d_2 are two positive constants. Then from the definition of K_β and (10)

$$(K_\beta u)(\omega'') = p(\varphi'', \lambda'') \cdot \int v(\Phi', \lambda', \varphi') p(\psi \mid \varphi', \lambda') \left| \frac{d\psi'}{d\varphi'} \right| \cdot e^{\beta F(\omega'', \omega')} d\Phi'$$

Put $v_1(\Phi, \varphi'', \lambda'') = 0$ if $p(\varphi'', \lambda'') = 0$ and

$$v_1(\omega'') = v_1(\Phi'', \varphi'', \lambda'')$$

$$= \int v(\Phi', \lambda', \varphi') p(\psi | \varphi', \lambda') \left| \frac{d\psi'}{d\varphi'} \right| e^{\beta F(\omega'', \omega')} d\Phi'.$$

Then v_1 satisfies 1. We may assume that

$$\frac{p(\psi', \varphi', \lambda')}{p(\psi'', \varphi', \lambda')} \leq d > 0$$

for some constant d. Otherwise we can pass to some power of the operator K_β for which it holds. For two different values

$$(\overline{\Phi}'', \overline{\varphi}'', \overline{\lambda}'') = \overline{\omega}'', \quad (\overline{\overline{\Phi}}'', \overline{\overline{\varphi}}'', \overline{\overline{\lambda}}'') = \overline{\overline{\omega}}''$$

we have

$$\frac{v_1(\overline{\omega}'')}{v_1(\overline{\overline{\omega}}'')} = \frac{\int v(\Phi', \lambda', \varphi') p(\overline{\psi}' \mid \varphi', \lambda') \left| \frac{d\overline{\psi}'}{d\varphi'} \right| e^{\beta F(\omega'', \omega')} d\Phi'}{\int v(\Phi', \lambda', \varphi') p(\overline{\overline{\psi}}' \mid \varphi', \lambda') \left| \frac{d\overline{\overline{\psi}}'}{d\varphi'} \right| e^{\beta F(\omega'', \omega')} d\Phi'}$$

$$\leq d \cdot \frac{\max \left| \frac{d\overline{\psi}'}{d\varphi'} \right|}{\min \left| \frac{d\overline{\overline{\psi}}'}{d\varphi'} \right|} \exp \beta \{\max F(\omega'', \omega') - \min F(\omega'', \omega')\}.$$

Here $\overline{\psi}', \overline{\overline{\psi}}''$ are the values of ψ' which correspond to $\overline{\omega}'', \overline{\overline{\omega}}''$ for the same ω'. Put $L_\beta v = v_1$. We see that the operator L_β is an operator with the positive kernel on a compact set and therefore by the Brouwer's fixed point theorem it has a positive eigenfunction and the corresponding positive eigenvalue.

The same arguments work for the operator K_β^*. The fact that the corresponding eigenvalues coincide is shown by simple direct arguments. □

Proof of Lemma 3. The statement of the Lemma is rather well-known in statistical mechanics. It means that β_0 is found from the condition that the expectation of $F(\omega_1 \mid \omega_0)$ with respect to the stationary Markov measure with the transition density $p_{\beta_0}(\omega'', \omega')$ is zero. It is easy to show that the derivative

$$\frac{d}{d\beta} \int F(\omega'', \omega') p_\beta(\omega'', \omega') \nu_\beta(\omega') d\omega' d\omega'' > 0 \tag{20}$$

because

$$\int F(\omega'',\omega')p_\beta(\omega'',\omega')\nu_\beta(\omega')d\omega' d\omega'' =$$

$$= \lim \frac{1}{n}\frac{\partial}{\partial\beta} \ln \int \pi_0(\omega_0) \prod_{m=1}^{n} \pi(\omega_m \mid \omega_{m-1}) e^{\beta \sum_{m=1}^{n} F(\omega_m,\omega_{m-1})} \prod_{m=0}^{n} d\omega_m \tag{21}$$

and therefore

$$\frac{\partial}{\partial\beta}\int F(\omega'',\omega')p_\beta(\omega'',\omega')\nu_\beta \omega' d\omega'' = \lim \frac{1}{n} \operatorname{Var}_\beta \left(\sum_{m=1}^{n} F(\omega_m,\omega_{m-1}) \right)$$

where Var_β is the variance which is found with the help of the distribution $P_n(\beta)$. Thus (20) is shown.

This yields that the expectation (21) is a monotone increasing function of β. It is easy to find periodic sequences $\{\omega_m\}$ for which the sums over a period

$$\sum_{m=1}^{t} F(\omega_m \mid \omega_{m-1})$$

are positive as well as periodic sequences for which this sum is negative. Then the limit of the l.h.r. of (21) is positive as $\beta \to \infty$ and is negative as $\beta \to -\infty$. Therefore there exists one and only one value of β_0 for which is it zero. □

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