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Some remarks on the geometry of austere manifolds

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Abstract. We prove several structure theorems about the special class of minimal submanifolds which Harvey and Lawson have called "austere" and which arose in connection with their foundational work on calibrations. The condition of austerity is a pontwise condition on the second fundamental form and essentially requires that the non-zero eigenvalues of the second fundamental form in any normal direction at any point occur in oppositely signed pairs. We solve the pointwise problem of describing the set of austere second fundamental forms in dimension at most four and the local problem of describing the austere three-folds in Euclidean space in all dimensions.

0. Introduction

This paper contains several new results on the geometry of a special class of minimal submanifolds of Euclidean space, the class of *austere* submanifolds, first introduced by Harvey and Lawson in their 1982 paper *Calibrated Geometries*. Austerity is formulated as an algebraic condition on the second fundamental form of a submanifold and essentially asserts that the eigenvalues of its second fundamental form, when measured in any normal direction, occur in oppositely signed pairs. This generalizes the well-known properties of the second fundamental forms of complex submanifolds of $\mathbb{C}^n$ (which are, of course, austere submanifolds of $\mathbb{E}^{2n}$).

In §1, we give the precise definition of austerity and give several examples to show that the class of austere manifolds is not trivial. Most of this section is drawn directly from the aforementioned work of Harvey and Lawson.

The algebraic conditions which express austerity are rather complex and it is

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not at all clear how one might classify all the possible austere second fundamental forms, even pointwise. In §2, we develop some algebra which allows us to classify all of the possible second fundamental forms for austere submanifolds of dimension at most four (Theorem 2.5). Along the way, we prove some algebraic results which are interesting in their own right regarding the common zero locus $\mathcal{A}_n$ of the $O(n)$-invariant polynomials of odd degree on the space $\mathcal{S}_n$ of symmetric n-by-n matrices. In particular, we construct several maximal linear subspaces of $\mathcal{A}_n$.

In §3, we classify the austere manifolds whose second fundamental forms belong to one of the types found in §2, namely the so-called *simple* austere second fundamental forms. Our result (Theorem 3.1) is that these austere submanifolds all belong to a family of "generalized helicoids". This result is, perhaps, the deepest result in the paper and is proved by a straightforward but rather involved analysis of the structure equations.

For submanifolds of dimensions one or two, it turns out that austerity is the same as minimality, so the first interesting case is that of austere three-folds. In §4, we classify all of the austere three-folds in $\mathbb{E}^{3+r}$. We find that, up to rigid motion, they are essentially of four kinds: (open subsets of) a linear $\mathbb{E}^3 \subset \mathbb{E}^{3+r}$, (open subsets of) generalized helicoids of dimension three in $\mathbb{E}^3$ (defined in §3), (open subsets of) orthogonal products of a line with minimal surfaces in $\mathbb{E}^{2+r}$, and the "twisted cones" (defined in §4) over minimal surfaces in $S^{2+r} \subset \mathbb{E}^{3+r}$. The later type is only classified locally, and we point out some of the difficulties of trying to extend this to a global classification.

1. Austere Submanifolds of Euclidean Space

Let n and r be positive integers and let $\mathbb{E}^{n+r}$ denote Euclidean $(n+r)$-space where the (positive definite) inner product is denoted by $\langle\ ,\ \rangle$. Let M^n be a smooth n-manifold and let $f\colon M^n \to \mathbb{E}^{n+r}$ an immersion. The *first fundamental form* (also called the *induced metric*) of f is given by $\mathrm{I} = \langle df, df\rangle$ and is a positive definite quadratic form on M. Let ∇ denote the associated Levi-Civita connection and let TM and NM denote the tangent and normal bundles respectively of M^n in $\mathbb{E}^{n+r}$. Then $M \times \mathbb{E}^{n+r} = TM \oplus NM$ and the differential df may be regarded as a $\mathbb{E}^{n+r}$-valued 1-form on M. The *second fundamental form* II of the immersion f is defined to be $\mathrm{II} = \nabla(df)$. It is well-known that II is a symmetric quadratic form on M with values in NM, i.e., $\mathrm{II} \in C^\infty(NM \otimes S^2(T^*M))$.

Now, there exists a canonical map

$$\Phi\colon C^\infty(S^2(T^*M)) \to C^\infty(S^2(\Lambda^n(T^*M)))$$

which is homogeneous of degree n and which is given in any local coframe by the formula

$$\Phi(h_{ij}\omega^i \circ \omega^j) = \det(h_{ij})(\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n)^2.$$

For example, $\Phi(\mathrm{I}) = (*1)^2$ where $*1 \in \Omega^n(M)$ is a unit volume form (well-defined up to a sign). If $\nu \in C^\infty(NM)$ is a normal vector field, then $\nu \cdot \mathrm{II} \in C^\infty(S^2(T * M))$. Hence, for each ν, the formula

$$\Phi(\mathrm{I} - t\nu \cdot \mathrm{II}) = \left(\sum_{k=0}^{n} (-t)^k \sigma_k(\nu)\right) \Phi(\mathrm{I})$$

defines a sequence $\sigma_k(\nu)$ of smooth function on M. Clearly, $\sigma_0(\nu) \equiv 1$ while $\sigma_1(\nu) = \mathrm{tr}_{\mathrm{I}}(\nu \cdot \mathrm{II})$. For $k > 1$ however, $\sigma_k(\nu)$ is a non-linear function of ν and is simply the k-th symmetric function of the eigenvalues of $\nu \cdot \mathrm{II}$ with respect to I.

The following definition is due to Harvey and Lawson and is to be found in their paper *Calibrated Geometries.*

Definition. An immersion (or submanifold) $f\colon M^n \to \mathbb{E}^{n+r}$ is said to be *austere* if, for every $\nu \in C^\infty(NM)$ and every integer k satisfying $0 \le k < n/2$, we have $\sigma_{2k+1}(\nu) = 0$.

Every austere immersion is clearly minimal. Moreover, in the case where n is equal to 1 or 2, the austere condition is simply $\sigma_1(\nu) = 0$ for all normal vector fields ν. Thus, for curves and surfaces, "austere" and "minimal" are equivalent.

For larger values of n, however, the austere condition is considerably stronger than minimality. In fact, it represents an overdetermined system of partial differential equations for the immersion f. The "generality" of the space of solutions of this system for given values of n and r is very poorly understood at this time. In fact, aside from the case $n = 3$ (treated in this paper) and a few scattered results mentioned below for general values of n, all that is known is a list of examples of austere submanifolds.

These examples include

- complex submanifolds (of any dimension) of $\mathbb{C}^m \simeq \mathbb{E}^{2m}$,
- minimal surfaces $M^2 \subset \mathbb{E}^{2+r}$,
- the cone $M^3 \subset \mathbb{E}^{3+r}$ on a minimal surface $\Sigma^2 \subset S^{2+r}$, and a few others constructed by Harvey and Lawson.

The orthogonal product of austere submanifolds is easily seen to be austere, so one can generate further examples (of rather high codimension) by taking orthogonal products of examples drawn from the above list.

We will close this section by briefly explaining the motivation behind the notion of austerity. For more details and proofs, the reader is referred to *Calibrated Geometries*.

Let $T^*\mathbb{E}^{n+r} \simeq \mathbb{E}^{2n+2r}$ denote the cotangent bundle of $\mathbb{E}^{n+r}$. Harvey and Lawson show that $T^*\mathbb{E}^{n+r}$ carries a canonical special Lagrangian calibration. If $f: M^n \to \mathbb{E}^{n+r}$ is any immersion, we define the *co-normal bundle* $N_f^* \subset M \times T^*\mathbb{E}^{n+r}$ by

$$N_f^* = \left\{(m,\xi) \in M \times T^*E^{n+r} | \xi \in T^*_{f(m)}\mathbb{E}^{n+r} \text{ and } \xi(f_*(T_mM)) = 0\right\}.$$

Then N_f^* is a smooth bundle of rank r over M. The projection on the second factor $\pi_2: N_f^* \to T^*\mathbb{E}^{n+r}$ is an immersion of N_f^* as a Lagrangian submanifold of $T^*\mathbb{E}^{n+r}$. This immersion is special Lagrangian if and only if the immersion f is austere. Thus, austere immersions allow us to construct special Lagrangian submanifolds of $T^*\mathbb{E}^{n+r}$. Special Lagrangian submanifolds are of interest because they are absolutely area minimizing (rather than just minimal).

2. Some Austere Algebra

As a first step in classifying the austere submanifolds, we will focus on the algebra problem of describing the possible second fundamental forms of austere submanifolds. For any submanifold $f: M^n \to \mathbb{E}^{n+r}$ and any $x \in M$, we let $|\,\mathrm{II}\,|_x \subset S^2(T_x^*M)$ denote the linear subspace spanned by the quadratic form $\nu \cdot \mathrm{II}$ as ν ranges over a basis of N_xM. According to the definition, the immersion is austere if, for any $x \in M$ and any $q \in |\,\mathrm{II}\,|_x$, the odd symmetric functions of the eigenvalues of q with respect to the quadratic form I_x are all zero. This motivates the following definition:

Definition. Let V be a real vector space of dimension n endowed with a positive definite inner product $\langle,\rangle$. A linear subspace $\mathcal{Q} \subset S^2(V^*)$ of the quadratic functions on V is said to be *austere* if the odd symmetric functions of the eigenvalues of any element of $\mathcal{Q}$ with respect to $\langle,\rangle$ are all zero.

This raises the following algebra problem: Classify the austere subspaces of $S^2(V^*)$. Since any isometry of V clearly carries an austere subspace of $S^2(V^*)$ to another austere subspace, it suffices to do this classification up to isometries of V. Moreover, since any subspace of an austere subspace is clearly austere, in order to find all of the austere subspaces it suffices to classify the *maximal* austere subspaces. Unfortunately, this classification problem is, as yet, unsolved in dimensions $n > 4$. We will present the solutions for $n = 2$, 3, or 4 below.

By choosing an orthonormal basis of $V \simeq \mathbb{E}^*$, we may regard the quadratic forms on V as the vector space $\mathcal{S}_n$ of symmetric n-by-n matrices. The action of the isometries of V on $S^2(V^*)$ then translates into the action of $\mathrm{O}(n)$ on $\mathcal{S}_n$ given by the formula $g \cdot m = gm^tg$ for $g \in \mathrm{O}(n)$ and $m \in \mathcal{S}_n$. Our problem is then seen to be the problem of classifying the (maximal) linear subspaces $\mathcal{Q} \subset \mathcal{S}_n$ on which the all of the functions σ_{2k+1} vanish identically. It is useful to note that the condition for a subspace $\mathcal{Q}$ to be austere is equivalent to the condition that all of the functions τ_{2k+1} vanish identically on $\mathcal{Q}$ where $\tau_j(a)$ is defined for $a \in \mathcal{S}_n$ by the formula $\tau_j(a) = \mathrm{tr}(a^j)$.

Example 1. Let $g \in \mathrm{O}(n)$ be any orthogonal matrix and let

$$\mathcal{Q}_g = \{m \in \mathcal{S}_n | gm + mg = 0\}.$$

Then $\mathcal{Q}_g$ is an austere subspace of $\mathcal{S}_n$. To see this, first note that, for $m \in \mathcal{Q}_g$, we have $g \cdot m = -m$. Since the function σ_l is $\mathrm{O}(n)$-invariant and homogeneous of degree l, we then have

$$\sigma_{2k+1}(m) = \sigma_{2k+1}(-m) = -\sigma_{2k+1}(m),$$

so $\sigma_{2k+1}(m) = 0$.

This allows us to construct some maximal austere subspaces of $\mathcal{S}_n$ explicitly.

Proposition 2.1. *Suppose that $n = 2p$ and that $J \in \mathrm{O}(n)$ satisfies $J^2 = -\mathrm{I}_n$. Then the space $\mathcal{Q}_J$ is a maximal austere subspace of $\mathcal{S}_n$ of dimension $p(p+1)$.*

Proof. We only need to show that $\mathcal{Q}_J$ is maximal. Conjugating by the appropriate matrix, we may assume that

$$J = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}.$$

It is then easy to see that $\mathcal{Q}_J$ consists of the matrices of the form

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & -m_1 \end{bmatrix}.$$

where m_1 and m_2 are p-by-p symmetric matrices.

Now suppose that P were a symmetric n-by-n matrix with the property that P and $\mathcal{Q}_J$ together spanned an austere subspace of S_n. We will show that P must already belong to $\mathcal{Q}_J$, thereby establishing our claim.

By subtracting an element of $\mathcal{Q}_J$ from P, we may assume that P is of the form

$$P = \begin{bmatrix} p_1 & p_2 \\ {}^tp_2 & p_1 \end{bmatrix},$$

where p_1 is a p-by-p symmetric matrix and p_2 is a p-by-p skew-symmetric matrix. The vector space of such matrices is, in fact, the orthogonal complement $\mathcal{Q}_J^{\perp}$ of $\mathcal{Q}_J$ under the natural $\mathrm{O}(n)$-invariant inner product on S_n given by the formula $\langle a,b\rangle = \mathrm{tr}(ab)$. Now the stabilizer of J in $\mathrm{O}(n) = \mathrm{O}(2p)$ is a group isomorphic to $\mathrm{U}(p)$ and we will use $\mathrm{U}(p)$ to normalize P. In fact, the $\mathrm{U}(p)$-representation $\mathcal{Q}_J^{\perp}$ is isomorphic to the representation of $\mathrm{U}(p)$ on Hermitian symmetric p-by-p matrices, so it follows that we may diagonalize P by such an action, i.e., we may assume that $p_2 = 0$ and that p_1 is diagonal, with its i'th diagonal entry equal to λ_i.

Now let m be a p-by-p diagonal matrix with its i'th diagonal entry equal to μ_i, and let $M \in \mathcal{Q}_J$ be given by the formula above with $m_1 = 0$ and $m_2 = m$. Then the eigenvalues of $P + M$ constitute the set $\Lambda = \{\lambda_i \pm \mu_i | 1 \le i \le p\}$. If any of the λ_i were non-zero, this latter set would not be symmetric about $0 \in \mathbb{R}$ for all choices of μ_i and hence it would be possible to choose m so that $P + M$ is not austere, contradicting our assumption that the space spanned by P and $\mathcal{Q}_J$ is austere. Thus, all of the λ_i must be zero, so $P = 0$, as we wished to show. □

The maximal austere subspace constructed in Proposition 2.1 is clearly associated to the "complex structure" J which we constructed on $\mathbb{R}^{2p} = \mathbb{C}^p$. It is easy to see that at each point of a complex p-dimensional submanifold of $\mathbb{C}^m$, the linear span of the quadratic forms in the second fundamental form yields an

austere subspace of the space of quadratic forms on the tangent space at that point which is $\mathrm{O}(2p)$-conjugate to a subspace of the space $\mathcal{Q}_J$.

By methods similar to those used in the proof of Proposition 2.1, we can prove the following analogous results for the case of odd n. We omit the proof.

Proposition 2.2. *Suppose that $n = 2p+1$. Then the space $\mathcal{Q} \subset S_n$ of matrices of the form*

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1 & m_1 \\ 0 & m_2 & -m_1 \end{bmatrix},$$

where m_1 and m_2 are p-by-p symmetric matrices is a maximal austere subspace of S_n whose dimension is $p(p+1)$. □

Clearly, this maximal austere subspace is closely related to the "complex" austere subspace given by the previous proposition.

Another family of maximal austere subspaces is furnished by the following result.

Proposition 2.3. *Suppose that $R \in \mathrm{O}(n)$ satisfies $R^2 = \mathrm{I}_n$, but $R \neq \pm \mathrm{I}_n$. Then the space $\mathcal{Q}_R$ is a maximal austere subspace of S_n except in the case where $\mathrm{tr}(R) = 0$, in which case the space $\mathcal{Q}_R^{+}$ which is spanned by R and $\mathcal{Q}_R$ is a maximal austere subspace of S_n.*

Proof. After conjugation by an orthogonal matrix, we may assume that R has the form

$$R = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix},$$

where p and q are positive integers satisfying $p + q = n$. It follows immediately that $\mathcal{Q}_R$ is the space of matrices of the form

$$A = \begin{bmatrix} 0 & a \\ {}^ta & 0 \end{bmatrix},$$

where a is an arbitrary p-by-q matrix.

Now suppose that M is any n-by-n symmetric matrix so that M and $\mathcal{Q}_R$ span an austere subspace of S_n. By subtracting an appropriate element of $\mathcal{Q}_R$ from M, we may assume that M takes the form

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

where m_1 and m_2 are symmetric matrices of dimensions p-by-p and q-by-q respectively. Using the fact that the stabilizer of R in $O(n)$ is clearly $O(p)\times O(q)$, we see that we may assume that m_1 and m_2 are diagonal. Let $\lambda_1, \dots, \lambda_p$ and $\mu_1, \dots, \mu_q$ denote the diagonal entries of m_1 and m_2 respectively. Now, for any real number s, and any integers i and j with $1 \le i \le p$ and $1 \le j \le q$, let a be the p-by-q matrix with an s in the i'th row and j'th column and zeros elsewhere and let $A \in \mathcal{Q}_R$ be the corresponding matrix given by the above formula. Now it is easy to compute that

$$\mathrm{tr}((M+A)^3) = \mathrm{tr}((M)^3) + 3s^2(\lambda_i + \mu_j).$$

Thus, in order that $M + A$ be austere for all choices of s, i and j, we must have $\lambda_i + \mu_j = 0$ for all i and j. Of course, this implies that M is a multiple of R. If $p \neq q$, then $\mathrm{tr}(R) \neq 0$, so M must be the zero multiple. Thus, $M = 0$ and we have shown that $\mathcal{Q}_R$ is maximal. On the other hand, if $p = q$, then it is easily seen that any matrix of the form $\lambda R + Q$ where $Q \in \mathcal{Q}_R$ is austere. Moreover, our analysis has shown that the space $\mathcal{Q}_R^+$ spanned by R and $\mathcal{Q}_R$ is indeed maximal. □

There is an analogue of Proposition 2.2 in this case. Although it is clear that $\mathcal{Q}_R$ is never maximal austere in S_{n+1} under the obvious inclusion of $S_n \subset S_{n+1}$, it turns out that, when $\mathrm{tr}(R) = 0$, the space $\mathcal{Q}_R^+$ remains maximal austere in S_{n+1}. The proof is similar to the ones we have already done, so we omit it.

Proposition 2.4. *Suppose that $n = 2p+1$. Then the space $\mathcal{Q} \subset S_n$ of matrices of the form*

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda \mathrm{I}_p & a \\ 0 & {}^t a & -\lambda \mathrm{I}_p \end{bmatrix},$$

where a is any p-by-p matrix and λ is any real number is a maximal austere subspace of S_{2p+1} whose dimension is $p^2 + 1$. □

We can now give a classification result for maximal austere subspaces of S_n for $n \le 4$.

Theorem 2.5. *For $n = 2$ or $n = 3$, any maximal austere subspace of S_n is conjugate under $O(n)$ to one of the subspaces constructed in Propositions 2.1, 2.2, 2.3, or 2.4. For $n = 4$, any maximal austere subspace of S_4 is conjugate under $O(4)$ to either the "complex" subspace of dimension 6 (described by*

Proposition 2.1), the space $\mathcal{Q}_R^+$ of dimension 5 associated to reflection R in a 2-plane (described by Proposition 2.3), or else precisely one of the spaces $\mathcal{Q}_\lambda$ of dimension 3 given by the formula

$$\mathcal{Q}_\lambda = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & \lambda_3 x_3 & \lambda_2 x_2 \\ x_2 & \lambda_3 x_3 & 0 & \lambda_1 x_1 \\ x_3 & \lambda_2 x_2 & \lambda_1 x_1 & 0 \end{pmatrix} \;\middle|\; x_i \in \mathbb{R} \right\}$$

where the constants λ_i satisfy the inequalities $\lambda_1 \ge \lambda_2 \ge 0 \ge \lambda_3$ and the relation

$$\lambda_1\lambda_2\lambda_3 + \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Proof. For the cases $n = 2$ or $n = 3$, the theorem is straightforward algebra and is left to the reader. The case $n = 4$ is difficult to do directly, but the following approach makes it manageable.

First, let $S_4' \subset S_4$ denote the nine-dimensional subspace which consists of the *trace-free* symmetric 4-by-4 matrices. This is well-known to be an irreducible representation of $O(4)$. Moreover, the element $-\mathrm{I}_4 \in SO(4) \subset O(4)$ acts trivially on S_4'. Now, the quotient group $SO(4)/\{\pm \mathrm{I}_4\}$ is isomorphic to $SO(3) \times SO(3)$. It follows that S_4' is, in particular, an irreducible representation of $SO(3)\times SO(3)$. Hence, it must be the tensor product of an irreducible representation of the first copy of $SO(3)$ with an irreducible representation of the second copy of $SO(3)$. Since conjugation by an element of $O(4) \setminus SO(4)$ exchanges the two copies of $SO(3)$, it follows that the two representations are isomorphic and hence must each be of dimension three. Thus, there must be an isomorphism of S_4' with the space of 3-by-3 matrices $\mathcal{M}_{3,3}$ which identifies the given action of $SO(3) \times SO(3)$ on S_4' with the action of $SO(3) \times SO(3)$ on $\mathcal{M}_{3,3}$ which, for $(g_1, g_2) \in SO(3) \times SO(3)$ acts on $m \in \mathcal{M}_{3,3}$ as $(g_1, g_2) \cdot m = g_1 m {}^t g_2$. This latter action preserves the cubic form $\delta(m) = \det(m)$ on $\mathcal{M}_{3,3}$. Since the action of $SO(4)$ on S_4' preserves only one non-trivial cubic form, namely σ_3, it follows that these two cubic forms must correspond under the isomorphism $S_4' \simeq \mathcal{M}_{3,3}$.

The problem of classifying the austere subspaces of S_4 under the action of $SO(4)$ is now seen to be equivalent to classifying the linear subspaces of singular matrices in $\mathcal{M}_{3,3}$ under the action of $SO(3) \times SO(3)$. Note also that the action on S_4' of an element of $O(4) \setminus SO(4)$ translates over into the action on $\mathcal{M}_{3,3}$ of a transposition followed by an element of $SO(3) \times SO(3)$.

Now, $\delta = \det$ on $\mathcal{M}_{3,3}$ is invariant under the action of the larger group G generated by $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ and transposition, so we may use G to reduce the singular subspaces of $\mathcal{M}_{3,3}$ to a normal form. After a little algebra, it turns out that every maximal singular subspace of $\mathcal{M}_{3,3}$ is conjugate under the action of G to one of the following three subspaces S_k (of dimension k) in $\mathcal{M}_{3,3}$:

- S_3, the space of skew-symmetric matrices,
- S_5, the subspace of matrices with zeros in the upper left-hand 2-by-2 block, and
- S_6, the subspace of matrices which have zeros in the last column.

The space S_5 is characterized geometrically as the space of linear maps from one copy of $\mathbb{R}^3$ to another which carry a fixed 2-plane in the first copy into a fixed line in the second copy. Since the group $SO(3)$ acts transitively on the space of linear subspaces of a fixed dimension in $\mathbb{R}^3$, it follows that the group $SO(3) \times SO(3)$ acts transitively on the G-orbit of S_5 and hence there is only one $SO(3) \times SO(3)$-conjugacy class of five dimensional maximal singular subspaces in $\mathcal{M}_{3,3}$. Of course, this implies that there is only one $O(4)$ conjugacy class of five dimensional maximal austere subspaces of S_4. Since we have already found one, namely $\mathcal{Q}_R^+$, in Proposition 2.3, we have found them all.

Similarly, S_6 is easily characterized geometrically as the space of linear maps from one copy of $\mathbb{R}^3$ to another which contain a fixed line in their kernels. Again, it follows that there is only one $O(4)$ conjugacy class of six dimensional maximal austere subspaces of S_4. Since we have already found one, namely $\mathcal{Q}_J$, in Proposition 2.1, we have found them all.

The case of three dimensional maximal austere subspaces is, perhaps, the most interesting. Since, for $h \in SL(3,\mathbb{R})$, we have $S_3 = hS_3{}^t h$, it follows that the diagonal subgroup $\Delta \subset SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ is a subgroup of the stabilizer of S_3. Moreover, S_3 is clearly invariant under the transposition mapping. It follows that the spaces $S_3(h) = hS_3$ for $h \in SL(3,\mathbb{R})$ run through the entire set of three dimensional maximal singular subspaces of $\mathcal{M}_{3,3}$. We must now determine the orbits of $SO(3) \times SO(3)$ in this space. Since, for $(g_1, g_2) \in SO(3) \times SO(3)$ and $h \in SL(3,\mathbb{R})$, we have

$$g_1(hS_3)^t g_2 = g_1 h^t g_2 (g_2 S_3{}^t g_2) = g_1 h^t g_2 S_3,$$

it follows that the space of $SO(3)\times SO(3)$-conjugacy classes of three dimensional maximal singular subspaces of $\mathcal{M}_{3,3}$ is parameterized by the space of orbits of $SO(3) \times SO(3)$ acting on $SL(3,\mathbb{R})$ via pre- and post-multiplication. As is well-known, this space of orbits is parameterized by the space $\mathcal{D}_3^+$ of diagonal 3-by-3 matrices with all of the diagonal entries positive and in (say) decreasing order. However, for $\delta \in \mathcal{D}_3^+$, the subspace δS_3 is clearly G-equivalent to $\delta^{-1}S_3$. Taking this into account and writing out explicitly the isomorphism of representations $S_4' \simeq \mathcal{M}_{3,3}$, we arrive at the normal form stated in the theorem. We leave further details to the interested reader. □

Note that the space $\mathcal{Q}_R$ where R is reflection in a hyperplane in $\mathbb{R}^4$ is one of the $\mathcal{Q}_\lambda$, namely $\lambda = 0$. That this conjugacy class is not an isolated point in the moduli space is something of a surprise.

For $n > 4$ the problem of classifying the maximal austere subspaces of S_n seems to be less tractable and deserves further investigation.

3. Simple Austere Submanifolds

In this section, we will classify the austere submanifolds associated to one of the types of austere subspaces found in the last section, namely the subspace $\mathcal{Q}_R$ where R is reflection in a hyperplane. This subspace can be described in terms of quadratic forms on the inner product space $V \simeq \mathbb{E}^n$ as follows: Let $x^1, x^2, \ldots, x^n$ be any orthonormal set of linear coordinates on V. Then the space $\mathcal{Q} \subset S^2(V^*)$ spanned by the quadratic forms $\{x^i x^n | 1 \le i < n\}$ is clearly austere and corresponds to the space $\mathcal{Q}_R$ where R is reflection in a hyperplane. Note that all of the elements of $\mathcal{Q}$ have a common linear factor.

We shall say that an austere subspace $S \subset S^2(V^*)$ is *simple* if it has dimension at least two and lies inside $\mathcal{Q} \subset S^2(V^*)$ for some choice of orthonormal linear coordinates $x^1, x^2, \ldots, x^n$. Analogously, we shall say that a (connected, smooth) austere submanifold $M^n \subset \mathbb{E}^{n+r}$ is *simple* if, on some open subset $U \subset M$, the subspace $|\mathrm{II}|_x \subset S^2(T^*M)$ is simple for all $x \in U$. Of course, since an austere submanifold is minimal and hence real analytic, the simplicity hypothesis implies that $|\mathrm{II}|_x \subset S^2(T^*M)$ is a simple subspace for all x in M outside of a closed set with no interior, namely, the set on which the dimension of $|\mathrm{II}|_x$ drops below 2.

We have the following classification theorem for simple austere submanifolds:

Theorem 3.1. *Suppose that $M^n \subset \mathbb{E}^{n+r}$ is a connected, simple, austere manifold. Then there exists an integer s with $2 \le s < n$ and constants $\lambda_0 \ge 0$ and $\lambda_1 \ge \cdots \ge \lambda_s > 0$ so that M^n is congruent to an open subset of the austere n-manifold $M(s,\lambda) \subset \mathbb{E}^{n+s}$ given by the parameterization*

$$M(s,\lambda) = \left\{ \begin{pmatrix} \lambda_0 x^0 \\ x^1\cos(\lambda_1 x^0) \\ x^1\sin(\lambda_1 x^0) \\ \vdots \\ x^s\cos(\lambda_s x^0) \\ x^s\sin(\lambda_s x^0) \\ x^{s+1} \\ \vdots \\ x^{n-1} \end{pmatrix} \;\middle|\; x^0, x^1, \ldots, x^{n-1} \in \mathbb{R} \right\}.$$

Moreover, the two n-manifolds $M(s,\lambda)$ and $M(s',\lambda')$ are congruent if and only if $s = s'$ and $[\lambda_0\colon\lambda_1\colon\cdots\colon\lambda_s] = [\lambda'_0\colon\lambda'_1\colon\cdots\colon\lambda'_s]$.

The manifolds $M(s,\lambda)$ may be thought of as "generalized helicoids". Note that $M(s,\lambda)$ is smooth if $\lambda_0 > 0$ but is singular along the locus $x^1 = x^2 = \cdots = x^s = 0$ (and is a cone to boot) if $\lambda_0 = 0$. Note also that, in order that $|\,\mathrm{II}\,|_x$ have dimension at least 2 generically on $M(s,\lambda)$, we must have either $\lambda_0 > 0$ or else $s > 2$.

Proof. We will use the method of the moving frame, with which we assume familiarity. We restrict our attention to an open set $U \subset M$ on which the rank of $|\,\mathrm{II}\,|_x$ is maximal and use the index ranges $1 \le i,j < n$ and $n < a,b \le n+r$. Since M is simple by assumption, it is possible to choose an orthonormal frame field $e_1,\ldots,e_n,\ldots,e_{n+r}$ locally on U so that $e_1,\ldots,e_n$ are tangent to M and so that, for the dual coframe field, ω_n is a linear divisor of the quadratic forms in $|\,\mathrm{II}\,|$. This implies that the second fundamental form takes the form (summing on repeated indices)

$$\mathrm{II} = 2h^a_i\omega^i \circ \omega^n \otimes e_a$$

for some functions h^a_i. We will regard the matrix $h = (h^a_i)$ as an r-by-$(n-1)$ matrix of functions whose rank, by hypothesis, is strictly greater than one.

We introduce the following matrix notation:

- ω represents the column of height $(n-1)$ of 1-forms (ω_i).
- θ represents the column of height r of 1-form (ω_a). In our framing, $\theta \equiv 0$.
- ρ represents the $(n-1)$–by–$(n-1)$ skew-symmetric matrix (ω_{ij}).
- κ represents the r-by-r skew-symmetric matrix (ω_{ab}).
- ϕ represents the row of length $(n-1)$ of 1-forms (ω_{ni}).
- η represents the r-by-$(n-1)$ matrix (ω_{aj}). In our framing, $\eta = h\omega_n$.
- η_n represents the column of height r of 1-forms (ω_{an}). In our framing, $\eta_n = h\omega$.

The structure equations may then be written in block form as

$$d\begin{bmatrix} \omega \\ \omega_n \\ \theta \end{bmatrix} = -\begin{bmatrix} \rho & -{}^t\phi & -{}^t\eta \\ \phi & 0 & -{}^t\eta_n \\ \eta & \eta_n & \kappa \end{bmatrix} \wedge \begin{bmatrix} \omega \\ \omega_n \\ \theta \end{bmatrix}$$

and

$$d\begin{bmatrix} \rho & -{}^t\phi & -{}^t\eta \\ \phi & 0 & -{}^t\eta_n \\ \eta & \eta_n & \kappa \end{bmatrix} = -\begin{bmatrix} \rho & -{}^t\phi & -{}^t\eta \\ \phi & 0 & -{}^t\eta_n \\ \eta & \eta_n & \kappa \end{bmatrix} \wedge \begin{bmatrix} \rho & -{}^t\phi & -{}^t\eta \\ \phi & 0 & -{}^t\eta_n \\ \eta & \eta_n & \kappa \end{bmatrix}.$$

Using these structure equations, we may take the exterior derivatives of the equations $\eta - h\omega_n = 0$ and $\eta_n - h\omega = 0$ and collect terms to give the equations

$$0 = h(\phi\wedge\omega) - (h\omega)\wedge\phi - (dh + \kappa h - h\rho)\wedge\omega_n$$
$$0 = -(dh + \kappa h - h\rho)\wedge\omega - 2(h\,{}^t\phi)\wedge\omega_n.$$

In particular, note that we have

$$h(\phi\wedge\omega) - (h\omega)\wedge\phi \equiv 0 \bmod \omega_n.$$

Since the rank of h is strictly greater than 1, and since all of the ω_i are linearly independent modulo ω_n, it easily follows that $\phi \equiv 0 \bmod \omega_n$. Thus, there exists a row $A = (A_i)$ of functions so that $\phi = A\omega_n$. Substituting this relation into our 2-form identities, we see that, if we set

$$\beta = dh + \kappa h - h\rho + h(A\omega) + (h\omega)A,$$

then our 2-form identities reduce to $\beta\wedge\omega = \beta\wedge\omega_n = 0$. Of course, this implies that we must have $\beta \equiv 0$, or, equivalently,

$$dh = -\kappa h + h\rho - h(A\omega) - (h\omega)A.$$

Now set

$$\alpha = dA - A\rho - {}^t\omega^t hh + A(A\omega).$$

Differentiating the relation $\phi = A\omega_n$, we get the formula $\alpha \wedge \omega_n = 0$. Thus, $\alpha = C\omega_n$ for some row function C. Moreover, if we now take the exterior derivative of our formula for dh and use the structure equations and the formulas derived so far, we get the formula

$$hC\omega \wedge \omega_n + h\omega C\omega_n = 0.$$

Using the linear independence of the ω_i and ω_n and the assumption that h has rank greater than 1, we conclude that $C \equiv 0$. In other words,

$$dA = A\rho + {}^t\omega^t hh - A(A\omega).$$

We now have formulas for the exterior derivatives of both of the functions h and A in terms of these quantities and the tautological forms. Moreover, computing the exterior derivative of our formula for dA only yields an identity, so there are no more identities to be found by exterior differentiation. We now proceed to integrate these equations.

First, note that our structure equations now imply that

$$d\rho + \rho \wedge \rho = {}^t\phi \wedge \phi + {}^t\eta \wedge \eta = 0.$$

By passing to a simply connected cover of U if necessary, we see that there must exist a smooth mapping $R: U \to \mathrm{SO}(n-1)$, unique up to left multiplication by a constant matrix in $\mathrm{SO}(n-1)$, so that $\rho = R^{-1}dR$. Rotating the frame $e = (e_i)$ by this matrix R then yields a new framing which still satisfies all our hypotheses, but which has $\rho = 0$. Thus, we shall assume that $\rho = 0$ from now on. Appealing to the structure equations again, we see that $d\omega = -\rho \wedge \omega + {}^t\phi \wedge \omega_n = 0$, so it follows that there exists a row of functions $x = (x_i)$ (unique up to an additive vector constant) on U so that $\omega = dx$.

We now compute that $d(A\omega) = 0$, so it follows that there exists a positive function q on U (unique up to a positive constant multiplicative factor) so that $A\omega = dq/(2q)$. (This form of the first integral helps in the calculations below.) Thus, $dq = (2qA)\omega = (2qA)dx$.

Set $B = {}^t hh$. Then B is a positive semidefinite, symmetric $(n-1)$-by-$(n-1)$ matrix of rank at least 2 at every point of U. We compute, as a consequence of our identities, that

$$d(2qA) = {}^t\omega(2q(B + {}^tAA)) = {}^t(dx)(2q(B + {}^tAA))$$

and finally that

$$d(2q(B + {}^tAA)) = 0.$$

Thus, $q(B + {}^tAA)$ is a constant positive semidefinite, symmetric $(n-1)$-by-$(n-1)$ matrix of rank at least 2. Let s be the rank of this matrix and let $\lambda_1^2 \geq \cdots \geq \lambda_s^2 > 0$ be its positive eigenvalues arranged in descending order (the constants λ_i are chosen to be positive). By rotating the tangent frame e by a constant orthogonal matrix, we may diagonalize $q(B + {}^tAA)$ in such a way that its eigenvalues are arranged in decreasing order down the diagonal.

By construction, $2q(B + {}^tAA)$ is the Hessian matrix of q in the "coordinate system" x so it follows that, after translating the coordinate system x if necessary, we may assume that q is expressed in the form

$$q = q_0 + (\lambda_1 x_1)^2 + (\lambda_2 x_2)^2 + \cdots + (\lambda_s x_s)^2 + 2\mu_{s+1}x_{s+1} + \cdots + 2\mu_{n-1}x_{n-1},$$

where q_0 and the μ_i are constants.

It then follows that the row of functions qA is given by

$$qA = [\lambda_1^2 x_1, \lambda_2^2 x_2, \ldots, \lambda_s^2 x_s, \mu_{s+1}, \ldots, \mu_{n-1}].$$

However, we now see that the constants μ_i for $i > s$ must all be zero since, for $i > s$, the i'th diagonal entry of the positive semidefinite symmetric matrix $q^2 B$ is seen to be $-\mu_i^2$. Thus, the formula for q simplifies to

$$q = q_0 + (\lambda_1 x_1)^2 + (\lambda_2 x_2)^2 + \cdots + (\lambda_s x_s)^2.$$

We will now show that $q_0 \geq 0$. For any vectors $y, z \in \mathbb{R}^{n-1}$, define the positive semidefinite symmetric bilinear form

$$Q(y,z) = \lambda_1^2 y_1 z_1 + \lambda_2^2 y_2 z_2 + \cdots + \lambda_s^2 y_s z_s.$$

Thus, $q = q_0 + Q(x,x)$. It is then easy to compute that for any vector $y \in \mathbb{R}^{n-1}$,

$$q^2\, {}^t yBy = q_0 Q(y,y) + Q(x,x)Q(y,y) - (Q(x,y))^2.$$

It now follows from the triangle inequality that, if q_0 were negative, then B would not be positive semi-definite. Thus, $q_0 \geq 0$. Set $q_0 = \lambda_0^2$ where $\lambda_0 \geq 0$.

Note that the rank of B (and hence of h) is equal to s if $q_0 > 0$ but is equal to $s-1$ if $q_0 = 0$.

Next, we note that the structure equations imply that $d(q^{-1/2}\omega_n) = 0$ so it follows that there exists a function x_n on U so that $\omega_n = a^{1/2}dx_n$ and that this function is uniquely defined up to an additive constant.

The functions, $(x_1, \ldots, x_{n-1}, x_n)$ clearly have linearly independent differentials on U so it follows that U is covered by open sets V for which the mapping $(x_1, \ldots, x_n)\colon V \to \mathbb{R}^n$ is a diffeomorphism onto an open coordinate rectangle in $\mathbb{R}^n$. To avoid minor technical difficulties, we restrict our attention to such a V for the rest of the argument.

The structure equations for the original immersion $f\colon M \to \mathbb{E}^{n+r}$ now read

$$\begin{aligned} df &= e_1\omega_1 + \cdots + e_n\omega_n \\ &= e_1 dx_1 + \cdots + e_{n-1}dx_{n-1} + e_n q^{1/2}dx_n \end{aligned}$$

Thus,

$$e_i = \partial f/\partial x_i \quad \text{and} \quad e_n = q^{-1/2}\partial f/\partial x_n.$$

Moreover, we also have

$$\begin{aligned} de_i &= e_j\omega_{ij} + e_n\omega_{nj} + e_a\omega_{aj} \\ &= (A_i e_n + h_i^a e_a)q^{1/2}dx_n \end{aligned}$$

(this uses the fact that $\rho = (\omega_{ij}) = 0$). In particular, it follows that e_i is a function of x_n alone. Thus, f is a linear function of the x_i so we may write f in the form

$$f = x_1e_1 + x_2e_2 + \cdots + x_{n-1}e_{n-1} + g$$

where g is a function of x_n alone.

For any function ϕ of x_n alone, we will denote its derivative with respect to x_n by a prime ϕ'. Thus, for example, $d(e_i) = e_i' dx_n$. It follows that

$$q^{1/2}e_n = x_1e_1' + \cdots + x_{n-1}e_{n-1}' + g'.$$

On the other hand, the structure equations imply that

$$\frac{\partial(q^{1/2}e_n)}{\partial x_n} = -(\lambda_1^2 x_1 e_1 + \cdots + \lambda_s^2 x_s e_s).$$

Comparing this with the derivative with respect to x_n of the previous equation yields that

$$e_i'' = -\lambda_i^2 e_i$$

for all $1 \le i \le s$ and $e_i'' = 0$ for all $s < i < n$. Moreover, we must have $g'' = 0$. Of course, these equations imply that, for $1 \le i \le s$, there is a fixed 2-plane $E_i \subset \mathbb{E}^{n+r}$ which contains e_i while, for $s < i < n$, the unit vector e_i must be constant. Moreover, the fact that

$$\langle e_i, e_j' \rangle dx_n = \langle e_i, de_j \rangle = \omega_{ij} = 0$$

for all i and j implies that any two of the planes E_j are mutually orthogonal. Of course, the equation $g'' = 0$ implies that g is a linear function of x_n. From this, it easily follows that g' is perpendicular to each of the e_i and has length λ_0.

Combining all of this information, we see that, after a rigid motion in $\mathbb{E}^{n+r}$, we can identify $f(V)$ with an open subset of $M(s,\lambda)$. By the connectedness of M and the real analyticity of minimal submanifolds, it follows that $f(M)$ itself must lie entirely in this $M(s,\lambda)$. □

4. Austere Three-Folds

In this section, we give a complete "pseudo-local" classification of the three dimensional austere submanifolds of $\mathbb{E}^{3+r}$. In rough outline, this classification can be described as follows: By Theorem 2.5, we know that, up to orthogonal equivalence, there are only four isomorphism classes of austere subspaces of S_3. Each of these isomorphism classes corresponds to a type of austere three-fold. For two of these four types, we determine the entire set of the corresponding austere three-folds. For the remaining two types, we show how locally to describe each such austere three-fold on a dense open set in terms of a minimal surface Σ in either $\mathbb{E}^{2+r}$ or S^{2+r} together with the additional data of a solution of a certain determined, linear second-order equation (to be described explicitly below) determined by the minimal surface Σ. The upshot of this "classification" is that the austere three-folds (originally defined in terms of an overdetermined system of partial differential equations) are described in terms of well-understood submanifolds in minimal surface theory.

We begin by explicitly recalling the four possible types of second fundamental form $|\,\mathrm{II}\,|_x$ available to austere three-folds. Let x^1, x^2 and x^3 denote an orthonormal coordinate system on $V \simeq \mathbb{E}^3$. It then follows from Theorem 2.5 that every austere subspace of $S^2(V^*) \simeq S_3$ is conjugate under $\mathrm{O}(3)$ to one of the following subspaces

1. $\mathcal{A}_0 = (0)$,
2. $\mathcal{A}_1 = ((x^1)^2 - (x^2)^2)$,
3. $\mathcal{A}_2 = ((x^1)^2 - (x^2)^2, 2x^1x^2)$,
4. $\mathcal{A}_2' = (x^1x^2, x^1x^3)$.

Now, if $M^3 \subset \mathbb{E}^{3+r}$ is any connected smooth austere submanifold, then M can be divided into four subsets according to the $O(3)$-conjugacy class of $|\,\mathrm{II}\,|$ at the various points of M. Since M is real analytic (as are all minimal submanifolds of Euclidean space), it easily follows that only one of these subsets can have a non-empty interior and that one is then open and dense in M. We may thus classify M as belonging to one of four types according to which of the four subsets has non-empty interior.

The first type, where $|\,\mathrm{II}\,|_x = 0$ for all x in a dense open set in M, clearly consists only of (open subsets of) affine subspaces $V \subset \mathbb{E}^{3+r}$ and merits no further comment.

Similarly, the fourth type is just the class of simple austere three-folds as defined in §3 and hence a connected smooth austere three-fold of this type is congruent to (an open subset of) a generalized helicoid $M(2,\lambda) \subset \mathbb{E}^5$ with $\lambda_0 > 0$ as described in Theorem 3.1.

It remains to describe the austere three-folds of the second and third types. These two types have an alternate characterization as the austere three-folds with degenerate Gauss mapping. Recall that the Gauss mapping $\gamma\colon M \to \mathrm{Gr}_3(\mathbb{E}^{3+r})$ is defined by the rule $\gamma(x) = T_xM$ for all $x \in M$. It is easily seen that the rank of the Jacobian mapping $\gamma'(x)$ is less than three if and only if $|\,\mathrm{II}\,|_x$ is equivalent to one of the subspaces $\mathcal{A}_0, \mathcal{A}_1$, or $\mathcal{A}_2$. Thus, the austere three-folds of the second and third types are simply the non-planar austere three-folds with degenerate Gauss map.

One class of examples of such austere three-folds consists of those of the form $M = \mathbb{R} \times \Sigma^2 \subset \mathbb{E}^{3+r}$ where $\Sigma^2 \subset \mathbb{E}^{2+r}$ is a (non-planar) minimal surface. A second class of examples of such austere three-folds consists of the "cones" $M^3 = \mathbb{R}^+ \times \Sigma^2 \subset \mathbb{E}^{3+r}$ where $\Sigma^2 \subset S^{2+r}$ is a minimal surface which is not totally geodesic. However, this latter type can be generalized to yield the following "twisted cone" construction:

Example 2. Let Σ^2 be a simply connected Riemann surface and let $u\colon \Sigma \to S^{2+r} \subset \mathbb{E}^{3+r}$ be a conformal minimal immersion. If $*$ denotes the Hodge star operator on Σ, then u satisfies the vector equation

$$d * du = -2\phi u$$

where ϕ is the induced area 2-form of the immersion u.

Now let b be a scalar-valued function on Σ which satisfies the second order linear equation

$$d * db = -2\phi b.$$

Then the vector-valued 1-form $\beta = u * db - b * du$ clearly satisfies $d\beta = 0$. Thus, there exists a vector-valued function v on Σ (unique up to an additive constant) so that $dv = \beta$. The mapping $f\colon \Sigma \times \mathbb{R} \to \mathbb{E}^{3+r}$ given by the formula $f = v + tu$ is then easily seen to be an austere immersion with degenerate Gauss map away from the locus in $\Sigma \times \mathbb{R}$ defined by the equations $b = t = 0$ (on which it fails to be an immersion). We call these austere three-folds "twisted cones" because taking $b \equiv 0$ in the construction simply yields the cone on Σ.

Our main result in this section is that the twisted cones essentially comprise all of the austere three-folds whose Gauss map has rank 2 and which are not orthogonal products of a line with a minimal surface. More precisely, we have the following theorem.

Theorem 4.1. *Suppose that $f\colon M^3 \to \mathbb{E}^{3+r}$ is an austere minimal immersion with degenerate Gauss map which is not a local orthogonal product. Then, at every point of a dense open subset $M^* \subset M$ (to be described more fully in the proof), the immersion f can be locally expressed as a twisted cone for some minimal immersion $u\colon \Sigma \to S^{2+r}$ with induced area 2-form ϕ and some solution b on Σ to the equation $d * db = -2\phi b$. Moreover, the pair (u, b) is unique up to replacement by $(-u, -b)$.*

Before beginning the proof, we should remark on the similarity of this result with that of Dajczer and Gromoll in their paper *Gauss Parameterizations and Rigidity Aspects of Submanifolds*. In this paper, they consider the case of hypersurfaces $M^n \subset \mathbb{E}^{n+1}$ for which the Gauss map is degenerate of constant rank and derive a (local) "Gauss parameterization" of M by the normal bundle Λ of the image $V^k \subset S^n = \mathrm{Gr}_n(\mathbb{E}^{n+1})$. They then use this parameterization to study the minimal hypersurfaces in $\mathbb{E}^{n+1}$ with degenerate Gauss map. The overlap of their theory and ours concerns the case of an austere hypersurface $M^3 \subset \mathbb{E}^4$. As will be seen, our parameterization turns out to be slightly different from theirs even in this case.

Proof. Let $M' \subset M$ be the dense open set consisting of those $x \in M$ where the rank of $\gamma'(x)$ reaches its maximal value $d \leq 2$. If $d = 0$, then γ is a constant mapping, so M is an open subset of a 3-plane. Of course, this falls into the "orthogonal product" case, so we need not discuss it further. In addition, it is well-known that the differential of the Gauss map of a minimal submanifold in Euclidean space cannot have $d = 1$. Thus, we may assume (as we do in the rest of the proof) that $d = 2$.

Let $e_I\colon U \to \mathbb{E}^{3+r}$ for $0 \leq I \leq 2+r$ denote an orthonormal frame field on an open subset $U \subset M'$ which satisfies the conditions that (e_0, e_1, e_2) is a tangential frame field for the immersion and that $e_0(x)$ spans the kernel of $\gamma'(x)$ for every $x \in U$. Clearly M' can be covered by open sets U on which such a frame field exists. As usual, we define the structure 1-form by the formulas $\omega_I = e_I \cdot df$ and $\omega_{IJ} = e_I \cdot de_J$. These forms are subject to the usual structure equations

$$df = e_I \omega_I$$
$$de_I = e_J \omega_{JI}$$

$$d\omega_I = -\omega_{IK} \wedge \omega_K$$
$$d\omega_{IJ} = -\omega_{IK} \wedge \omega_{KJ}$$

In what follows, we will use lower case Roman letters for the "normal" index range $3 \leq a, b, c \leq 2 + r$. Note that because the e_a are normal vector fields we have

$$\omega_a = 0$$

for all a. Moreover, the forms $\omega_0, \omega_1, \omega_2$ form a coframing of U, so all of the other ω's are linear combinations of these forms. It will be convenient in what follows to use complex notation and set

$$\omega = \omega_1 + i\omega_2.$$

The Gauss map γ is represented (up to an orientation sign) by $\gamma = e_0 \wedge e_1 \wedge e_2$. Our assumption that e_0 is tangent to the fibers of the Gauss map implies that the all of the forms $\omega_{a0}, \omega_{a1}, \omega_{a2}$ are linear combinations of ω_1 and ω_2. This, coupled

with the structure equation

$$0 = d\omega_a = -\omega_{a0} \wedge \omega_0 - \omega_{a1} \wedge \omega_1 - \omega_{a2} \wedge \omega_2$$

implies that we must have $\omega_{a0} = 0$. It follows that, if we set $\pi_a = \omega_{a1} - i\omega_{a2}$ and use the assumption of minimality, then we have $\pi_a = z_a\omega$ for some complex functions z_a on U. Moreover, the structure equations for $d\omega_{a1}$ and $d\omega_{a2}$ can now be combined into

$$d\pi_a = i\omega_{21} \wedge \pi_a - \omega_{ab} \wedge \pi_b.$$

Note that not all of the z_a can vanish simultaneously on U since the pull-back under γ of the natural invariant metric on $\mathrm{Gr}_3(\mathbb{E}^{3+r})$ (i.e., the so-called *third fundamental form* of f) is

$$\sum_a (\omega_{a0})^2 + (\omega_{a1})^2 + (\omega_{a2})^2 = \left(\sum_a \|z_a\|^2\right) \left((\omega_1)^2 + (\omega_2)^2\right).$$

Now, we have

$$0 = d\omega_{a0} = -\omega_{a1} \wedge \omega_{10} - \omega_{a2} \wedge \omega_{20}$$

which implies that, on the open set $U_a \subset U$ where $z_a \neq 0$, the forms ω_{10}, ω_{20} must be linear combinations of ω_{a1}, ω_{a2} and hence are linear combinations of ω_1, ω_2. Since U is covered by the open sets U_a, it follows that ω_{10}, ω_{20} must be linear combinations of ω_1, ω_2 on all of U. If we set $\pi_0 = \omega_{01} - i\omega_{02}$, we then have that

$$\pi_0 = z_0\omega + h\overline{\omega}$$

for some complex functions z_0 and h on U. Moreover, the structure equations for the exterior derivatives of ω_{10}, ω_{20} can be combined into

$$d\pi_0 = i\omega_{21} \wedge \pi_0.$$

On the other hand, the structure equation for the exterior derivatives of ω_1 and ω_2 can be written in the form

$$d\omega = -i\omega_{21} \wedge \omega + \overline{\pi_0} \wedge \omega_0.$$

Using the equations so far, we see that $0 = d\pi_a \wedge \omega$ and this yields

$$0 = d\pi_a \wedge \omega = (z_a d\omega) \wedge \omega = -z_a(\overline{z_0}\,\overline{\omega} \wedge \omega_0) \wedge \omega.$$

It follows that $z_a\overline{z_0} = 0$ for all a. Thus, $z_0 \equiv 0$. Our formulae for π_0 and $d\omega$ now simplify to

$$\pi_0 = \overline{h\omega}$$
$$d\omega = -(h\omega_0 + i\omega_{21}) \wedge \omega$$

At this point, we can already seen that $de_0 \equiv 0 \bmod \omega_1, \omega_2$ so it follows that e_0 is constant along the fibers of $\gamma: U \to \mathrm{Gr}_3(\mathbb{E}^{3+r})$. Since e_0 is tangent to these fibers, the connected components of the γ-fibers are open subsets of straight lines in $\mathbb{E}^3$.

There are two cases:

If $h \equiv 0$ on U, then $de_0 \equiv 0$ on U and hence U is locally ruled by a family of parallel straight lines. Of course, this implies that U is locally an orthogonal product of a minimal surface in a hyperplane in $\mathbb{E}^{3+r}$ with a normal line to the hypersurface. We will not consider this case any further.

If h does not vanish identically, then, again due to the real analytic structure of minimal submanifolds on Euclidean space, it follows that h is non-zero on an open dense subset of U, say $U^* \subset U$. It is easy to see that whether or not h is zero at a point of U is independent of which adapted orthonormal frame field we choose, so there is a well-defined open dense subset $M^* \subset M$ which intersects each $U \subset M$ in the set U^*. For the remainder of the proof, we restrict our attention to M^*.

Now consider the mapping $e_0: U^* \to S^{2+r}$. We have

$$de_0 \cdot de_0 = (\omega_{10})^2 + (\omega_{20})^2 = |h|^2 \omega \circ \overline{\omega}.$$

Since h never vanishes on U^*, it follows that the differential of e_0 has rank 2 at every point of U^*. In particular, every point $x \in U^*$ has an open neighborhood V with the property that $e_0: V \to S^{2+r}$ is a submersion with connected fibers onto an analytic embedded disk $\Sigma^2 \subset S^{2+r}$. We thus may regard V analytically as $\mathbb{R} \times \Sigma$ and think of e_0 as actually being well defined on Σ. It is then easy to see that there exists a complex 1-form η on Σ which satisfies

$$\eta \circ \overline{\eta} = |h|^2 \omega \circ \overline{\omega} \quad \text{and} \quad \frac{i}{2}\eta \wedge \overline{\eta} = \frac{i}{2}|h|^2 \omega \wedge \overline{\omega}.$$

This implies that $h\omega = e^{i\theta}\eta$ for some function θ on V. Rotating the pair e_1, e_2 by an angle of θ then yields a new adapted framing which satisfies $h\omega = \eta$, in particular, $h\omega$ is now well-defined on Σ. Henceforth, we regard Σ as a Riemann surface endowed with conformal metric $\eta \circ \overline{\eta}$ and associated area form $\phi = \frac{i}{2}\eta \wedge \overline{\eta}$.

The second fundamental form II_Σ of the surface $\Sigma \subset S^{2+r}$ is now easily computed from the structure equations and we get that

$$\begin{aligned}
\mathrm{II}_\Sigma &= \sum_a e_a \otimes (\omega_{a1} \circ \omega_{10} + \omega_{a2} \circ \omega_{20}) \\
&= -\sum_a e_a \otimes \mathrm{Re}(\pi_a \circ \overline{\pi_0}) \\
&= -\sum_a e_a \otimes \mathrm{Re}(z_a h \omega^2) \\
&= -\sum_a e_a \otimes \mathrm{Re}(z_a h^{-1} \eta^2).
\end{aligned}$$

Of course, this implies that Σ is a minimal surface in S^{2+r}.

We can now rewrite the structure equations for $d\pi_0$ and $d\omega$ in the form

$$d\eta = i\omega_{21} \wedge \eta \quad \text{and} \quad d(h^{-1}\eta) = -(h\omega_0 + i\omega_{21}) \wedge (h^{-1}\eta).$$

In particular, these two equations imply that

$$d(h^{-1}) = -\omega_0 + 2g\eta$$

for some complex function g on V. Now let us write $h^{-1} = -(A + Bi)$ where A and B are real functions on V. The above equation then separates into the two real equations

$$dA = \omega_0 + g\eta + \overline{g}\,\overline{\eta}$$
$$dB = -i(g\eta - \overline{g}\,\overline{\eta})$$

In particular, it follows that B is locally constant on the fibers of the projection $e_0: V \to \Sigma$. Since the fibers of this map are connected, it follows that B is actually well defined on Σ. Thus, $*dB$ makes sense and is equal to $-(g\eta + \overline{g}\,\overline{\eta})$. In particular, we have

$$dA = \omega_0 - *dB.$$

Note that this implies that the function A restricted to each fiber of e_0 to be "arc length" along that line. Now the structure equation for $d\omega_0$ yields

$$\begin{aligned}
d\omega_0 &= -\omega_{01} \wedge \omega_1 - \omega_{02} \wedge \omega_2 = -\mathrm{Re}(\pi_0 \wedge \omega) \\
&= -\frac{1}{2}(\overline{h\omega} \wedge \omega + h\omega \wedge \overline{\omega}) = \frac{1}{2}(\overline{h} - h)\omega \wedge \overline{\omega} \\
&= \frac{1}{2}(h^{-1} - \overline{h}^{-1})\eta \wedge \overline{\eta} = \frac{1}{2}(-2iB)\eta \wedge \overline{\eta}
\end{aligned}$$

Thus, we get $d * dB = d(\omega_0 - dA) = -2\phi B$. Moreover, the structure equations give

$$de_0 = e_1\omega_{10} + e_2\omega_{20} = -e_1\omega_{01} - e_2\omega_{02}$$
$$= -\operatorname{Re}((e_1 - ie_2)\overline{\pi_0}) = -\operatorname{Re}((e_1 - ie_2)\eta)$$

and, finally,

$$df = e_0\omega_0 + e_1\omega_1 + e_2\omega_2 = e_0\omega_0 + \operatorname{Re}((e_1 - ie_2)\omega)$$
$$= e_0(dA + *dB) - \operatorname{Re}((e_1 - ie_2)(A + Bi)\eta)$$
$$= d(Ae_0) + e_0 * dB - B * de_0.$$

Thus, $f = v + Ae_0$ where $v: V \to \mathbb{E}^{3+r}$ satisfies $dv = e_0 * dB - B * de_0$. Of course, this is what we were trying to prove. □

It is natural to ask whether there is a global version of Theorem 4.1 which does not need to restrict to a dense open set in order to achieve the "twisted cone" normal form. It is possible that such a global result is derivable, but there are two sources of complication which require further analysis.

The first source, which also seems the more difficult to deal with, is the problem of showing that the one-dimensional distribution defined by the kernel of γ' on M' extends smoothly across the locus Z on which the rank of γ' drops to zero. It is not difficult to show that the kernel distribution can be extended over the complement of a real analytic locus W of codimension at least two, but removing the locus W seems to be a problem.

The second source of difficulty arises from considering the part of M' which does not lie in M^* (as defined in the proof). In the notation of the proof above, this is the locus in U where h vanishes. In fact, it is not difficult to show that these points can also be locally parameterized by the twisted cone construction, but one has to allow *branched* minimal surfaces Σ and auxiliary functions b which have singularities at the branch points of Σ. Specifically, here is how one can generalize the twisted cone construction to the case of branched minimal surfaces in S^{2+r}: If $u: \Sigma \to S^{2+r}$ is a conformal minimal branched immersion of a Riemann surface Σ, then near any branch point $p \in \Sigma$, there is a local holomorphic coordinate z centered on p and an integer $k > 0$ so that the induced metric on Σ is given by $|z|^{2k} F dz \circ d\bar{z}$ where the function F does not vanish at p (i.e., at $z = 0$). Thus, the induced area form ϕ is given by $\phi = \frac{i}{2}|z|^{2k} F dz \wedge d\bar{z}$. We then require that b (the auxiliary function) be a solution of $d * db = -2\phi b$ which is smooth on a deleted neighborhood of each branch point and which, at the branch point p, can be written in the form $b = \operatorname{Re}(f(z)/z^k) + s$ where f is a non-vanishing holomorphic function of z and s is a smooth function on a neighborhood of p. We then proceed with the twisted cone construction as before, except that the vector valued function v is constructed so that it satisfies

$$dv = u * db - b * du - d(\operatorname{Im}(f(z)/z^k)u)$$

(this correction term is essential to ensure that the right hand side is smooth on a neighborhood of p). It is not difficult to verify that every point of M' has a neighborhood which can be parameterized by a generalized twisted cone in this sense. Details will be left to the reader.

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