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Linear foliations of T^n

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—To the memory of J. Martinet

Abstract. We present a definition of diophantine matrix and use this concept to distinguish two classes of minimal linear foliations of T^n, the diophantine and the Liouville one. Let $\mathcal{E}_p$, $1 \leq p \leq n-1$, denote a minimal (all leaves are dense) linear p-dimensional foliation of T^n, and $H^{om}(T^n, \mathcal{E}_p)$, $1 \leq m \leq p$, the cohomology group of type $(0, m)$ of the foliated manifold $(T^n, \mathcal{E}_p)$. Our main result is the computation of these groups. $H^{om}(T^n, \mathcal{E}_p)$ is isomorphic to $\mathbb{R}^{\binom{n}{m}}$ if $\mathcal{E}_p$ is diophantine and is an infinite dimensional non-Hausdorff vector space if $\mathcal{E}_p$ is Liouville. Some of these groups were computed before, see [**4**], [**6**] and [**9**].

0. Introduction

In this paper we present a definition of diophantine $(q \times p)$-matrix; when $p = 1$ it coincides with the usual notion of diophantine vector. We use this definition to distinguish two classes of minimal linear foliations of T^n, the diophantine and the Liouville one. Let $\mathcal{E}_p, 1 \leq p \leq n-1$, denote a minimal linear p-dimensional foliation of T^n, minimal in the sense that every leave is dense, and $H^{om}(T^n, \mathcal{E}_p), 1 \leq m \leq p$, the cohomology group of type (o, m) of the foliated manifold $(T^n, \mathcal{E}_p)$ [**10**].

Our main result is the computation of $H^{om}(T^n, \mathcal{E}_p), 1 \leq m \leq p$. Assume, without loss of generality, that $\mathcal{E}_p$ is transversal to the fibers of the projection $T^q \to T^{p+q} \to T^p, p + q = n$, then $\mathcal{E}_p$ is defined by a linear q-form w on T^n which induces the canonical volume element on each fiber T^q. Consider the map

$$\wp \colon (\Lambda^{om}(T^n, \mathcal{E}_p), d_e) \to (\Lambda^m(T^n), d), \quad 0 \leq m \leq p$$

of differential complexes, given by $\wp(\mu) = \fint \mu \wedge w$, where $\fint$ denotes integration

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along the fibers and write $\Lambda^{om}(\ker)$ for the kernel of $\wp$. We first show that

2.1 $$H^{om}(T^n, \mathcal{E}_p) = H^m(T^p) \oplus H^{om}(\ker)$$

and after that

2.2 (i) If $\mathcal{E}_p$ is diophantine, then $H^{om}(\ker) = 0$

(ii) If $\mathcal{E}_p$ is Liouville, then $H^{om}(\ker)$ is infinite dimensional, and

2.4 $$H^{om}(\ker) = \frac{CL(d_e\Lambda^{o(m-1)}(\ker))}{d_e\Lambda^{o(m-1)}(ker)}$$

where CL denotes the closure in the C^∞-topology.

Some of these groups were computed before, $H^{01}(T^2, \mathcal{E}_1)$ by Heitsch [6] in the diophantine case and by Roger [9] in the Liouville's case. Later Alaoui and Tihami [4] computed $H^{om}(T^n, \mathcal{E}_p)$ for some diophantine foliations, in our sense, and $H^{01}(T^n, \mathcal{E}_1)$ for Liouville's foliations. In [2] the study of locally free actions of $\mathbb{R}^p$ on T^n led us to compute $H^{om}(T^n, \mathcal{E}_{n-1})$.

In section 3 we give as an application, a proof of the vanishing of the characteristic mapping [1] for locally free actions of $\mathbb{R}^p$ on T^n whose underlying foliation is linear and minimal.

1. Diophantine and Liouville foliations

For $1 \leq p \leq n-1$ write T^n as $T^p \times T^q$ and let $\exp\colon \mathbb{R}^p \times \mathbb{R}^q \to T^p \times T^q$ be the universal covering map i.e.,

$$\exp(x_1, \ldots, x_p, y_1, \ldots, y_q) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_p}, e^{2\pi i y_1}, \ldots, e^{2\pi i y_q}).$$

Let $\mathcal{E}_p$ denote a p-dimensional linear foliation of T^n and assume, without loss of generality, that $\mathcal{E}_p$ is transversal to the projection $T^q \to T^{p+q} \xrightarrow{P} T^p$. Under this assumption $\mathcal{E}_p$ determines uniquely the linear 1-forms

$$\begin{aligned} w_1 &= a_{11}dx_1 + \cdots + a_{1p}dx_p + dy_1 \\ w_2 &= a_{21}dx_1 + \cdots + a_{2p}dx_p + dy_2 \\ &\vdots \\ w_q &= a_{q1}dx_1 + \cdots + a_{qp}dx_p + dy_q \end{aligned} \tag{1.1}$$

which in turn define $\mathcal{E}_p$, i.e.

$$T\mathcal{E}_p = \bigcap_{j=1}^{q} \ker w_j.$$

Therefore, there is a bijection between the set of these foliations and the vector space of $q \times p$ real matrices $A = [a_{ij}]$. We recall that:

1.1 $\mathcal{E}_p$ is a minimal foliation if and only if $r_1 w_1 + \cdots + r_q w_q$, with $r_j \in Q$, is a rational form, implies $r_1 = \cdots = r_q = 0$.

Let

$$\begin{aligned} k &= [k_1 \cdots k_q] \in M(1,q;\mathbb{Z}), \\ \ell &= [\ell_1 \cdots \ell_p] \in M(1,p;\mathbb{Z}), \\ x &= [x_1 \cdots x_p] \in M(1,p;\mathbb{R}) \end{aligned}$$

and

$$\begin{aligned} A &= [a_{ij}] \in M(q,p;\mathbb{R}), \\ |k| &= \sup_i |k_i|, \\ |\ell| &= \sup_j |\ell_j|, \\ |x| &= \sup_j |x_j| \\ \|x\| &= \inf_\ell |x - \ell|; \end{aligned}$$

$\|x\|$ defines a metric on T^p.

1.2 Definition. We say that a $(q \times p)$-matrix A

(i) verifies a *diophantine condition* and write $A \in D_\beta$, if there exists $\beta \geq 0$ and $c > 0$ such that for every $k \in \mathbb{Z}^q, k \neq 0$, we have

$$\|kA\| > \frac{c}{|k|^{q+\beta}}$$

(ii) is diophantine if

$$A \in D = \bigcup_{\beta \geq 0} D_\beta$$

(iii) is Liouville, and write $A \in L$, if A is not diophantine and the lines $A_1, \ldots, A_q$ of A are linearly independent as elements of the vector space $\mathbb{R}^p(Q)$. If $A \in L$ there exists a sequence $\{k^s = [k_1^s \ldots k_q^s]\}$ with $|k^s| \to \infty$ as $s \to \infty$

such that

$$\|k^s A\| < \frac{1}{|k^s|^s}$$

Observe that definition 1.2 produces for $p = 1$ the usual concepts of diophantine and Liouville numbers $(q = 1)$ and vectors $(q > 1)$ [7]. For $q = 1$ and $p > 1$ the concepts obtained are not the standard ones and we used them in [2] to define diophantine and Liouville linear forms of T^n. For example, let $A = [a\ b]$. If a is a diophantine number then $A \in D$. If a is a Liouville number and $b = a^2$, then A is a Liouville matrix.

1.3 Remarks. If a column of A, say A^j, is diophantine, then A is diophantine. In fact

$$\|kA\| \geq \left\|kA^j\right\| > \frac{c}{|k|^{q+\beta}}.$$

If a line of A, say A_i, is Liouville, then A is Liouville. In fact, let $\{k_i^s\}$ be a sequence of positive integers such that $k_i^s \to \infty$ with $s \to \infty$ and that

$$\|k_i^s A_i\| < \frac{1}{|k_i^s|^s}.$$

Take $k^s = [0 \ldots k_i^s \ldots 0]$. Then

$$\|k^s A\| = \|k_i^s A_i\| < \frac{1}{|k_i^s|^s} = \frac{1}{|k^s|^s}.$$

A matrix made of Liouville columns may be a diophantine matrix.

We obtain examples of this situation by combining the following two statements:

(i) Let $a, b \in \mathbb{R}$ such that $a + b \in D_\beta$; then the matrix $[a\ b] \in D_\beta$. By definition

$$\|[ka\ kb]\| = \inf_{[\ell_1\ \ell_2]} \sup\{|ka - \ell_1|,\ |kb - \ell_2|\}.$$

Assume that

$$\|[ka\ kb]\| = |ka - \ell_1|;$$

then

$$\begin{aligned} 2|ka - \ell_1| &\geq |ka - \ell_1| + |kb - \ell_2| \\ &\geq |ka - \ell_1 + kb - \ell_2| \\ &= |k(a+b) - (\ell_1 + \ell_2)| \\ &> \frac{c}{|k|^{1+\beta}}. \end{aligned}$$

Therefore

$$\|[ka\ kb]\| > \frac{1}{2} \cdot \frac{c}{|k|^{1+\beta}}$$

(ii) If $c \in \mathbb{R} - Q$, then $c = a + b$ where a and b are Liouville numbers, see [5].

$$\begin{aligned} c &= [c].c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11} \ldots c_{33}c_{34}c_{35} \ldots \\ a &= [c].c_1 00 c_4c_5c_6c_7c_8c_9 00 \ldots 0c_{34}c_{35} \ldots \\ b &= 0.0c_2c_3 000000 c_{10}c_{11} \ldots c_{33} 00 \ldots \end{aligned}$$

$$2! \qquad 3! \qquad 4!$$

It is easy to check that a and b are Liouville numbers. It may also be the case that a matrix made of diophantine lines is a Liouville matrix, but we do not know of any example.

1.4 Definition. A linear foliation $\mathcal{E}_p$ of T^n is said to be *diophantine* or *Liouville* if the corresponding matrix A has such a property.

It follows directly from definitions 1.2 and 1.4 that linear diophantine and Liouville foliations are minimal and that every linear minimal foliation is either diophantine or Liouville.

We want to point out that the contents of definitions 1.2 and 1.4 are essentially the same as the one of definition 2.1 in [8]. In fact we could say that a linear foliation $\mathcal{E}_p$ of T^n is diophantine or Liouville if the vector subspace of $\mathbb{R}^n$ given by the rows of the matrix $[AI_q]$, A determined by $\mathcal{E}_p$, is diophantine or Liouville in the sense of J. Moser. We thank the referee for calling our attention to this coincidence.

2. The Cohomology of Linear Foliations

Let $\mathcal{E}_p$ be a minimal linear p-dimensional foliation of $T^n, 1 \leq p \leq n-1$, transversal to the fibration

$$T^q \to T^{p+q} \xrightarrow{P} T^p$$

where P is the projection

$$(x_1, \ldots, x_p, y_1, \ldots, y_q) \to (x_1, \ldots, x_p)$$

and $w_1, \ldots, w_q$ the linear 1-forms uniquely determined by $\mathcal{E}_p$, as in (1.1). Denote by $\mathcal{F}_q$ the foliation by tori given by the fibers of P. The coframe

$$\{dx_1, \ldots, dx_p, w_1, \ldots, w_q\}$$

of T^n is well adapted to the decomposition $T^n = T\mathcal{E}_p \oplus T\mathcal{F}_q$. Its dual frame is

$$\{E_1, \ldots, E_p, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_q}\}$$

where

$$E_j = \frac{\partial}{\partial x_j} - a_{1j}\frac{\partial}{\partial y_1} - \cdots - a_{qj}\frac{\partial}{\partial y_q}.$$

Denote by $\Lambda^{om}(T^n, \mathcal{E}_p)$ then C^∞ forms of type (o, m) of the foliated manifold $(T^n, \mathcal{E}_p)$. If $\mu \in \Lambda^{om}(T^n, \mathcal{E}_p)$ then

$$\mu = \sum f_{i_1 \ldots i_m} dx_{i_1} \wedge \ldots \wedge dx_{i_m}.$$

The *foliated exterior derivative*

$$d_e \colon \Lambda^{om}(T^n, \mathcal{E}_p) \to \Lambda^{o(m+1)}(T^n, \mathcal{E}_p)$$

is given by

$$d_e(f dx_{i_1} \wedge \ldots \wedge dx_{i_m}) = \sum_j E_j(f) dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_m} \tag{2.1}$$

$(\Lambda^{o*}(T^n, \mathcal{E}_p), d_e)$ is a differential complex and the associated cohomology is denoted by $H^{o*}(T^n, \mathcal{E}_p)$.

Now, let $w = w_1 \wedge \ldots \wedge w_q$ and for each $0 \le m \le p$

$$\Lambda^{om}(T^n, \mathcal{E}_p) \overset{\wp}{\rightleftarrows} \Lambda^m(T^p) \tag{2.2}$$

be given by $\wp(\mu) = ⨍ \mu \wedge w$, where $⨍$ denotes integration along the fibers of P. $\wp$ is an epimorphism of differential complexes and

$$\frac{(-1)^m}{(2\pi)^m}\wp \circ P^* = \mathrm{id}.$$

Denote by $\Lambda^{om}(\ker)$ the kernel of $\wp$ and let d_f be the exterior foliated derivative of the foliated manifold $(T^n, \mathcal{F}_q)$. Direct computation gives:

$$\mu \in \ker^{om} \Leftrightarrow \mu \wedge w \quad \text{is} \quad d_f\text{-exact}. \tag{2.3}$$

Associated to $\wp$ we have a long exact sequence of cohomology groups

$$\cdots \to H^{om}(\ker) \to H^{om}(T^n, \mathcal{E}_p) \xrightarrow{\wp^*} H^m(T^n) \xrightarrow{\delta} H^{om+1}(\ker) \to \cdots \tag{2.4}$$

and, since the connecting morphisms δ are zero maps, the sequence (2.4) breaks into short exact sequences

$$0 \to H^{om}(\ker) \to H^{om}(T^n, \mathcal{E}_p) \xrightarrow{\wp^*} H^m(T^n) \to 0 \tag{2.5}$$

It follows from (2.5) that

2.1 $$H^{om}(T^n, \mathcal{E}_p) = H^{om}(\ker) \oplus H^m(T^n) = H^{om}(\ker) \oplus \mathbb{R}^{\binom{n}{m}}$$

therefore to know $H^{om}(T^n, \mathcal{E}_p)$ it remains to compute $H^{om}(\ker)$.

2.2 Theorem. *Let $\mathcal{E}_p$ be a minimal linear p-dimensional foliation of T^n, $1 \le p \le n-1$. Then $H^{00}(\ker) = 0$ and for every $1 \le m \le p$*

(i) *$H^{om}(\ker) = 0$ if $\mathcal{E}_p$ is diophantine*

(ii) *$H^{om}(\ker)$ is infinite dimensional if $\mathcal{E}_p$ is Liouville.*

For the sake of clarity we give the proof in the particular case $n = 4$ and $p = q = 2$. Associated to the decomposition $TT^4 = T\mathcal{E}_2 \oplus T\mathcal{F}_2$ one has the frame and dual coframe of T^4:

$$E_1 = \frac{\partial}{\partial x_1} - a_{11}\frac{\partial}{\partial y_1} - a_{21}\frac{\partial}{\partial y_2} \qquad dx_1$$

$$E_2 = \frac{\partial}{\partial x_2} - a_{12}\frac{\partial}{\partial y_1} - a_{22}\frac{\partial}{\partial y_2} \qquad dx_2$$

$$\frac{\partial}{\partial y_1} \qquad \text{and} \qquad w_1 = a_{11}dx_1 + a_{12}dx_2 + dy_1$$

$$\frac{\partial}{\partial y_2} \qquad\qquad w_2 = a_{21}dx_1 + a_{22}dx_2 + dy_2$$

Let f be a 0-cycle i.e.,

$$d_e f = E_1 f dx_1 + E_2 f dx_2 = 0;$$

then $E_1 f = E_2 f = 0$ and, since $\mathcal{E}_2$ is minimal, f is a constant. Furthermore, since $f \in \Lambda^{00}(\ker)$ then we have necessarily $f \equiv 0$. Therefore $H^{00}(\ker) = 0$. We will denote the liftings of objects from T^4 to $\mathbb{R}^4$ with the same letters. The lifting of a function $f \in C^\infty(T^4)$ has a Fourier expansion

$$f = \sum_{(\ell,k)} f_{\ell k} e^{2\pi i(\ell.x + k.y)}$$

with $\ell = [\ell_1 \ell_2], k = [k_1 k_2], x = [x_1 x_2]$ and $y = [y_1 y_2]$. If

$$\mu = f^1 dx_1 + f^2 dx_2$$

and

$$\nu = f dx_1 \wedge dx_2,$$

it follows from (2.1) that:

$$\mu \in \Lambda^{01}(\ker) \quad \Leftrightarrow \quad f^1_{\ell 0} = f^2_{\ell 0} = 0 \text{ for every } \ell. \tag{2.6}$$

$$\mu \in Z^{01}(\ker) \quad \Leftrightarrow \quad E_1 f^2 = E_2 f^1 \tag{2.7}$$

or

$$(\ell_1 - k_1 a_{11} - k_2 a_{21}) f^2_{\ell k} = (\ell_2 - k_1 a_{12} - k_2 a_{22}) f^1_{\ell k}$$

for every $k \neq [0\ 0]$.

$$\mu \in B^{01}(\ker) \quad \Leftrightarrow \quad \text{there exists } h \in \Lambda^{00}(\ker) \text{ such that}$$

$$E_1 h = f^1 \qquad 2\pi i(\ell_1 - k_1 a_{11} - k_2 a_{21}) h_{\ell k} = f^1_{\ell k} \tag{2.8.1}$$

or

$$E_2 h = f^2 \qquad 2\pi i(\ell_2 - k_1 a_{12} - k_2 a_{22}) h_{\ell k} = f^2_{\ell k} \tag{2.8.2}$$

$$\nu \in \Lambda^{02}(\ker) \quad \Leftrightarrow \quad f_{\ell 0} = 0 \text{ for all } \ell \text{ and } Z^{02}(\ker) = \Lambda^{02}(\ker) \tag{2.9}$$

$$\nu \in B^{02}(\ker) \quad \Leftrightarrow \quad \text{there exists } h^1 dx_1 + h^2 dx_2 \in \Lambda^{01}(\ker) \text{ such that } E_1 h^2 - E_2 h^1 = f$$

or

$$2\pi i[(\ell_1 - k_1 a_{11} - k_2 a_{21}) h^2_{\ell k}] - (\ell_2 - k_1 a_{12} - k_2 a_{22}) h^1_{\ell k} = f_{\ell k} \tag{2.10}$$

Proof of (i). Since $\mathcal{E}_2$ is diophantine, $A \in D_\beta$ for some $\beta \geq 0$. Let

$$K_1 = \{k \neq 0 \colon \|kA^1\| = \|kA\|\}$$

and

$$K_2 = \{k \in M(1,2;\mathbb{Z}) - K_1, k \neq 0 \colon \|kA^2\| = \|kA\|\}$$

Then $M(1,2;\mathbb{Z}) - \{0\} = K_1 \cup K_2$ and $K_1 \cap K_2 = \phi$.

We show first that every (0,1)-cocycle $\mu = f^1 dx_1 + f^2 dx_2$ is d_e-exact. Define complex numbers $h_{\ell k}$ by (2.8.j) if $k \in K_j, j = 1,2$ and $h_{\ell 0} = 0$. Since by hypothesis $f^1_{\ell k}$ and $f^2_{\ell k}$ satisfy the relations (2.6) and (2.7) it follows that every $h_{\ell k}$, just defined, satisfies both (2.8.1) and (2.8.2). Thus, for $k \in K_j$ one has

$$|h_{\ell k}| = \frac{1}{2\pi} \frac{|f^j_{\ell k}|}{|\ell_j - k_1 a_{1j} - k_2 a_{2j}|} < \frac{1}{2\pi} \frac{|f^j_{\ell k}|}{\|kA^j\|} = \frac{1}{2\pi} \frac{|f^j_{\ell k}|}{\|kA\|}$$

and by 1.2 there exists $c > 0$ such that

$$|h_{\ell k}| < \frac{1}{2\pi c} |k|^{2+\beta} \left| f^j_{\ell k} \right|.$$

Therefore the $h_{\ell k}$'s are the Fourier coefficients of a C^∞-function h such that $d_e h = \mu$. Now, take a (0,2)-cocycle $\gamma = f dx_1 dx_2$ and define complex numbers $h^1_{\ell k}$ and $h^2_{\ell k}$ by:

$$h^1_{\ell k} = \begin{cases} 0, & \text{if } k \in K_1 \\ \dfrac{-f_{\ell k}}{(\ell_2 - k_1 a_{12} - k_2 a_{22})}, & \text{if } k \in K_2 \end{cases}$$

$$h^2_{\ell k} = \begin{cases} 0, & \text{if } k \in K_2 \\ \dfrac{f_{\ell k}}{(\ell_1 - k_1 a_{11} - k_2 a_{21})}, & \text{if } k \in K_1 \end{cases}$$

From the diophantine character of A it follows that the $h^1_{\ell k}$'s and $h^2_{\ell k}$'s are the Fourier coefficients of C^∞-functions h^1 and h^2. Besides, since they satisfy relations (2.10),

$$\gamma = f dx_1 dx_2 = d_e(h^1 dx_1 + h^2 dx_2)$$

and the proof of (i) is complete.

Proof of (ii). Since $\mathcal{E}_2$ is Liouville there exists a sequence $k^s = [k^s_1\ k^s_2]$ with $|k^s| \to \infty$ such that

$$\|k^s A\| < \frac{1}{|k^s|^s}.$$

For every s let $n_s \geq s$ be the greatest integer such that

$$\left\|k^s A^j\right\| < \frac{1}{|k^s|^{n_s}}, \quad j = 1,2.$$

Since $\mathcal{E}_2$ is minimal one obtains from 1.1 that $\|kA\| \neq 0$ for every $k \neq 0$, thus for each s there exists $j(s) \in \{1,2\}$ such that

$$\frac{1}{|k^s|^{n_s+1}} < \|k^s A^{j(s)}\| < \frac{1}{|k^s|^{n_s}}.$$

Therefore, by taking a subsequence one can assume, for example, that for every s

$$\frac{1}{|k^s|^{n_s+1}} < \|k^s A^1\| < \frac{1}{|k^s|^{n_s}} \tag{2.11}$$

$$\left\|k^s A^2\right\| < \frac{1}{|k^s|^{n_s}} \tag{2.12}$$

Let $\ell^s = [\ell^s{}_1 \ell^s{}_2]$ be such that

$$\left\|k^s A^j\right\| = \left|\ell^s_j - k^s A^j\right| = \left|\ell^s_j - k^s_1 a_{1j} - k^s_2 a_{2j}\right| \qquad j = 1,2. \tag{2.13}$$

Now, we are going to construct a sequence of d_e-closed forms

$$\{\mu_n = f^n dx_1 + g^n dx_2\}$$

such that the subset $\{[\mu_n]\} \subset H^{01}(\ker)$ is linearly independent. We construct the sequence $\{\mu_n\}$ inductively.

Define

$$f^1_{\ell k} = \begin{cases} \dfrac{1}{|\ell^s_1|^{n_s/3}|\ell^s_2|^{n_s/3}|k^s|^{n_s/3}}, & \text{if } (\ell,k) = \pm(\ell^s,k^s) \\ 0, & \text{otherwise.} \end{cases}$$

The $f^1_{\ell k}$'s are clearly the Fourier coefficients of a C^∞ real function f^1. Define the $g^1_{\ell k}$'s through relations 2.6 and 2.7. Then $g^1_{\ell k} = 0$ if $(\ell,k) \neq (\ell^s,k^s)$ and

$$g^1_{\ell^s k^s} = \frac{\ell^s_2 - k^s_1 a_{12} - k^s_2 a_{22}}{\ell^s_1 - k^s_1 a_{11} - k^s_2 a_{21}} f^1_{\ell^s k^s}$$

By taking absolute values on both sides and by using (2.13), (2.11) and (2.12) one obtains

$$\left|g^1_{\ell^s k^s}\right| = \frac{\|k^s A^2\|}{\|k^s A^1\|}\left|f^1_{\ell^s k^s}\right| < |k^s|\left|f^1_{\ell^s k^s}\right|$$

Therefore the $g^1_{\ell_k}$'s are the coefficients of a C^∞ real function g^1 and $\mu_1 = f^1 dx_1 + g^1 dx_2$ belongs to $\mathbb{Z}^{01}(\ker)$. Now we show that μ_1 is not d_e-exact. In fact, assume $\mu_1 = d_e h$ with $h \in C^\infty$. Then the Fourier coefficients of h must satisfy relations (2.8.1) and in particular

$$2\pi i h_{\ell^s k^s} = \frac{f^1_{\ell^s k^s}}{(\ell^s_1 - k^s_1 a_{11} - k^s_2 a_{21})}.$$

From (2.13) and (2.11) one obtains

$$2\pi\left|h_{\ell^s k^s}\right| > |k^s|^{n_s}\left|f^1_{\ell^s k^s}\right| = \left[\frac{|k^s|}{|\ell^s_1|}\right]^{n_s/3}\left[\frac{|k^s|}{|\ell^s_2|}\right]^{n_s/3}$$

But

$$\left|\ell^s_j - k^s a^j\right| < \frac{1}{|k^s|^{n_s}}$$

implies

$$\frac{|\ell^s_j|}{|k^s|} < \left|A^j\right| + \frac{1}{|k^s|^{n_s+1}} < \left|A^j\right| + 1$$

which in turn yields

$$\left|h_{\ell^s k^s}\right| = \frac{1}{2\pi}\frac{\left|f^1_{\ell^s k^s}\right|}{\|k^s A^1\|} > \frac{1}{2_\pi}\frac{1}{|A^j|+1} \tag{2.14}$$

By the Riemann-Lebesgue theorem h could not be a L^1-function. We conclude that $0 \neq [\mu_1] \in H^{01}(\ker)$. To construct μ_2 take a proper sequence of $\{(\ell^s,k^s)\}$ which contains infinitely many terms and disregard infinitely many of them. Starting with the new sequence, the same construction made for μ_1 gives

$$\mu_2 = f^2 dx_1 + g^2 dx_2.$$

Following this procedure one constructs inductively the sequence $\{\mu_n\}$. Now assume

$$c_1[\mu_1] + c_2[\mu_2] + \cdots + c_n[\mu_n] = 0,$$

with $c_j \in \mathbb{R}$. Then, there exists a C^∞-real function h such that

$$d_e h = c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n.$$

The Fourier coefficients of h must satisfy relations (2.8.1) and in particular

$$2\pi i(\ell^s_1 - k^s_1 a_{11} - k^s_2 a_{21})h_{\ell^s k^s} = c_1 f^1_{\ell^s k^s} + c_2 f^2_{\ell^s k^s} + \cdots + c_n f^n_{\ell^s k^s} \tag{2.15}$$

From the construction of $\{\mu_n\}$ it follows that there are infinite subsequences of (ℓ^s,k^s) for which the following relations are satisfied:

$$f^2_{\ell^s k^s} = \cdots = f^n_{\ell^s k^s} = 0 \tag{2.16.1}$$

$$f^1_{\ell^s k^s} = f^2_{\ell^s k^s} \quad \text{and} \quad f^3_{\ell^s k^s} = f^n_{\ell^s k^s} = 0 \tag{2.16.2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f^1_{\ell^s k^s} = f^{n-1}_{\ell^s k^s} \quad \text{and} \quad f^n_{\ell^s k^s} = 0 \tag{2.16.n}$$

Considering (2.15) together with (2.16.j) $j = 1,\ldots,n$ and (2.14) one obtains $c_1 = c_2 = \cdots = c_n = 0$ and this completes the proof of (ii).

Now, we take care of $H^{02}(\ker)$. Let $\nu = f dx_1 \wedge dx_2$ with $f = f^1$ previously defined. Assume ν is d_e-exact i.e., there exists $h^1 dx_1 + h^2 dx_2 \in \Lambda^{01}(\ker)$ such that $E_1 h^2 - E_2 h^1 = f$. From (2.10), (2.11) and (2.12) one gets

$$\frac{1}{2\pi i}\frac{f_{\ell^s k^s}}{\ell^s_1 - k^s_1 a_{11} - k^s_2 a_{21}} = \frac{\ell^s_2 - k^s_1 a_{12} - k^s_2 a_{22}}{\ell^s_1 - k^s_1 a_{11} - k^s_2 a_{21}} h^1_{\ell^s k^s} + h^2_{\ell^s k^s}$$

and

$$\frac{1}{2\pi}\frac{\left|f_{\ell^s k^s}\right|}{\|k^s A^1\|} < |k^s|\left|h^1_{\ell^s k^s}\right| + \left|h^2_{\ell^s k^s}\right|$$

Since h^1 and h^2 are C^∞-functions this last inequality implies

$$\frac{1}{2\pi}\frac{|f_{\ell^s k^s}|}{||k^s A^1||} \to \infty \quad \text{as} \quad s \to \infty$$

which contradicts (2.14).

We conclude that $0 \neq [\nu] \in H^{02}(\ker)$. The proof continue as in the case of $H^{01}(\ker)$.

To prove (i) for arbitrary n and p we first define

$$K_1 = \{k \in M(1,q;\mathbb{Z}), k \neq 0: ||kA^1|| = ||kA||\}$$
$$K_2 = \{k \in M(1,q;\mathbb{Z}) - K_1, k \neq 0: ||kA^2|| = ||kA||\}$$
$$\vdots$$
$$K_q = \{k \in M(1,q;\mathbb{Z}) - (K_1 \cup \cdots \cup K_q), k \neq 0: ||kA^q|| = ||kA||\}$$

Then

$$M(1,q;\mathbb{Z}) - \{0\} = K_1 \cup K_2 \cup \cdots \cup K_q$$

and $K_i \cap K_j = \varnothing$ for $i \neq j$. Next, imitate the steps given above.

To prove (ii) for arbitrary n and p we can assume, by a permutation of indices if necessary, that there exists a sequence $k^s = [k_1^s \cdots k_q^s]$ with $|k^s| \to \infty$ such that

$$\frac{1}{|k^s|^{n_s+1}} < \left\|k^s A^1\right\| < \frac{1}{|k^s|^{n_s}}$$

$$\left\|k^s A^j\right\| < \frac{1}{|k^s|^{n_s}}, \quad 2 \leq j \leq q$$

Next, define

$$f^1_{lk} = \frac{1}{|l_1|^{n_s/q+1}\cdots|l_q^s|^{n_s/q+1}|k^s|^{n_s/q+1}}$$

if $(l,k) = \pm(l^s,k^s)$ and 0 otherwise. Next imitate the steps given above.

2.3 Remark. In the proof of 2.2 part (ii) we constructed C^∞-functions f^1 and f^2 such that $E_1 f^2 = E_2 f^1$ and the pair of equations

$$E_1 h = f^1, \quad E_2 h = f^2 \tag{2.17}$$

had no L^1-solutions. However, let

$$\mu = f^1 dx_1 + f^2 dx_2 \in \mathbb{Z}^{01}(\ker)$$

and

$$f_n^j = \sum_{|(\ell,k)|\leq n} f_{(\ell k)} e^{2\pi i(\ell\cdot x + k\cdot y)} \quad j = 1,2.$$

Then for each n the form $\mu_n = f_n^1 dx_1 + f_n^2 dx_2$ is d_e-closed and equations (2.17) with f_n^j in place of f^j have analytic solutions h_n. In other words, given any $\mu \in \mathbb{Z}^{01}(\ker)$ there exists a sequence $\{\mu_n\} \subset B^{01}(\ker)$ which converges in $\Lambda^{01}(\ker)$ with the C^∞-topology to μ. An analogous consideration applies to 2-forms ν. We have proved

2.4 Proposition. *Let $\mathcal{E}_p$ be a minimal linear p-dimensional Liouville foliation of $T^n, 1 \leq p \leq n-1$. Then*

$$H^{0m}(\ker) = \frac{CL(d_e\Lambda^{o(m-1)}(\ker))}{d_e\Lambda^{o(m-1)}(\ker)} \quad 1 \leq m \leq p.$$

Here CL means the closure in the C^∞ topology.

3. Applications to Actions of $\mathbb{R}^p$

Let M be a closed orientable connected m-dimensional manifold and $F: \mathbb{R}^p \times M \to M$ be a non-singular C^r-action, $r \geq 2$. To study the space $A^r(\mathbb{R}^p, M)$ of all non-singular C^r-actions, $r \geq 2$, of $\mathbb{R}^p$ on M we introduced in [1] a *characteristic mapping* which associates to each action F in $A^r(\mathbb{R}^p, M)$ a $(q+1)$-linear mapping α_F of $\mathbb{R}^p$ on the de Rham cohomology group $H^{2q+1}(M), q = m-p$ being the codimension of the underlying foliation $\mathcal{F}$ of F. We proved in [1]:

2.2 The characteristic mapping α_F vanishes if F is a non-singular action of T^p with $1 \leq p \leq m-1$.

2.8 α_F is degenerate if F is a non-singular action of $T^{m-2} \times \mathbb{R}$ on M.

4.2 Let F be a $C^r, r \geq 2$ non-singular action of $\mathbb{R}^2$ on a closed orientable connected 3-manifold $M \neq T^3$. If α_F is non-degenerate, then F has a compact orbit and so does any action G in $A^r(\mathbb{R}^2, M)$ which is sufficiently C^1 close to F.

See [3] for further results.

In this section we prove the vanishing of the characteristic mapping of a

$C^r, r \geq 2$ non-singular action whose underlying foliation is a minimal linear foliation of T^n. This generalizes Theorem 4.6 in [1]. We now recall the definition of the characteristic mapping α_F as given in [1]. The orbit of $x \in M$ under the non-singular action $F: \mathbb{R}^p \times M \to M$ is the mapping $F_x: \mathbb{R}^p \to M$ defined by $F_x(v) = F(v, x)$. Consider the vector fields on M given by

$$X_j(x) = DF_x(0) \cdot e_j, \quad 1 \leq j \leq p$$

where $e_1, \dots, e_m$ is the natural basis of $\mathbb{R}^m$. $X = \{X_1, \dots, X_p\}$ is a commuting p-frame of the underlying foliation $\mathcal{F}$ of F. Any ordered set of 1-forms $\xi = \{\xi_1, \dots, \xi_p\}$ such that $\xi_j(X_i) = \delta_{ij}$ is called a *p-coframe adapted to F*. Let $\Lambda(M)$ be the graded algebra of smooth forms on M and $I(\mathcal{F})$ be the annihilating ideal of $\mathcal{F}$ i.e., a j-form belongs to $I(\mathcal{F})$ if $W_x(v_1, \dots, v_j) = 0$ whenever $v_1, \dots, v_j$ are all tangent to $\mathcal{F}$ at x. Thus $dI(\mathcal{F}) \subset I(\mathcal{F})$ and $I(\mathcal{F})^{q+1} = 0$, $q = m - p$. The *characteristic mapping* of F is the $(q+1)$-linear symmetric mapping

$$\alpha_F: \mathbb{R}^p \times \cdots \times \mathbb{R}^p \to H^{2q+1}(M) \tag{3.1}$$

defined by

$$\alpha_F(e_{i_1}, \dots, e_{i_{q+1}}) = [\xi_{i_1} \wedge d\xi_{i_2} \wedge \ldots \wedge d\xi_{i_{q+1}}]$$

where $\xi = \{\xi_1, \dots, \xi_p\}$ is any C^r p-coframe adapted to F and $[w]$ denotes the de Rham cohomology class of a closed form w in $\Lambda(M)$. It is shown in [1] that α_F does not depend on the choice of ξ and is $(q+1)$-linear and symmetric. For example: if $m = 3$ and $p = 2$ then $\alpha_F: \mathbb{R}^2 \times \mathbb{R}^2 \to H^3(M) = \mathbb{R}$ is the symmetric bilinear form given by

$$\alpha_F(e_i, e_j) = \int_M \xi_i \wedge d\xi_j, \quad 1 \leq i, \ j \leq 2$$

Let $A^r(\mathbb{R}^p, T^n, \mathcal{E}_p), n = p + q$ be the space of all non-singular $C^r, r \geq 2$ actions of $\mathbb{R}^p$ whose underlying foliation is a linear foliation $\mathcal{E}_p$ of T^n. Here we prove

3.1 Theorem. *If $\mathcal{E}_p$ is a minimal linear foliation of $T^n, n = p + q$, then $\alpha_F = 0$ for every action F in $A^r(\mathbb{R}^p, T^n, \mathcal{E}_p)$.*

Proof. Let $\mathbb{Z}^{01}(T^n, \mathcal{E}_p)$ denote the linear subspace of all d_e-closed 1-forms in $\Lambda^{01}(T^n, \mathcal{E}_p)$. In ([1], 1.11) it is shown that the $(q+1)$-linear mapping

$$\beta: \mathbb{Z}^{01}(T^n, \mathcal{E}_p) \times \ldots \times \mathbb{Z}^{01}(T^n, \mathcal{E}_p) \to H_{DR}^{2q+1}(T^n)$$

given by

$$\beta(\eta_1, \dots, \eta_{q+1}) = [\eta_1 \wedge d\eta_2 \wedge \ldots \wedge d\eta_{q+1}] \tag{3.2}$$

is continuous in the C^1-topology.

To prove 3.1 we show that β is the zero mapping.

It follows from (2.1) and proposition 2.4 that

$$\eta_j = P^*\eta_j^0 + \eta_j^1 + d_e h^j \tag{3.3}$$

where η_j^0 is closed in $\Lambda^1(T^p), \eta_j^1 \in \mathbb{Z}^{01}(\ker)$ and

$$\eta_j^1 = \lim_{n \to \infty} d_e h_n^j$$

in the C^∞ topology, $1 \leq j \leq q+1$. Actually by Theorem 2.2 i) $\eta_j^1 = 0$ if $\mathcal{E}_p$ is diophantine. Thus, by the continuity of β, it suffices to consider

$$\eta_j = P^*\eta_j^0 + d_e h^j, \quad 1 \leq j \leq q+1 \tag{3.4}$$

In this case we have

$$\beta(\eta_1, \dots, \eta_{q+1}) = [d_e h^1 \wedge dd_e h^2 \wedge \ldots \wedge dd_e h^{q+1}] \tag{3.5}$$

Since $dd_e h^j \in \mathrm{I}(\mathcal{E}_p), 1 \leq j \leq q+1$, and $\mathrm{I}(\mathcal{E}_p)^{q+1} = 0$, it follows that

$$d_e h^1 \wedge dd_e h^2 \wedge \cdots \wedge dd_e h^{q+1} = d(h^1 dd_e h^2 \wedge \cdots \wedge dd_e h^{q+1}) \tag{3.6}$$

From (3.5) and (3.6) we see that $\beta(\eta_1, \dots, \eta_{q+1}) = 0$ when the η_j's are as in (3.4). This proves the theorem.

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