In memory of Heinz Hopf

by

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Heinz Hopf died on June 3 at the age of 76, after a long illness. From 1931 to 1965, he was Professor of Mathematics at the *Eidgenossische Technische Hochschule.* In Zollikon, where he lived, the friendly, kind and modest scientist was well known; yet how many knew of his world-wide scientific reputation? It isn't only his relatives, friends, colleagues and students who now mourn for him, but also the whole mathematical world who deeply admired him for his work. Heinz Hopf was one of the most important mathematicians of our time. His name is well known in all mathematical centers as a research worker and teacher whose work strongly influenced science and will survive in the cultural heritage of coming generations.

Heinz Hopf was born on November 19, 1984 in Breslau. There he spent his youth and began his mathematical studies, interrupted shortly after, during the four years of the First World War, when he was at the Front. With yet greater intensity, he resumed his studies after the war at the universities of Berlin, Heidelberg and Göttingen. He got his degree in Berlin with Erhard Schmidt, to whom he owed his first and most enduring scientific stimulus. In 1925 in Göttingen he met the Russian mathematician Paul Aleksandrov, with whom he established an intimate friendship of life – long duration, both on its human and its scientific sides. They spent the year 1927 together at Princeton on fellowships from the Rockefeller Foundation. The papers of Hopf which appeared at that time were received with great enthusiasm; through them Topology became a new branch of Mathematics occupying an increasingly central position. Aleksandrov and Hopf registered their ideas about the new field in an important book on the subject.

In the year 1931, the ETH invited the young *Privatdozent* to a professorship, where he succeeded Hermann Weyl and revealed himself up to his illustrious predecessor. We cannot help but admire the vision and courage of his collegues and the administration for this choice, which was of the greatest importance for the future of the Zürich mathematical school. Heinz Hopf remained faithful to the ETH for the rest of his life. He devoted himself with like intensity and passion to training the new generation and to research. And there is hardly anybody in all Switzerland connected with research or teaching, at the secondary or university level, who was not directly or indirectly influenced by him. Hopf trained numerous mathematicians or was for them a source of inspiration. A surprising number of these teach at universities and not only in Switzerland, but throughout the world. He was many times Visiting Professor at such places as Princeton, New York, Stanford, Berkeley, Rome and many others. Already in 1947, on the occasion of the 200th Anniversary, Princeton University awarded him an *honoris causa* Doctor's degree – and this was the first of a series of high distinctions conferred on him by the scientific world, at which occasions his great influence on contemporary scientific thought was emphasized. We mention here only that he was a member of the National Academy of Sciences, USA, of the Accademia dei Lincei, doctor *honoris causa* from the Sorbonne and holder of the Lobatschewski Prize from the University of Moscow. From 1955 to 1958 he was President of the International Mathematical Union.

In 1964, on the occasion of this 70th birthday, the ETH published a volume of "Selecta" as a homage, which contained a selection of his papers chosen by the author himelf. This book unveils a perpective not only of his work but also of several lines of development of Mathetics in the last decades. Each of these works, in view of their depth and originality, opened up new theories which now dominate most fields of Mathematics. The book contains also a complete list of Heinz Hopf's publications; their number is suprisingly small. With moderation most unusual now-a-days – he chose deep problems with a sure instinct, letting them ripen till suddenly a solution would be given in which new ideas and new methods are to be found. With this same tranquillity, this same control and coherence which characterized his whole scientific life, he proceeded from one truth to another.

In his mathematical work Heinz Hopf devoted himself mainly to Topology, a field formerly called Analysis Situs. This discipline is frequently described as "Continuous Geometry"; it refers to those types of geometrical arguments in which spatial concepts are treated not in the sense of measure, movement, similarity, etc., but rather with respect to the notions of neighborhood, continuity and deformation. Here space is, to begin with, the space of our everyday experience, that

is, three dimensional Euclidean space in its mathematical abstraction. In this setting we already find an abundance of intuitive material and difficult problems (links, knots, closed surfaces, vector fields, flows, fixed points, etc.) The spaces which mathematicians study (higher dimensional spaces, function spaces, phase spaces, etc.) are generalizations of the previous ones in which analogous notions are presented. These spaces appear in Álgebra, in Analysis and in the varied applications to Mechanics, Physics and Technology. Thanks to the analogies with three dimensional space, the full force of geometrical intuition can also be applied to these abstract spaces, though naturally rigorous proofs must be given independently of intuition. In spite of this, the process of "geometrization", which in particular led to the notion of topological structure, represents an essential step in the development of the subject since the beginning of the century, as important heuristically as it is conceptually.

The first papers of Hopf, on problems suggested by Erhard Schmidt, follow the line of Brouwer's famous theorem on the topological invariance of the dimension of Euclidean spaces (1912), and also important results of Poincaré on Analysis Situs. Hopf dealt with mappings of n-dimensional manifolds, vector fields on manifolds and "Curvatura integra" of closed hypersurfaces. The most significant aspect of this work – apart from the remarkable power of understanding and insight of the young mathematician is the "algebraization" of the problems. Certainly this reduction of geometrical phenomena to algebraic (i.e. calculable) relations already existed in embryonic form in the earlier stages of Analysis Situs from Euler (in its simplest aspect, in the form of Euler's theorem on polyhedra) to Poincaré; however, it is only in a work like "A generalization of the Euler-Poincaré formula" by Hopf that it finally appears in its definite algebraic form, formalized in the concept of homology group.

This grasping of topological properties through auxiliary algebraic methods was decisve for Topology and constituted a critical turn in modern mathematics. The true starting point came only in 1925, while Hopf was in Gottingen under the influence of Emmy Noether and her school. The concept of abstract algebraic structure was then being formulated, in the form in which it has become current in mathematics, having penetrated even into elementary teaching. Its success was impressive. Hopf realized that the decomposition of the classical mathematical concepts into their algebraic and topological components would be specially fruitful wherever intuitive notions and rigorous results, originating from different sources, can be applied to one another. In this manner, there appeared Algebraic Topology in its modern sense, first intensively developed by Hopf himself, and then by a growing number of his students and followers. Besides its own successes, Topology became very important because through the cross-fertilization of algebraic and topological ideas there arose through this remarkable interaction a whole series of new fields which have become now-a-days practically essential in all branches of mathematics and its aplications. These go from number theory to the theory of nets, from complex analysis to theoretical physics. We mention only a few of these new fields: homological algebra, sheaf theory, complex analytic spaces, fibrations and vector bundles, homotopy theory, categories and functors. In general, we are not aware of their origins; much of it can be traced back to the work of Hopf himself. His discovery of the essential maps of spheres into spheres of lower dimension was the starting point of the theory of homotopy groups and fibrations; the relations between the fundamental group and the homology groups of a space led to homological algebra; his famous paper on the global properties of Lie groups led to the theory of "Hopf algebras".

In this famous work, in a single stroke there were clarified the nature of certain global properties of Lie groups (such as, for example, the classical groups: rotation groups, unitary groups, etc.) which were earlier verified by explicit computations. The deep reason behind these properties lies in the structure of algebras which admit a "co-product", now known as "Hopf algebras". This is a central and fertile idea. One would like to compare the explanation of these regularities with Newton's law which, in a single stroke, explained the already well known behavior of planetary orbits and much else besides. Finally we have to mention the great achievements of Heinz Hopf in global differential geometry: with the algebraic-topological notion of global differential concepts, it became possible to connect differential properties, i.e. local and infinitesimal properties such as curvature and length, tensor fields, with global properties of the underlying manifold. Hence recognition that local solutions are only possible if they satisfy certain global properties that are fundamental to the problem as a whole. Shouldn't one be tempted to see here a parallel with the most difficult problems of our world?

Certainly we cannot attribute all these developments to a single person; Hopf would have energetically rejected this. This would contradict one of the tendencies of Mathematics, which might be defined as collaboration.

The trend of Hopf's ideas stood in interaction with many others, who contributed and received from him, one should be here excused for mentionning no other names; their number is so big, if for no other

reason that he had unusually many students who carried on with the development of his ideas. He was particularly glad when they brought him new ideas and perspectives that he had not thought of. And also because of his intensive international exchange of ideas which fertilized many other schools, which trailed quite different routes, contributing to their success. In a report elaborated by Hopf in 1966, coverring the period up to 1942, he says: "Here I stop. At this time it seems to me that the pre-history of modern topology was finished; history began. A young generation came into the scene. Successfully, they carried forward our old ideas and more than this, with their own new ideas they solved many of the old problems and they gave an unexpected *façade* to Topology". Of course, one can doubt whether the "heroic" times before 1942, should be considered as pre-history.

In an analysis of this work – brief as it may be – several thoughts impose themselves. One aspect is related to the concept of structure. Besides an extraordinary geometrical intuition, it is characteristic of the works of Hopf the confrontation of the algebraic and topological aspects of the problem. The isolation of the divers algebraic, topological or other structures for independent observation is now commonplace; this has penetrated under the guise of 'modern mathematics" even into elementary teaching. This discrimination of divers structures leads to the desired conceptual clarification and by the synthesis of several structures there arise complicated mathematical theories, including those of classical mathematics. The passage from one structure to another, which is characteristic of Algebraic Topology, finds a generalization of extraordinary relevance in the theory of categories and functors, where apparently very different structures reveal themselves similar or dual to one another. We should consider, however, that the first papers of Hopf appeared at a time when the structural approach (axiomatic) was in its beginning; one of the greatest revolutions in mathematical thought, and perhaps also in all intellectual disciplines, was under preparation; classical mathematics based on integers, real numbers and space was replaced by the modern version based on the concept of structure. From the historical standpoint the achievements of Hopf should be valued even more; in several ways they anticipated this development and also contributed essentially to it by exhibiting "ad oculos' the full force of the new abstract methods in producing efficient solutions to concrete problems.

Thus we arrive at a second aspect, the conspicuous fact that Heinz Hopf always concerned himself with particular and explicit problems, which in the context of his ideas must be considered "concrete" (vector fields on spheres and manifolds, prime ends of spaces and groups, essential mappings, etc,). In a form which is certainly difficult to unravel – this belongs to the field of heuristcs – he always gave the solution to the particular problem and simultaneously created a method to deal with it, which clarified the guiding idea, the *deep reason* behind it and ulterior possibilities. Who could decide whether he first arrived intuitively at the solution and then extracted the idea for the abstraction and the method or whether he first hit upon the method which led to the solution by analogies and trials? Certainly both aspects must have occurred. Surely, if we consider the depth of the problems posed and the intrinsic necessity of new methods, something would have to come up come up embodying the force of generalization and development. The abstraction thus attained opened new pathaways to knowledge, deeper understanding and applications; it generated new problems which, in turn, are concrete in their context – and the whole process begins again.

However, the "old" concepts. problems and viewpoints are not thereby surpassed. On the contrary, they may reappear in a new light and suddenly become very actual, in the same way in which new abstractions may produce unexpected applications, which could not be attained before or for which there did not exist adequate algorithms. Such developments can hardly be foreseen; they show how imprecise and transitory really are the boundaries between pure and applied mathematics.

Isolated structural concepts, however elegant and convincing they may be, without roots in some legitimate, concrete problem will hardly do justice to the essence of mathematics., The same is true of the simple holding on to classical objects and immediate applications without conceptual motivation. Mathematical works derive their force from the depth of the underlying problems and the conceptual clarification which they seek; from the intuitive understanding of its solution, from the struggle for its formulation within the rigorous limits of the abstraction which is adequate for them. In the works of Hopf, this is very clearly apparent, for they were written in an admirably lucid style which rendered simple the most complicated things.

The extraordinary influence of the scientific work of Heinz Hopf must not be attributed exclusively to his great gifts, but also to certain singular traits of his personality. It was marked by such traits as human warmth, modesty and rectitude, by serious objectivity and a sense of humour; everyone liked him. Everywhere one felt his peculiar manner: in teaching, to all aspects of which he dedicated himself with affectionate care; in his manner of expounding mathematics, clear and lively, free from every pose; in the manner of advising his students and in the manner in which he could both listen and pose relevant question;

and last but not least in the manner in which he could deal with human problems, always available with his sincere advice. His home, created by his wife in Zollikon, was for him an ideal place in which to live and work; but it was also a meeting place where friends and colleagues from near and far away would come together, where his students would receive their first suggestions for research and where they would celebrate their doctorate and introduce their fiancées. There in the garden or walks through the forest, arose plans ideas and theorems; these personal relationships developed in which not only advice, criticism and encouragement were given, but also a beautiful attitude toward life. A happy word of recognition or a delicate skepticism expressed were often sufficient to provoke decisive changes. In the period before and during World War II, many fugitives and victims of perscution found in his house disinterested help and hospitality. The long illness of his wife, Mrs. Anja Hopf, extended a somber shadow over their home. Even so the strands of friendship and collaboration continued to converge there.

Hopf's conception of life and of man cannot be separated from his conception of science. He was fascinated by the spiritual communion without which there is no Mathematics, a communion which will connect strangers in a most unexpected way. It binds them in the search for intuitive understanding and the joy before the unknown, the courage in face of the new, but also in the struggle for mathematical rigour whose setting must be constantly amplified. In such traits he saw the model for each human spiritual enterprise and the path toward mutual understanding. It was also this conviction that led him to accept in 1955 the position of President of the International Mathematical Union, which he exercised in the foregoing sense. Thanks to his popularity and his rectitude, he succeeded in transforming the Union into a truly worldwide organization, which like him personnally, established scientific and human contacts with the new generations of research workers over all frontiers and between all continents.

Like other scientific societies, the Mathematical Union was at that time under reconstruction; undoubtedly it owes much to Heinz Hopf for the role it plays in the world today. He was a citizen of the world in the best sense of the word. However, it was in Switzerland, in Zürich at the ETH that he was really at home. Here was the atmosphere in which he wanted to live and accomplish his mission. We are grateful for that. His friendship, his work, the image of his personality will remain alive, though we had to part full of grief.