Immersions of manifolds with non-negative sectional curvatures

by

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DEDICADO A BENEDITO DE MORAIS

1. We give a proof of the following

Theorem: *Let* $x : M \to \mathbb{R}^{n+1}$, $n > 1$, *be a* C^∞ *isometric immersion, in euclidean space* $\mathbb{R}^{n+1}$ *of an n-dimensional, complete, orientable,* C^∞ *Riemannian manifold M, whose sectional curvatures are non-negative and, at one point, are all positive. Then*:

i) *M is either homeomorphic to a sphere* S^n *or to* $\mathbb{R}^n$,

ii) $x(M) \subset \mathbb{R}^{n+1}$ *is the boundary of a convex body in particular x imbeds M topologically as a closed subset of* $\mathbb{R}^{n+1}$,

iii) *For almost all points* $v \in S^n \subset \mathbb{R}^{n+1}$, *the hyperplanes which are normal to v intersect x(M) in a set which, when non-empty, is either a point or homeomorphic to* S^n.

iv) *The total curvature of* $x(M) \subset \mathbb{R}^{n+1}$ *is either* 2π *(if M is compact) or* $\leq \pi$.

If, in particular, M is non-compact and all sectional curvatures are positive, then:

v) *The normal map* $v : M \to S^n$ *is a diffeomorphism onto an open set contained in a hemisphere of* S^n

vi) *A point* $v_0 \in S^n$ *can be so chosen that x(M) is the graph of a convex function defined on a set contained in a hyperplane normal to* v_0; *in particular, the volume of x(M) is infinite.*

i) and (ii) follow from a paper of Sacksteder [9] combined with a result of Heijenoort [7], and generalize previous result of Hadamard [6], Stoker [10] and Chern-Lashof [5]. We will say a few words about Heijenoort-Sacksteder proof.

According to Heijenoort, $x(M) \subset \mathbb{R}^{n+1}$ is called *locally convex* at *p* if there exists a neighborhood *U* of *p* such that $x(U)$ is contained in one of the closed half-spaces *H* determined by the hyperplane of $x(M)$ at $x(p)$. If, in addition, $x(U)$ has only one common point with the boundary of *H*, then $x(M)$ is called *strictly locally convex* at *p*. Heijenoort proved in [7] that if $x(M)$ is locally convex everywhere, strictly locally convex at some point and complete, then $x(M)$ is (globally) convex. Local convexity (strict local convexity) implies that the eigenvalues of the second fundamental form of the immersion x are non-negative (positive), which by its turn is equivalent to the fact that the sectional curvatures of M are non-negative (positive).

The following example, pointed out by Sacksteder, shows that the converse is not true. Let $z = x^3(1 + y^2)$ be a surface defined in the neighborhood $y^2 < 1/2$ of $(0,0)$. It is easily seen that the curvature is non-negative in this neighborhood and the surface is not locally convex at $(0,0)$.

Sacksteder proved in [9] the striking fact that the above example cannot coexist with completeness. More precisely, it was proved that if M is complete, has sectional curvatures which are non-negative and, at one point, are all positive, then $x(M)$ is locally convex everywhere and stricly convex at some point, thus reducing the problem to Heijenoort's theorem.

Heijenoort-Sacksteder proof is rather long. We present here an independent proof. It should be remarked that Sacksteder proves in [9] a more general result, but we will not go into that.

(iii) - (vi) generalize results of Stoker [10] (for $n = 2$, M non-compact and positive curvature) and were obtained independently by H. Wu [11]. We are much indebted to Wu for pointing us that lemma 3 below is not true without the closure condition. It should also be remarked that a version of (vi) can be obtained in the general hypothesis of the theorem. We refer to Wu's paper [11] for details.

This paper is a reorganization of an earlier preprint which was critically read by F. Warner. Thanks are also due to H. Wu for drawing our attention back to the subject.

2. Manifolds are supposed to be connected, unless explicitly stated, and differentiable means C^∞. We denote by $S^n \subset \mathbb{R}^{n+1}$ the sphere of unit vectors of $\mathbb{R}^{n+1}$. The choice of an orientation for M defines, for each $p \in M$, a unique normal unit vector $\nu(p)$ of $x(M)$ at $x(p)$, and this gives rise to an obvious (called *normal*) *map* $\nu : M \to S^n$. Whenever we speak of a normal map, we assume that an orientation of M has been chosen. The *total curvature* of $x(M)$ is the integral over M of the absolute value of the determinant of $d\nu(p)$, $p \in M$.

For each $v_0 \in S^n$, we define a *height function* $h : M \to \mathbb{R}$ by $h(p) = \langle x(p), v_0 \rangle$, $p \in M$, where $\langle\, ,\rangle$ is the usual scalar product of $\mathbb{R}^{n+1}$.

A *level surface* of h is a connected component of the set $\{p \in M;\ h(p) = \text{constant}\}$. We denote by grad h the vector field in M defined by

$$\langle \operatorname{grad} h(p), v \rangle_p = dh(p)\cdot v, \qquad v \in M_p,$$

where M_p denotes the tangent space of M at $p \in M$, and $\langle\, ,\rangle$ is the inner product in M_p given by the Riemannian metric in M.

A *trajectory* of grad h is a curve $\varphi(t)$ in M with $\varphi'(t) = \operatorname{grad} h(\varphi(t))$; a trajectory is *issuing form* (respect. *going into*) a point $p \in M$ if there exists a neighborhood $V \subset M$ of p such that $\varphi(0) \in V$ and $\lim_{t\to-\infty} \varphi(t) = p$ (respect. $\lim_{t\to\infty} \varphi(t) = p$).

Lemma 1: Assume the hypothesis of the theorem. A point $p \in M$ is a critical point of the height function $h = \langle x, v_0 \rangle$, $v_0 \in S^n$, if and only if v_0 is a normal vector at $x(p)$. If v_0 is a regular value of ν, all critical points of h are non-degenerate and are either maxima or minima.

Proof: The first statement comes immediately from the relation

$$dh(p)\cdot v = \langle dx(p)\cdot v, v_0 \rangle = 0,$$

for all $v \in M_p$. Observe that the hessian of h at a critical point $p \in M$

$$d^2 h(p) = \langle d^2 x(p), v_0 \rangle \tag{1}$$

is the second quadratic form of $x(M)$ in the normal direction v_0. Since the determinant of this quadratic form is, except for a sign, the determinant of the differential $d\nu$ at p, we conclude that p is a non degenerate critical point.

Of course, the above conclusions do not depend on the hypothesis on the curvature. Now, because the sectional curvatures of M are non-negative, we see that the second member of (1) maintains a fixed sign for all vectors of M_p. From this and the fact that p is a non degenerate critical point, it follows that p is either a maximum or a minimum, which finishes the proof.

The following lemma was proved in [2].

Lemma 2: Let $x : M \to \mathbb{R}^{n+1}$ be an isometric immersion of an n-dimensional complete riemannian manifold. Let $h(p) = \langle x(p), v \rangle$, $v \in S^n$, be a height function on M and let $\varphi(t)$ be a trajectory of grad h. Then $\varphi(t)$ is defined for all $t \in (-\infty, \infty)$.

Proof: We first observe that $\|\operatorname{grad} h\| = 1$. In fact, since x is a local isometry

$$\frac{d}{dt}(h \circ \varphi(t)) = \langle dx(\varphi'(t)), v\rangle = \langle dx(\text{grad } h),\ v\rangle = \langle \text{grad } h, v\rangle ;$$

on the other hand, by definition of grad h,

$$\frac{d}{dt}(h \circ \varphi(t)) = dh(\varphi(t)) = dh(\text{grad } h) = \|\text{grad } h\|^2 .$$

Hence $\|\text{grad } h\|^2 = \langle \text{grad } h,\ v\rangle$ and the assertion follows.

Now suppose that $\varphi(t)$ is defined for $t < t_0$ but not for $t = t_0$. Then there exists a sequence $\{t_i\}$, $i = 1, \ldots, n, \ldots$ converging to t_0 such that $\{\varphi(t_i)\}$ does not converge. Since $\|\text{grad } h\| \leq 1$, we obtain

$$d(\varphi(t_i),\ \varphi(t_j)) \leq \int_{t_i}^{t_j} \|\text{grad } h(\varphi(t))\|\, dt \leq |t_i - t_j| ,$$

where d is the distance in the intrinsec metric of M. If follows that $\{\varphi(t_i)\}$ is a Cauchy sequence, and this contradicts the completeness of M q.e.d.

The following lemma was proved in [4]. A proof in the more general context of continuous convex surfaces can be found in [11].

Lemma 3: *Let* $x\colon M \to \mathbb{R}^{n+1}$ *be an isometric immersion of an n-di-mensional complete and orientable riemannian manifold M. Assume that* $x(M) \subset \mathbb{R}^{n+1}$ *is the boundary of a convex body. Then the closure* $\overline{v(M)}$ *of the image of the normal map* $v: M \to S^n$ *is convex in* S^n.

Proof: Consider first the case where $v_0 = v(p_0)$, $v_1 \in v(p_1)$ are two non antipodal points of S^n. Let v be a point in the smallest arc of the sphere joining v_0 and v_1. Consider the height function $h(p) = \langle x(p), v\rangle$. Since $x(M)$ is contained in the convex intersection of the half-spaces bounded by the (non-parallel) tangent hyperplane at $x(p_0)$ and $x(p_1)$, there exists a hyperplane H normal to v such that $x(M)$ is in the same side of H. Thus h is bounded.

Now let $p \in M$ and $\varphi(t)$ be the trajectory of grad h with $\varphi(0) = p$. By lemma 2, $\varphi(t)$ is defined for all $t \in (-\infty, \infty)$. We claim that $\|\text{grad } h(\varphi'(t)\|$ is not bounded away from zero in $[0, \infty)$. Otherwise, we have

$$\lim_{t\to\infty} h(\varphi(t)) - h(\varphi(0)) = \int_0^{\infty} \frac{d}{dt} h(\varphi(t))\, dt = -\int_0^{\infty} \|\text{grad } h(\varphi(t))\|^2\, dt = -\infty ,$$

which contradicts the fact that $h(\varphi(t))$ is bounded below.

It follows that either there is a critical point $\varphi(t_0)$, of h, $t_0 \in [0, \infty)$, and then the normal vector at $\varphi(t_0)$ is v, or there exists a sequence of points in M whose normal vectors converge to v. In any case, v belong to the closure of $v(M)$.

We now consider the remaining cases. If some or both v_0 and v_1 belong to the boundary of $v(M)$ and are not antipodal, we take the limit of minimal geodesics joining v_n^0 to v_n^1, where $\{v_n^0\} \to v_0$, $\{v_n^1\} \to v_1$ and no pair v_n^0, v_n^1 is antipodal. If v_0 and v_1 belong to $\overline{v(M)}$ and are anti-podal, we take a third point $v_2 = v(p_2)$ $v_2 \neq v_1$, $v_2 \neq v_0$ (which exists because M is connected). By the previous argument, the minimal geo-desics $\widehat{v_0 v_2}$, $\widehat{v_1 v_2}$ belong to $\overline{v(M)}$. Taking sequences of points on $\widehat{v_0 v_2}$ and $\widehat{v_1 v_2}$, which converge to v_0 and v_1, respectively, and considering the minimal geodesics joining these points, we easily see that $\widehat{v_0 v_1}$, belong to $\overline{v(M)}$, q.e.d.

The following proposition shows that the theorem is true if M is assumed to be compact.

Proposition 1: *Let* $x: M \to \mathbb{R}^{n+1}$, $n > 1$, *be an isometric immer-sion of an n-dimensional, compact, orientable, riemannian manifold M, with the property that the second quadratic forms are semi-definite. Then* $x(M)$ *is boundary of a convex body and the total curvature of* $x(M)$ *is equal to* 2π; *in particular M is homeomorphic to a sphere.*

Proof: The proof is easy and is essentialy contained in [3]; for sake of completeness, we will give the details.

By compactness of $x(M)$, there exists a point $r \in M$, such that the second quadratic form of $x(r)$ is definite. This means that $dv(r)$ is non singular and, by Sard's theorem, there is a point $p \in M$ is a neighborhood of r, such that $v(p) = v_0$ is a regular value of v. It follows from lemma 1 that p is a non degenerate critical point of the height function $h = \langle x, v_0\rangle$, and it is either a maximum or a minimum.

For definiteness, let us assume that $h(p)$ is a minimum. Let $\varphi(t)$ be a trajectory of grad h issuing from p. By compactness, $\varphi(t)$ is defined for all $t > 0$ and h is bounded above. It follows that $\|\text{grad}\|$ is not bounded away from zero on $\varphi(t)$, $t > 0$. Therefore there exists a critical point q in the (compact) closure of the trajectory i.e., $\lim_{t\to+\infty} \varphi(t) = q$.

Now, let S be a level surface of h sufficiently closure to p. Let $A \subset S$ the set of points in S which are intersections of trajectories of grad h issuing from p and going into q. By continuity and the fact that p and q are not saddles, A is an open set in S. On the other hand, a trajectory which issues from p and intersects S in a point belonging to the com-plement of A, goes into a critical point, say r. By the above argument, the complement of A is open in S. Since S is connected, $A = S$. By a similar argument, these trajectories cover an open and closed subset of M, hence the entire manifold M. Therefore h has exactly two critical points p and q.

It follows that the inverse image of the regular values of ν contain only one element. The degree of ν is then ± 1, and the proposition follows from Theorem 4 of [5].

Proposition 2: Let $x : M \to \mathbb{R}^{n+1}$ *be an isometric immersion of an n-dimensional riemannian manifold M (not necessarily connected) complete and orientable in each connected component. Assume that* $x(M) \subset \subset \mathbb{R}^{n+1}$ *is contained in no hyperplane of* $\mathbb{R}^{n+1}$ *and, for each* $p \in M$, $x(M)$ *is entirely contained in one of the closed half-spaces bounded by the tangent hyperplane* $dx(p)(M_p) = T_p$. *Then* $x(M)$ *is the boundary of a convex body. If, in addition, the second quadratic form of x is definite at some point of M, then x is a homeomorphism and the total curvature of* $x(M)$ *is either* 2π *(if M is compact) or* $\leq \pi$.

Proof: The intersections of all closed half-spaces, bounded by T_p, $p \in M$, and containing $x(M)$, is a closed convex set K of $\mathbb{R}^{n+1}$. Because $x(M)$ is contained in no hyperplane of $\mathbb{R}^{n+1}$, K contains interior points, hence is a convex body. Clearly $x(M) \subset K'$ the boundary of K. We want to show that $x(M) = K'$.

By the classification of boundaries of convex bodies ([1], p. 3), K' is either connected or the union of two parallel hyperplanes; in the latter case, there are points of $x(M)$ in both components of K'. $x(M)$ is clearly open in K'. We will show that $x(M)$ is closed in each component of K'.

We first consider K' connected and remark that as the boundary of a convex body, K' has an intrinsic metric defined as the infimum of the arc length of all rectifiable curves in K', joining points of K'. It is clear that this metric is complete and it is an elementary fact ([1], p. 78) that it is topologically equivalent to the metric induced in K' by $\mathbb{R}^{n+1} \supset K'$. Since K' is complete in the intrinsic metric, given two points $x(p)$, $x(q) \in K'$, $p, q \in M$, there exists a segment (shortest geodesic) γ in K' joining them ([1], p. 79). Because x is a local isometry and $x(M)$ is open in K', there is a neighborhood V of p in M such that $(x/V)^{-1} \circ \gamma$ is a geodesic in M; here x/V is the restriction of x to V. If this "lifting" cannot be extended to the whole geodesic γ, there exists a geodesic in M which cannot be defined for all values of the parameter, and this contradicts the completeness of M. It follows that γ is entirely contained in $x(M)$, and is a segment in $x(M)$. Therefore $x(M) \subset K'$ is a metric subspace of K'. Since $x(M)$ is complete, it is closed in K'. Because K' was assumed to be connected, we have $x(M) = K'$.

If K' is not connected, we repeat the above argument for each connected component, which proves the first part of the proposition.

To prove the second statement, we observe that the hypothesis on the second quadratic form implies that K' is neither a cylinder nor the union of two parallel hyperplanes. By the classification quoted above, K' is then simply connected. On the other hand, since $x : M \to K'$ is a local isometry onto K', and M is complete, x is a covering map ([6], p. 74). It follows that x is a homeomorphism. The assertion on the total curvature of $x(M)$ is then a consequence of lemma 3, and this finishes the proof of proposition 2.

3. In this section we prove part (i) of the theorem, which will be a simple consequence of proposition 3 below. The proof of proposition 3 has some points of contact with [8]; the proposition itself will also play a vital role in the proof of the remaining items of the theorem.

Proposition 3: Assume the hypothesis of the theorem and let $\nu(p) = \nu_0$, $p \in M$, *be a regular value of* ν. *Then either the height function* $h = \langle x, \nu_0 \rangle$ *has only one critical point p, and the trajectories of* grad h *issuing from p cover M, or h has two critical points p and q, and the trajectories of* grad h *issuing from p go all into q and cover M. In any case, the hyperplanes which are normal to* ν_0 *intersect* $x(M)$ *in a set which, when nonempty, is either a point or homeomorphic to* S^n.

Proof: By lemma 1, h has non degenerate critical points which are either maxima or minima, and p is a critical point of h. For definiteness, let us assume that p is a minimum and that $h(p) = 0$.

We shall say that a level surface S_λ of h, where λ denote the value of h at S_λ, is *normal* at p if the following conditions are satisfied:

1) S_λ is homeomorphic to a sphere and bounds an open region $\Sigma_\lambda \subset M$ containing p as the only critical point in Σ_λ.

2) There exists a homeomorphism θ from the closed ball $B_\lambda = \{x \in \mathbb{R}^n ; \|x\| \leq \lambda\} \subset \mathbb{R}^n$ onto the closure $\overline{\Sigma}_\lambda$ of Σ_λ, such that the image of the sphere $\{x \in \mathbb{R}^n ; \|x\| = \alpha\}$, $\alpha \leq \lambda$, by θ is a level surface S_α.

Σ_λ is then called a *normal region* of p and λ a *normal value* of h relative to p.

We observe that the level surface of h near p are normal at p. From (2), it follows that if a level surface S has one point in a normal region Σ, then $S \subset \Sigma$, and S is a normal level surface.

It is also easy to see, using the trajectories of grad h and the fact that the critical points of h are isolated, that if S_λ is a normal level surface, which contains no critical point of h, there exists a normal level surface S_{λ_1}, $\lambda_1 > \lambda$.

Now, let Σ be the union of all normal region of p. Σ is a non void open subset of M, and we have the following possibilities: a) $\Sigma = M$; b) $\Sigma \neq M$, and the boundary Σ' of Σ contains a critical point q of h; c) $\Sigma \neq M$, and Σ' contains no critical point of h.

Suppose that (a) *holds.* Then all level surfaces of h are normal at p and p is the only critical point of h in M. Let $r \neq p$ be an arbitrary point of M. Let S be the level surface of h which passes through r and $\varphi(t)$ a trajectory of grad h, with $\varphi(0) = r$. It is clear, that, for $t \leq 0$, $\varphi(t)$ belongs to the (compact) closure $\bar{\Sigma}$ of the region Σ bounded by S. By compactness $\varphi(t)$ is defined for all $t \leq 0$ and, since h is bounded on $\bar{\Sigma}$, $\|\text{grad}\, h\|$ is not bounded away from zero on $\varphi((-\infty, 0])$. It follows that $\lim_{t \to -\infty} \varphi(t) = p$, i.e., $\varphi(t)$ is issuing from p. Since r is arbitrary, we conclude that M is covered by the trajectories of grad h issuing from p.

Suppose that (b) *holds*: Then, using the above argument and the fact that the level surfaces near q are again homeomorphic to spheres, we obtains that any trajectory of grad h which goes into q is issuing from p. Therefore q is a maximum and, exchanging the roles of p and q, we see that the trajectories of grad h issuing from p go all into q. By the same argument, it is easily seen that these trajectories cover an open and closed set of M, hence the entire manifold M.

Suppose that (c) *holds*: We will show that this assumption leads to a contradiction and, by the above, this will prove proposition 3.

A sketch of the proof is as follows. We first show that Σ' is a union of level surfaces. Next we show that Σ' is not compact and its total curvature (restricting the immersion) is 2π. On the other hand Σ' is arbitrarily close to the normal level surfaces, which are shown to be convex. It follows that Σ' is convex and has a point of strict local convexity. Being non-compact, its curvature is, by Proposition 2, $\leq \pi$, and this is a contradiction.

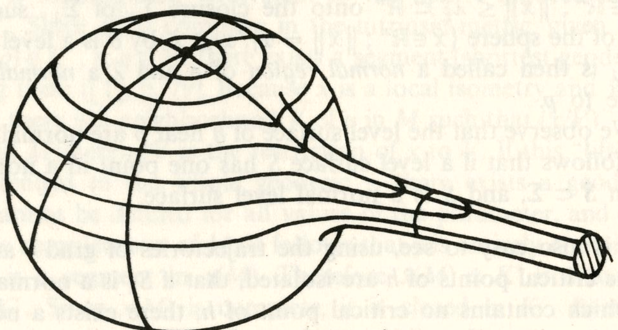

FIG. 1

We now give the details. A good example to keep in mind during the proof is a sphere with an infinite tube of negative curvature "attached" to its side (see preceding figure).

Let λ^* be the supremum of the normal values of h relative to p. We first show that if $q \in \Sigma'$, then $h(q) = \lambda = \lambda^*$. Clearly $h(q) \leq \lambda^*$. Assume that $h(q) = \lambda < \lambda^*$ and consider the level surface S_λ through q. Since λ is a normal value, there exists a normal level surface $S'_\lambda \subset \Sigma$, which contains no critical points. It follows that there is a neighborhood of S'_λ in M, which contains all points of Σ whose levels are close enough to λ, hence, by a limit argument, S'_λ passes through q. By connectedness $S'_\lambda = S_\lambda$, and this contradicts the fact that q is a boundary point of Σ.

Now, let $S_{\lambda*}$ be a level surface passing through $q \in \Sigma'$. We assert that $S_{\lambda*} \subset \Sigma'$. Clearly the set $A \subset S_{\lambda*}$ which has the property $A \subset \Sigma'$, is a closed set in $S_{\lambda*}$. To show that A is open in $S_{\lambda*}$, let $r \in A$. Because r is not a critical point, we can choose a neighborhood V of r in M such that the closure $\bar{V}$ of V contains no critical points, and that all points in V at the level λ^* are in $V \cap S_{\lambda*}$. Now, using the trajectories of grad h, we fill up a neighborhood W of $V \cap S_{\lambda*}$ in M. Since $r \in V \cap S_{\lambda*}$ and $r \in \Sigma'$, there is a point $s \in \Sigma \cap W$ in the trajectory passing through r, and therefore there is a neighborhood U of s, $U \subset \Sigma \cap M$. Projecting U onto $V \cap S_{\lambda*}$ along the trajectories, we find a neighborhood of r in $S_{\lambda*}$ which is entirely contained in Σ'. It follows that A is open and by connectedness $A = S_{\lambda*}$ as we asserted.

From the above we conclude that Σ' is a union of level surfaces at the level λ^*. Observe that if $q \in \Sigma'$, there exists a trajectory $\varphi_q(t)$ of grad h, with $\varphi_q(0) = q$. For small $t < 0$, $\varphi_q(t)$ will certainly intersect a level surface through a point of Σ sufficiently near q, hence $\varphi_q(t) \in \Sigma$. By the argument in (a), it follows that $\varphi_q(t)$ is issuing from p. Due to the absence of critical points in $\bar{\Sigma} - \{p\}$, it follows that we may parametrize such a trajectory in an interval $[\lambda_0, \lambda^*]$, $\lambda_0 \neq 0$, such that $h(\varphi_q(\lambda)) = \lambda$, $\lambda \in [\lambda_0, \lambda^*]$. Since $\varphi_q(\lambda)$ is clearly differentiable in $[\lambda_0, \lambda^*]$ and $\varphi'_q(\lambda)$ is orthogonal to the tangent space of S_λ at $\varphi_q(\lambda)$, it follows that the tangent space of Σ' at q is the limit as $\lambda \to \lambda^*$, of the tangent spaces of S_λ at $\varphi_q(\lambda)$.

Now, let $S_\lambda \subset \Sigma$, $0 \neq \lambda \leq \lambda^*$, be a level surface in Σ, and denote by $y : S_\lambda \to \mathbb{R}^n_\lambda \subset \mathbb{R}^{n+1}$ the restriction of the immersion $x : M \to \mathbb{R}^{n+1}$ to S_λ, where $\mathbb{R}^n_\lambda$ denotes the hyperplane of $\mathbb{R}^{n+1}$ which contains $x(S_\lambda)$. It will be convenient to organize the rest of the proof into some lemmas.

Lemma 4: *The second quadratic forms of the immersion* $y : S_\lambda \to \mathbb{R}^n_\lambda$, $0 \neq \lambda \leq \lambda^*$, *are semi-definite.*

Proof: Let $q \in S_\lambda$. Then

$$\langle dy(q) \cdot v,\ v_0 \rangle = 0$$

and

$$\langle d^2\, y(q) \cdot v,\ v_0 \rangle = 0$$

for any vector $v \in (S_\lambda)_q$, the tangent space of S_λ at q. Let ν be the unit normal vector of $x(M) \subset \mathbb{R}^{n+1}$ at $x(q)$, and ν_y be the unit normal vector of $y(S_\lambda) \subset \mathbb{R}^n_\lambda$ at $y(q) = x(q)$. It is clear that ν, ν_y and v_0 are orthogonal to $dx((S_\lambda)_q)$ and that $\langle v_0, \nu_y \rangle = 0$. It follows that

$$\nu = \alpha \nu_y + \beta v_0 ,$$

where $\alpha = \langle \nu, \nu_y \rangle$ is non zero because q is not a critical point. Therefore, for any $v \in (S_\lambda)_q$,

$$\langle d^2\, y(q) \cdot v,\ \nu_y \rangle = \frac{1}{\alpha} \langle d^2\, y(q) \cdot v,\ \nu \rangle .$$

Since α is a non zero continuous function in the connected set $\overline{\Sigma} - \{p\}$, the lemma follows from the hypothesis of the theorem.

Lemma 5: Σ' *is not compact and the total curvature of* $y(\Sigma') \subset \mathbb{R}^n_{\lambda^*}$ *is equal to* 2π.

Proof: We first consider the case $n > 2$. By lemma 2, the fact that S_λ is compact for $\lambda < \lambda^*$, and proposition 1, it follows that the total curvature of $y(S_\lambda) \subset \mathbb{R}^n_\lambda$, $\lambda < \lambda^*$ is 2π. As an integral, the total curvature is a continuous function of the parameter λ, hence the total curvature of $y(\Sigma')$ in 2π. Moreover a level surface $S_{\lambda^*} \subset \Sigma'$ is not compact; otherwise, by proposition 1, it is homeomorphic to a sphere and, since Σ' contains no critical points there is a normal value $\lambda > \lambda^*$, which is a contradiction. Therefore Σ' is not compact, and this proves the lemma for $n > 2$.

For $n = 2$, proposition 1 cannot be applied and we must argue directly. All we have to show is that, for any circle S_λ, $\lambda < \lambda^*$, the total curvature of the closed plane curve $y(S_\lambda)$ is equal to 2π. This is clearly true near p. On the other hand, the only other possible values are integer multiples of 2π. By continuity, the value must be 2π for all λ, which finishes the lemma.

Lemma 6: *For each* $q \in \Sigma'$, $y(\Sigma') \subset \mathbb{R}^n_{\lambda^*}$ *is entirely contained in one of the closed half-spaces of* $\mathbb{R}^n_{\lambda^*}$ *bounded by the tangent hyperplane* $dy(q)(\Sigma'_q) = T$ *of* $y(\Sigma')$ *at* $y(q)$.

Proof: Project the sets $y(S_\lambda) \subset \mathbb{R}^n_\lambda$, $\lambda \in [\lambda_0, \lambda^*]$, onto $\mathbb{R}^n_{\lambda^*}$, using the common normal v_0, and denote the projected sets by the same letters as before. For each $x(q) \in x(\Sigma')$, denote by $\psi_q(\lambda)$, $\lambda \in [\lambda_0, \lambda^*]$ the *continuous* curve obtained by projecting $x(\varphi_q(\lambda))$ onto $\mathbb{R}^n_{\lambda^*}$.

Now, assume that there are points $x(q_1)$ and $x(q_2)$, $q_1, q_2 \in \Sigma'$, in both sides of T. By continuity and the remarks made before lemma 4, T is the limit as $\lambda \to \lambda^*$, of the tangent spaces T_λ of $x(S_\lambda)$ at $\psi_q(\lambda)$. Therefore, we may choose a $\lambda' \in [\lambda_0, \lambda^*]$, close to λ^*, such that $x(q_1)$ and $x(q_2)$ are in different sides of T_λ, for $\lambda > \lambda'$. Because $\psi_{q_1}(\lambda)$ and $\psi_{q_2}(\lambda)$ converge to $x(q_1)$ and $x(q_2)$, respectively, as $\lambda \to \lambda^*$, there is a $\lambda'' > \lambda'$ such that $\psi_{q_1}(\lambda'')$ and $\psi_{q_2}(\lambda'')$ are in different sides of $T_{\lambda''}$. But this means that $x(S_{\lambda''})$ will have points in different sides of some of its tangent spaces, which contradicts the convexity of $x(S_{\lambda''})$ and proves the lemma.

We now finish the proof of proposition 3. $x(\Sigma')$ is not contained in a hyperplane of $\mathbb{R}^n$; otherwise its total curvature is zero which contradicts lemma 5. By the same reason, the second quadratic form of $y : \Sigma' \to \mathbb{R}^n_{\lambda^*}$ is definite at some point of Σ'. Since Σ' is not compact, and is complete in each connected component, it follows from lemma 6 and proposition 2 that the total curvature $x(\Sigma')$ is $\leq \pi$, which is again a contradiction to lemma 5, and finishes the proof of proposition 3.

The proof of (i) of the theorem now follows easily. Let $r \in M$ be a point where the sectional curvature are all positive. By Sard's theorem. there exists a point $p \in M$, near r, such that $\nu(p) = v_0$ is a regular value of ν (cf. proof of proposition 1). Now applying proposition 3 we can easily construct the homeomorphisms claimed in (i).

Remark: Part (i) of the theorem may also be obtained as a corollary of part (ii), which will be proved in the next section.

4. Before proving part (ii) of the theorem, we need another lemma. Lemma 7 below is an adaptation of an argument in [5] to our non (necessarily) compact case. In fact, use is made of lemma 2 of [5], which is clearly local.

Lemma 7: *With the hypothesis of the theorem, the set of regular values of the normal map* $\nu : M \to S^n$ *is dense in the image* $\nu(M)$. *Moreover, if* $\nu(q)$ *is a critical value of* ν, *there exists a sequence of regular values* $\nu(p_1), \ldots, \nu(p_m), \ldots$ *converging to* $\nu(q)$, *such that the tangent hyperplanes of* $x(M)$ at $x(p_1), \ldots, x(p_m) \ldots$ *converge to the tangent hyperplane at* q.

Proof: Let $\nu(q)$, $q \in M$, be a critical value of ν and V a neighborhood of $\nu(q)$ in S^n. We first show that there is a point $\nu(p) \in V$, such that rank $(d\nu(p)) = n$. We may assume that rank $(d\nu(q)) = k < n$ and it will be convenient to denote by U_l the set of points in M in which rank (dv) is equal to l.

Since rank $(dv(q)) = k$, either there is a neighborhood of q in M contained in U_k or in every neighborhood of q there are points of U_m, $m > k$. Repeating this argument, if necessary, we find, in any case, a point $p_1 \in M$, with $v(p_1) \in V$, and such that a neighborhood W of p_1 is contained in U_m, $m \geq k$. By lemma 2 of [5], the image $x(W) \subset \mathbb{R}^{n+1}$ of this neighborhood is generated by $(n-m)$-dimensional planes, and the normal map in the $(n-m)$-plane π_1, passing through p_1, is constant, hence equal to $v(p_1)$.

We claim that π_1 is not entirely contained in $x(M)$. To see this, let $r \in M$ be such that $v(r)$ is a regular value of v, and let h be the height function $h = \langle x, v(r) \rangle$. By proposition 3, given any point $s \in M$, there exists a trajectory of grad h issuing from r and passing through (or going into) s. It follows that $x(M)$ belongs entirely to one side of the tangent hyperplane π at $x(r)$. If π_1 intersects π, then π_1 is not entirely contained in $x(M)$. If π_1 is parallel to π, it belongs to a level surface of h; by proposition 3, the level surfaces of h are compact, and again π_1 is not entirely contained in $x(M)$, which proves our claim.

It follows that the set of points in π_1 which belong to $x(M)$ has a boundary point which, by completeness, belongs to $x(M)$; denotes it by $x(p_2)$. It is clear that $v(p_2) = v(p_1)$ and that p_2 is a boundary point of U_m. By lemma 2 of [5], $p_2 \in U_m$. Therefore, in every neighborhood of p_2 there are points of U_l, $l > m \geq k$. It follows that there is a point $p_3 \in M$, such that $v(p_3) \in V$ and $p_3 \in U_l$. Proceeding with this argument, we arrive at a point $p \in M$, such that $v(p) \in V$ and $p \in U_n$, which proves the assertion made at the beginning of the proof. We remark that the construction used, makes it possible to choose such a p in a way that the tangent hyperplane of $x(M)$ at $x(p)$ is arbitrarily near to the tangent hyperplane at $x(q)$.

The first statement of the lemma follows immediately if we observe that arbitrarily close to a p whith rank $(dv(p)) = n$, there is a point $r \in M$, the image of which is a regular value of v (cf. proof of proposition 1). The second statement follows from the above remark on the tangent hyperplane at $x(p)$, and this completes the proof of lemma 7.

We now prove part (ii) of the theorem.

Let $p \in M$. If $v(p)$ is a regular value of v, it follows from proposition (cf. proof of lemma 7) that $x(M)$ lies in one side of the tangent hyperplane of $x(M)$ at $x(p)$. If $v(p)$ is a critical value of v, then, by lemma 7, $v(p)$ is the limit of a sequence of regular values $v(p_1), \ldots, v(p_m), \ldots$, and the tangent hyperplane π at $x(p)$ is the limit of the tangent hyperplanes at $x(p_1), \ldots, x(p_m), \ldots$. It follows that $x(M)$ lies in one side of

each of its tangent hyperplanes. Because there exists one point at which the second quadratic form of $x : M \to \mathbb{R}^{n+1}$ is definite, the set $x(M)$ is contained in no hyperplane of $\mathbb{R}^{n+1}$. Now, we apply proposition 2 to obtain (ii) of the theorem, and this finishes the proof.

(iii) and (iv) are immediate consequences of Proposition 3 and 2.

Now let M be non-compact and have positive sectional curvatures. Then all height functions have exactly one non-degenerate critical point. Thus the map $v : M \to S^n$ is injective, hence a diffeomorphism into its image, and the image $v(M) \subset S^n$ is an open set containing no antipodal points. By lemma 3, $v(M)$ is contained in a hemisphere of S^n, and this proves (v).

To prove (vi), take the pole of the hemisphere which contains $v(M)$ and denote it by v_0. We claim that any line l in $\mathbb{R}^{n+1}$ parallel to v_0 intersects $x(M)$ at most once. Assume that l intersects $x(M)$ in two distinct points $x(p_1)$ and $x(p_2)$. Observe that the segment $\omega = \overline{x(p_1)\,x(p_2)}$ belongs to the convex body $K \subset \mathbb{R}^{n+1}$ whose boundary is $x(M)$ and set the convention that normal vectors to $x(M)$ point away from K. Let π_1 and π_2 be the hyperplanes normal to v_0 and passing through $x(p_1)$ and $x(p_2)$, respectively. By convexity, $v(p_1)$ and $v(p_2)$ point to the sides of π_1 and π_2 which do not contain the segment ω. This means that either $v(p_1)$ or $v(p_2)$ do not belong to Σ, which is a contradiction.

This clearly establishes $x(M)$ as the graph of a function f defined in a set contained in a hyperplane π normal to v_0. From the assumptions on the curvature, the hessian of f is positive definite, hence f is convex. f is unbounded in one direction, otherwise by the argument used in the proof of lemma 3, we would have that both v_0 and $-v_0$ belong to the closure of the spherical image, which is a contradiction. From that, it easily follows that the volume of $x(M)$ is infinite.

Remark: It is possible to choose the pole v_0 such that $v_0 \in v(M)$. This will force the function f above to be non-negative. For details, see [11].

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