On isometric immersions of riemannian manifolds

by
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1. It has been long known that given two quadratic forms in an open set $U \subset \mathbb{R}^2$ which satisfy the Gauss and Codazzi-Mainard equations, there exists a unique local immersion $f : V \subset U \longrightarrow \mathbb{R}^3$, with the prescribed forms as induced metric and second fundamental form. The first proof of this fact was given by Ossian Bonnet [2] as early as 1867.

With the advent of multi-dimensional geometry, in the beginning of this century, it became natural to observe that a similar fact holds for immersions with arbitrary codimension. The only new fact is that, besides the induced metric and the second fundamental forms, another assumption should be added, which has to do with the induced connection in the normal bundle. This however introduces some technicalities in the statement and proof of the general result and accounts for different presentations (see [1], [3], [5], [6]) of a theorem, which is otherwise simple and should find its way in courses on Riemannian Geometry.

In this note we present a detailed and elementary proof of this theorem, which uses essentially the theorem of Frobenius. It requires only elementary notions of Riemannian Geometry (§8 and §9 of [4], for example) and a certain knowledge of differential forms. The idea of the proof is essentially due to E. Cartan.

In section 2 we obtain necessary conditions to the existence of an isometric immersion of a Riemannian n-manifold M in Euclidean space $\mathbb{R}^{n+k}$.

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In section 3 we prove that the necessary conditions are also sufficient for the existence of a local isometric immersion $f: U \subset M \to \mathbb{R}^{n+k}$, which is unique up to a rigid motion. The proof depends on two lemmas proved in the same section. Finally we remark that if M is a simply connected Riemannian manifold, the local immersion can be extended to M.

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2. Let M be a Riemannian n-manifold and $f: M \longrightarrow \mathbb{R}^{n+k}$ an isometric immersion of M into euclidean space $\mathbb{R}^{n+k}$. If E is the normal bundle of the immersion, then E has a bundle metric induced from the usual metric in $\mathbb{R}^{n+k}$ Moreover if T is the tangent bundle of M, we can define a connection $D: T \otimes E \longrightarrow E$ and a second fundamental form $S: T \otimes E \longrightarrow T$ by projecting the usual connection $\bar{D}$ in $\mathbb{R}^{n+k}$, onto the normal and tangent spaces of M respectively, i.e.

$$(1) \qquad D_X N = (\bar{D}_X N)^N$$

and

$$(2) \qquad S_X N = (\bar{D}_X N)^T$$

where $X \in T$ and $N \in E$.

The connection D is compatible with the bundle metric and S is self adjoint. In fact

$$(D_X N, N') + (N, D_X N') = ((\bar{D}_X N)^N, N') + (N, (\bar{D}_X N')^N) =$$
$$= (\bar{D}_X N, N') + (N, \bar{D}_X N') =$$
$$(3) \qquad = X(N, N'),$$

and

$$(S_X N, Y) - (X, S_Y N) = (\bar{D}_X N, Y) - (X, \bar{D}_Y N) =$$
$$= X(N, Y) - (N, \bar{D}_X Y) - Y(X, N) + (N, \bar{D}_Y X) =$$
$$(4) \qquad = -(N, [X, Y]) = 0,$$

where we used the symmetry of $\bar{D}$.

We define the second fundamental tensor associated with S to be the homomorphism $B: T \otimes T \longrightarrow E$ given by

$$(5) \qquad (B(X, Y), N) = (S_X N, Y).$$

From (5) we can see that

$$(6) \qquad B(X, Y) = -(\bar{D}_X Y)^N.$$

In fact

$$(B(X, Y), N) = (S_X N, Y) = (\bar{D}_X N, Y) =$$
$$= -(N, \bar{D}_X Y) + X(N, Y) = -(N, \bar{D}_X Y)^N).$$

If we denote the Riemannian curvature tensor by R, we know that

$$(7) \qquad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where $X, Y, Z \in T$ and ∇ is the symmetric connection, compatible with the Riemannian metric on M. We define $\bar{R}$, the curvature of E relative to D, by the equation

$$(8) \qquad \bar{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[X, Y]} N,$$

where $X, Y \in T$ and $N \in E$. With this notation, we get the Gauss equations

$$(9) \qquad R(X, Y)Z = S_X B(Y, Z) - S_Y B(X, Z)$$

and

$$(10) \qquad \bar{R}(X, Y)N = B(X, S_Y N) - B(S_X N, Y)$$

and the Codazzi-Mainard equation

$$(11) \qquad \nabla_X S_Y N - \nabla_Y S_X N - S_{[X, Y]} N = S_Y D_X N - S_X D_Y N.$$

In fact, we know that

$$0 = \bar{D}_X \bar{D}_Y Z - \bar{D}_Y \bar{D}_X Z - \bar{D}_{[X, Y]} Z,$$

hence

$$0 = \bar{D}_X((\bar{D}_Y Z)^T + (\bar{D}_Y Z)^N) - \bar{D}_Y((\bar{D}_X Z)^T +$$
$$+ (\bar{D}_X Z)^N) - (\bar{D}_{[X, Y]} Z)^T - (\bar{D}_{[X, Y]} Z)^N =$$
$$= \bar{D}_X(\bar{D}_Y Z)^T + \bar{D}_X(\bar{D}_Y Z)^N - \bar{D}_Y(\bar{D}_X Z)^T -$$
$$- \bar{D}_Y(\bar{D}_X Z)^T - \bar{D}_{[X, Y]} Z)^T - (\bar{D}_{[X, Y]} Z)^N =$$
$$= \bar{D}_X(\bar{D}_Y Z)^T)^T + (\bar{D}_X(\bar{D}_Y Z)^T)^N + (\bar{D}_X(\bar{D}_Y Z)^N)^T +$$
$$+ (\bar{D}_X(\bar{D}_Y Z)^N)^N - (\bar{D}_Y(\bar{D}_X Z)^T)^T - (\bar{D}_Y(\bar{D}_X Z)^T)^N -$$
$$(12) \qquad - (\bar{D}_Y(\bar{D}_X Z)^N)^T - (\bar{D}_Y(\bar{D}_X Z)^N)^N - (\bar{D}_{[X, Y]} Z)^T - (\bar{D}_{[X, Y]} Z)^N.$$

If $X, Y, Z \in T$, we use (2), (6) and the fact that

(13) $$\nabla_X Y = (\bar{D}_X Y)^T$$

in the tangencial component of (12) and we get

$$0 = \nabla_X \nabla_Y Z - S_Y B(Y, Z) - \nabla_Y \nabla_X Z + S_Y B(X, Z) - \nabla_{[X,Y]} Z,$$

which is the Gauss equation (9).

If $X, Y \in T$ and $Z = N \in E$, using (1), (2) and (13) in the tangencial component of (12) we get the Codazzi-Mainard equation; and using (1), (2) and (6) in the normal component of (12) we get

$$0 = -B(X, S_Y N) + D_X D_Y N + B(Y, S_X N) - D_Y D_X N - D_{[X,Y]} N,$$

which is the Gauss equation (10).

Therefore, we proved that if the Riemannian manifold M is isometrically immersed in $\mathbb{R}^{n+k}$, then there is a k-plane bundle E over M with an induced structure for which the Gauss and Codazzi-Mainard equations hold. We will prove that this condition is also sufficient for the existence of such immersion locally.

3. Let E be a k-plane bundle over M equipped with a bundle metric and a compatible connection D. We define a *second fundamental form* S in E to be a homomorphism $S : T \otimes E \longrightarrow T$ satisfying (4), and the *second fundamental tensor* B *associated* with S to be the homomorphism $B : T \otimes T \longrightarrow E$ defined by (5). (We will use the same notation (,) for the bundle metric, the Riemannian metric on M and the usual metric in $\mathbb{R}^{n+k}$).

Theorem: Let M be a Riemannian n-manifold and E a k-plane bundle over M equipped with a bundle metric and a compatible connection D. Let S be a second fundamental form in E and B the associated second fundamental tensor. Then if the Gauss and Codazzi-Mainard equations (9), (10), (11) are satisfied, there is a local isometric immersion $f : V \subset M \longrightarrow \mathbb{R}^{n+k}$, in a such way that we may identify the normal bundle of the immersion with the bundle E. Then the metric induced on the normal bundle coincides with the given bundle metric on E, and the second fundamental form and the connection of the immersion coincide with S and D respectively. Moreover the immersion is unique up to a rigid motion.

In order to prove this theorem we need the following lemmas.

Lemma 1. Let M be a Riemannian n-manifold and E a k-plane bundle over M equipped with a bundle metric and a compatible connection D. Let S be a second fundamental form in E and B its associated second fundamental tensor. Assume that the Gauss and Codazzi-Mainard equations (9), (10), (11) are satisfied. Then there exist differential 1-forms w_i and w_b^a, $i = 1, \ldots, n;\ a, b = 1, \ldots, n + k$, defined locally in M, satisfying the following conditions:

(14) $$w_b^a = -w_a^b$$

(15) $$dw_i = \sum_j w_j \wedge w_i^j, \quad j = 1, \ldots, n,$$

(16) $$dw_b^a = \sum_c w_c^a \wedge w_b^c, \quad c = 1, \ldots, n + k.$$

Proof. We will make use of the following convention on the ranges of indices:

$$1 \le a, b, c, d \le n + k,$$
$$1 \le i, j, l, s, t \le n,$$
$$n + 1 \le \alpha, \beta, \gamma \le n + k,$$

and we shall agree that repeated indices are summed over the respective ranges.

Let U be a coordinate neighborhood in M. Choose a frame field $\partial^i, \ldots, \partial^n$ in U such that $(\partial^i, \partial^j) = \delta^{ij}$. Let $\{w_i\}$ be the dual coframe of ∂^j, i.e., $w_i(\partial^j) = \delta_i^j$ and define $w_\alpha(\partial^i) = 0$.

In order to get the other differential forms we choose an orthonormal frame $Y^{n+1}, \ldots, Y^{n+k}$ in $U \times R^k \subset E$, and we define

(17) $$w_j^i(\partial^l) = (\nabla_{\partial^l} \partial^i, \partial^j)$$

(18) $$w_\alpha^i(\partial^j) = (S_{\partial^i} Y^\alpha, \partial^j)$$

(19) $$w_\beta^\alpha(\partial^i) = (D_{\partial^i} Y^\alpha, Y^\beta)$$
$$w_i^\alpha = -w_\alpha^i$$

where ∇ is the Riemannian connection of M.

Since D is compatible with the bundle metric, we have $w_\beta^\alpha = -w_\alpha^\beta$; similarly from the fact that ∇ is compatible with the Riemannian metric we get $w_j^i = -w_i^j$; and by definition $w_i^\alpha = -w_\alpha^i$. Hence $w_b^a = -w_a^b$ and $w_a^a = 0$.

It remains to prove equations (15) and (16). In order to prove equation (15) we remark that equation (17) is equivalent to

(20) $$\nabla_{\partial^i}\partial^j = w_l^j(\partial^i)\partial^l.$$

Moreover we can see that

(21) $$\partial^i = w_j(\partial^i)\partial^j.$$

Since ∇ is symmetric, i.e.,

$$\nabla_{\partial^i}\partial^j - \nabla_{\partial^j}\partial^i = [\partial^i, \partial^j],$$

we obtain from (21)

$$^f\nabla_{\partial^i} w_l(\partial^j)\partial^l - \nabla_{\partial^i} w_l(\partial^i)\partial^l = w_l[\partial^i, \partial^j]\partial^l.$$

Using the fact that a connection is a derivation and (20) we get

$$(\partial^i w_l(\partial^j) - \partial^j w_l(\partial^i) - w_l[\partial^i, \partial^j])\partial^l + (w_l(\partial^j)w_s^l(\partial^i) - w_l(\partial^i)w_s^l(\partial^j))\partial^s = 0,$$

which is equivalent to

$$dw_l(\partial^i, \partial^j)\partial^l = (w_s(\partial^i)w_l^s(\partial^j) - w_s(\partial^j)w_l^s(\partial^i))\partial^l,$$

or

$$dw_l = \sum_s w_s \wedge w_l^s.$$

Next we will prove that equation (16) holds. This will follow from the Gauss and Codazzi-Mainard equations. But first we remark that (18) and (19) are respectively equivalent to

(22) $$S_{\partial^i} Y^\alpha = \sum_j w_\alpha^i(\partial^j)\partial^j$$

and

(23) $$D_{\partial^i} Y^\alpha = w_\beta^\alpha(\partial^i)Y^\beta.$$

Moreover using (4) and (5) we get

(24) $$w_\alpha^i(\partial^j) = w_\alpha^j(\partial^i)$$

and

(25) $$B(\partial^i, \partial^j) = w_\alpha^j(\partial^i)Y^\alpha.$$

Now consider Gauss equation (9)

$$R(\partial^i, \partial^j)\partial^l = S_{\partial^i}B(\partial^j, \partial^l) - S_{\partial^j}B(\partial^i, \partial^l),$$

and use (25) to obtain

$$R(\partial^i, \partial^j)\partial^l = S_{\partial^i}(w_\alpha^l(\partial^j)Y^\alpha) - S_{\partial^j}(w_\alpha^l(\partial^i)Y^\alpha).$$

From the fact that S satisfies (22), it follows

$$R(\partial^i, \partial^j)\partial^l = \sum_s (\sum_\alpha w_\alpha^l(\partial^i)w_\alpha^i(\partial^s) - \sum_\alpha w_\alpha^l(\partial^i)w_\alpha^j(\partial^s))\partial^s$$

which, according to (24), is equivalent to

$$R(\partial^i, \partial^j)\partial^l = \sum_s (\sum_\alpha w_\alpha^l(\partial^j)w_\alpha^s(\partial^i) - \sum_\alpha w_\alpha^l(\partial^i)w_\alpha^s(\partial^j))\partial^s =$$

$$= -\sum_s (\sum_\alpha w_\alpha^l \wedge w_\alpha^s(\partial^i, \partial^j))\partial^s.$$

Hence

(26) $$R(\partial^i, \partial^j)\partial^l = (w_\alpha^l \wedge w_s^\alpha(\partial^i, \partial^j))\partial^s.$$

On the other hand, R satisfies equation (7), i.e.,

$$R(\partial^i, \partial^j)\partial^l = \nabla_{\partial^i}\nabla_{\partial^j}\partial^l - \nabla_{\partial^j}\nabla_{\partial^i}\partial^l - \nabla_{[\partial^i, \partial^j]}\partial^l.$$

Using (20) and the fact that ∇ is a derivation, we have that

$$R(\partial^i, \partial^j)\partial^l = (\partial^i(w_s^l(\partial^j)) - \partial^j(w_s^l(\partial^i)) - w_s^l([\partial^i, \partial^j]))\partial^s +$$
$$+ (w_t^s(\partial^i)w_s^l(\partial^j) - w_t^s(\partial^j)w_s^l(\partial^i))\partial^t =$$
$$= (dw_s^l(\partial^i, \partial^j))\partial^s + (w_t^s \wedge w_s^l(\partial^i, \partial^j))\partial^t =$$
$$(dw_s^l(\partial^i, \partial^j))\partial^s + (w_s^t \wedge w_t^l(\partial^i, \partial^j))\partial^s.$$

Hence,

(27) $$R(\partial^i, \partial^j)\partial^l = (dw_s^l(\partial^i, \partial^j) - w_t^l \quad w_s^t(\partial^i, \partial^j))\partial^s.$$

Comparing expressions (26) and (27) we obtain

$$w_\alpha^l \wedge w_s^\alpha = dw_s^l - w_t^l \wedge w_s^t,$$

or

(28) $$dw_s^l = w_a^l \wedge w_s^a.$$

Similarly, if we take Gauss equation (10) and expression (8) we get

(29) $$dw_\beta^\alpha = w_a^\alpha \wedge w_\beta^a.$$

This follows from (4), (5), (23), (24) and from the fact that

$$B(\partial^i, S_{\partial^j} Y^\alpha) = \sum_l w^i_s(\partial^l) w^j_\alpha(\partial^l) Y^\beta.$$

If we take the Codazzi-Mainard equation (11) we obtain

(30) $$dw^i_\alpha = w^i_a \wedge w^a_\alpha ,$$

which follows from (20), (22), . . . , (25) and the fact that

$$S_{[\partial^i, \partial^j]} Y^\alpha = w^l_\alpha([\partial^i, \partial^j])\partial^l.$$

Equation (16) follows from (28), (29) and (30). This concludes the proof of lemma 1.

Lemma 2. Let w_i and w^q_b be differential 1-forms defined on a manifold M of dimension n satisfying (14), (15) and (16). Then there exist an orthogonal matrix $A = (A^a_b)$ of functions defined on a neighborhood V of a point $q_\circ \in M$, and a map $f : V \longrightarrow \mathbb{R}^{n+k}$ such that

(31) $$W = dA \cdot {}^tA,$$

(32) $$df = w \cdot A,$$

where W represents the matrix (w^a_b), $w = (w_1, \dots, w_n, \underbrace{0, \dots, 0}_{k})$ and ${}^tA$ is the transpose matrix of A.

Proof. We have to solve the differential equation $W = dA \cdot {}^tA$, for a given initial condition $A(q_\circ)$, where $A(q_\circ)$ is an orthogonal matrix. Let U be a coordinate neighborhood in M, such that $q_\circ \in U$, and y_1, . . ., y_n local coordinates in , with $q_\circ = (0, \dots, 0)$. Let z^q_b be the standard coordinate system on $\mathbb{R}^{(n+k)^2}$. We may identify the set of $n + k$ by $n + k$ real matrices with $\mathbb{R}^{(n+k)^2}$, and hence the set of orthogonal matrices $0(\mathbb{R}^{n+k})$ with a subset of $\mathbb{R}^{(n+k)^2}$. Let Z be the matrix of functions $Z = (z^a_b)$ defined on this subset.

Consider the matrix of 1-forms in $U \times 0(\mathbb{R}^{n+k})$

(33) $$\Lambda = dZ - W \cdot Z.$$

We define an n-dimensional distribution in $U \times 0(\mathbb{R}^{n+k})$ as follows: for each $(q, Z) \in U \times 0(\mathbb{R}^{n+k})$ we associate

$$D(q, Z) = \ker \Lambda_{(q,z)}.$$

We will prove that D actually defines an n-dimensional distribution. First we remark that for each (q, Z)

$$\Lambda_{(q,Z)} : T_qU \times T_Z0(\mathbb{R}^{n+k}) \longrightarrow T_Z0(\mathbb{R}^{n+k});$$

in fact if $(v, X) \in T_qU \times T_Z0(\mathbb{R}^{n+k})$, then

$$\begin{aligned}\Lambda_{(q,Z)}(v, X) \cdot {}^tZ + Z \cdot {}^t\Lambda_{(q,Z)}(v, X) &= \\ = (dZ(X) - W(v) \cdot Z)^tZ + Z(d^tZ(X) - {}^tZ^tW(v)) &= \\ = dZ(X)^tZ + Z\, d^tZ(X) - W(v) - {}^tW(v) &= 0.\end{aligned}$$

Last equality follows from (14) and the fact that $Z \in 0(\mathbb{R}^{n+k})$. Now the linear map $\Lambda_{(q,Z)}$ is onto $T_Z0(\mathbb{R}^{n+k})$, since for each $X = X^a_b \dfrac{\partial}{\partial z^a_b} \in T_Z0(\mathbb{R}^{n+k})$, $\Lambda_{(q,Z)}(0, X) = X$. Hence, $\dim \ker \Lambda_{(q,Z)} = n$.

The distribution D is involutive. In fact, if (v_1, X_1) and $(v_2, X_2) \in D(q, Z)$, then for each a, b, $\Lambda^a_b(v_1, X_1) = 0$ and $\Lambda^a_b(v_2, X_2) = 0$. From (33), we have

$$d\Lambda = -dW \cdot Z + W \wedge dZ,$$

from (16) and (33) we get

$$\begin{aligned}d\Lambda &= -(W \wedge W) \cdot Z + W \wedge (\Lambda + W \cdot Z) = \\ &= W \wedge \Lambda.\end{aligned}$$

Hence, for each a, b

$$\begin{aligned}\Lambda^a_b[(v_1, X_1), (v_2, X_2)] &= (v_1, X_1)\Lambda^a_b(v_2, X_2) - (v_2, X_2)\Lambda^a_b(v_1, X_1) - \\ - d\Lambda^a_b((v_1, X_1), (v_2, X_2)) &= -W^a_c \wedge \Lambda^c_b((v_1, X_1), (v_2, X_2)) = 0.\end{aligned}$$

So $[(v_1, X_1), (v_2, X_2)] \in D(q, Z)$, and the distribution is integrable.

Given a point $(0, Z) \in U \times 0(\mathbb{R}^{n+k})$, we claim that $D(0, Z) \cap (0 \times T_Z0(\mathbb{R}^{n+k})) = \{0\}$. In fact, if $(0, X)$ belongs to this intersection, then

$$\Lambda(0, X) = dZ(X) = X = 0.$$

By the implicit function theorem, the integral manifold through a point $(0, Z)$ is locally the graph of a function $q \longrightarrow A(q) \in 0(\mathbb{R}^{n+k})$ with $A(q_\circ) = A(0) = Z$. Since $Z = A$ along the graph and $D = 0$ on this graph, equation (33) gives

$$dA = W \cdot A$$

and hence

$$W = dA \cdot {}^tA.$$

From this we get an orthogonal matrix defined on a neighborhood $V_1 \subset U$ of the point $q_\circ$.

We now solve the differential equation (32), with a given initial condition $f(q_\circ)$. The proof is similar to the one above. Let $y_1, \dots y_n$ be the local coordinates in U, with $q_\circ = (0, \dots, 0)$. Let z_a be the usual coordinate system in $\mathbb{R}^{n+k}$, Z the matrix (z_a) and consider the matrix of 1-forms in $V_1 \times \mathbb{R}^{n+k}$

$$\Lambda = dZ - w \cdot A \tag{34}$$

We define an n-dimensional distribution in $V_1 \times \mathbb{R}^{n+k}$ as follows: for each $p \in V_1 \times \mathbb{R}^{n+k}$ we associate

$$D(p) = \ker \Lambda_p$$

This distribution is involutive. In fact, from (34) we have

$$d\Lambda = -dw\, A + w \wedge dA$$

from (15) and (31) we get

$$d\Lambda = -w \wedge WA + w \wedge WA = 0.$$

Hence if $X, Y \in D(p)$, then $[X, Y] \in D(p)$, and the distribution is integrable.

Given a point $q = (0, \dots, 0, z_a)$, $D(q) \cap (0 \times \mathbb{R}^{n+k}) = \{0\}$. By the implicit function theorem the integral manifold through any point q is locally the graph of a function $p \longrightarrow f(p) \in \mathbb{R}^{n+k}$ with $f(q_\circ) = f(0) = z_a$. Since $Z = f$ and $D = 0$ on this graph, equation (34) gives

$$df = w \cdot A.$$

Hence there exist a map $f : V \longrightarrow \mathbb{R}^{n+k}$, where $V \subset V_1 \subset U$, and an orthogonal matrix A of functions defined on V satisfying (31) and (32). This concludes the proof of lemma 2.

We now prove our theorem.

Proof. Using lemma 1 we obtain differential forms w_i and w^a_b, defined on a coordinate neighborhood U of M, satisfying equations (14), (15), (16). From lemma 2, for a given initial condition, we have a map $f : V \longrightarrow \mathbb{R}^{n+k}$ and an orthogonal matrix A defined on V, such that

$$W = dA \cdot {}^tA$$

and

$$df = w \cdot A$$

Since A is a non-singular matrix and, from the proof of lemma 1, w_i are linearly independent, df is a one-to-one map, and hence f is an immersion. Moreover, f induces on V the initial metric, and thus is an isometric immersion. In fact, if $\partial^1, \dots, \partial^n$ is the frame field in U of lemma 1, from (32) we have that

$$(df(\partial^i),\ df(\partial^j)) = (w(\partial^i)A,\ w(\partial^j)A) = \sum_a A^i_a A^j_a = \delta^{ij} = (\partial^i, \partial^j).$$

Next we will identify E with the normal bundle of the immersion, and we will show that the metric induced on the normal bundle coincides with the given bundle metric on E. Moreover, we will prove that the second fundamental form and the connection of the immersion coincide with S and D respectively. We will define a diffeomorphism, as a linear extension of f, which will identify locally E with the normal bundle of the immersion.

We identify $V \subset U$ with $f(V)$, and consider the orthonormal system $\bar{Y}^{n+1}, \dots, \bar{Y}^{n+k}$ defined by

$$\bar{Y}^\alpha = \sum_a A^\alpha_a \frac{\partial}{\partial x^a}$$

where $x^1, \dots, x^{n+k}$ are the usual coordinates in $\mathbb{R}^{n+k}$. $\bar{Y}^\alpha$ is a normal vector to $f(V)$. In fact, for each i, $df(\partial^i)$ is tangent to $f(V)$ and

$$(\bar{Y}^\alpha,\ df(\partial^i)) = \left(A^\alpha_a \frac{\partial}{\partial x^a},\ A^i_b \frac{\partial}{\partial x^b} \right) = \sum_a A^\alpha_a A^i_a = \delta^{\alpha i} = 0.$$

If $\pi : E \longrightarrow M$ is the projection map, and $Y^{n+1}, \dots, Y^{n+k}$ is the orthonormal frame in E of lemma 1, we choose local coordinates in $\pi^{-1}(U)$ given by $(u, y) = (u_1, \dots, u_n, y_{n+1}, \dots, y_{n+k})$ where $(u_1, \dots, u_n)$ are the local coordinates in U and $y = y_\alpha Y^\alpha$. Let ψ be the extension of the immersion f defined as follows:

$$\psi : V \times \mathbb{R}^k \longrightarrow \mathbb{R}^{n+k}$$

$$(u, y) \longrightarrow \sum_a f^a(u) \frac{\partial}{\partial x^a} + \sum_\alpha y_\alpha \bar{Y}^\alpha(u),$$

i.e.,

$$\psi^a(u, y) = f^a(u) + \sum_\alpha y_\alpha A^\alpha_a. \tag{35}$$

ψ is a diffeomorphism onto $\psi(V \times \mathbb{R}^k)$, since f is a diffeomorphism onto $f(V)$ and ψ is a linear extension of f. Moreover $d\psi(Y^\alpha) = \overline{Y^\alpha}$.

The map ψ induces on $T(V \times \mathbb{R}^k)$ (tangent space of $V \times \mathbb{R}^k$) a metric which restricted to V coincides with the initial metric. Moreover the metric induced on the bundle also coincides with the given bundle metric since $d\psi(Y^\alpha) = \bar{Y}^\alpha$. Hence we identify $V \times R^k$ with the normal bundle of the immersion and the metric induced on the normal bundle coincides with the given bundle metric on E.

Now we will prove that the second fundamental form and the connection of the immersion coincide with S and D respectively. We choose a frame $e_1, \ldots, e_{n+k}$ in a neighborhood of $\mathbb{R}^{n+k}$ which contains $f(V)$ as follows:

$$e_i = df(\partial^i), \qquad e_\alpha = \bar{Y}^\alpha = \sum_a A_a^\alpha \frac{\partial}{\partial x^a}$$

We denote the coframe associated to this frame by θ_n and the corresponding connection forms by θ_b^a, defined by the equation $de_a = \sum_b \theta_b^a e_b$. These forms restricted to $f(V)$ are such that $\theta_\alpha = 0$, and since $e_a = \sum_b A_b^a \frac{\partial}{\partial x^b}$ we have that

$$\theta_b^a = \sum_c dA_c^a A_c^b. \tag{36}$$

Using f, the forms θ_a and θ_b^a will induce forms $f^*\theta_a$ and $f^*\theta_b^a$ on V. We claim that $f^*\theta_a = w_a$ and $f^*\theta_b^a = w_b^a$. In fact,

$$f^*\theta_i(\partial^j) = \theta_i(df(\partial^j)) = \theta_i(e_j) = \delta_{ij} = w_i(\partial^j)$$
$$f^*\theta_\alpha = 0 = w_\alpha$$

and

$$f^*\theta_b^a(\partial^j) = \theta_b^a(df(\partial^j)) = \theta_b^a(e_j) = \sum_c dA_c^a(e_j)A_c^b = w_b^a(\partial^j).$$

In these equalities we used (36) and the identification of $A(u)$ with $A(f(u))$.

Since the induced forms $f^*\theta_\alpha^i$ and $f^*\theta_\beta^\alpha$ coincide with w_α^i and w_β^α, they define the same connection D and the second fundamental form S in the normal bundle of the immersion by the equations (23) and (22). This proves the first part of the theorem.

If remains to prove that the local immersion is unique up to a rigid motion of $\mathbb{R}^{n+k}$. Let $\bar{f} : V \longrightarrow \mathbb{R}^{n+k}$ be another map satisfying the theorem, and let $\{\bar{e}_a\}$ be the corresponding frame. If $q \in V$, using a rigid motion, we can take $f(q)$ onto $\bar{f}(q)$ and e_a onto $\bar{e}_a$ on $f(q)$.

From the uniqueness of the solutions of the equations (31) and (32), the frame $\{\bar{e}_a\}$ coincides with $\{e_a\}$ and the map $\bar{f}$ coincides with f in a neighborhood of q. Since V is connected, $f = \bar{f}$ on V.

Remark: If we make the additional assumption that M is a simply connected Riemannian manifold, then the local immersion obtained in the theorem above can be extended to M, and this extension is unique up to a rigid motion. We consider $p \in M$, $p \neq q_\circ$, and a simple curve $\alpha : [0, 1] \longrightarrow M$ joining $q_\circ$ to $p \cdot$ Let $\{V_i\}_{i=1}^n$ be a collection of neighborhoods which cover $\alpha[0, 1]$, such that $V_1 = V$. We can extend f to $\bigcup_{i=1}^n V_i$, and hence we get $f(p)$. Since M is simply connected, $f(p)$ is independent of the curve α. Thus we have an isometric imersion $f : M \longrightarrow \mathbb{R}^{n+k}$, and we can see from this construction that f is unique up to a rigid motion.

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