Steady motions of Lagrangian systems

by

W. M. Oliva.-

Let V_m be a connected smooth manifold with a Riemannian metric $<,>$. Consider also on V_m a Pfaffian $\mathscr{L}$ – a smooth differential form of degree one – and a real smooth function π, called the *potential energy*.

By $J_0^r(R, V_m)$ we denote the manifold of all r-jets ($r \geq 1$), at the origin of the set of real numbers, of functions defined in a neighborhood of the origin with values on V_m. The canonical projection $\rho_r^{r+1}(j_0^{r+1}\varphi(t)) = j_0^r\varphi(t)$ is a smooth fibration map and $J_0^1(R, V_m)$ is precisely the tangent bundle of the manifold V_m. A second order normal system of ordinary differential equations on V_m is a map $\sigma : J_0^1(R, V_m) \to J_0^2(R, V_m)$ such that $\rho_1^2 \circ \sigma = \mathrm{id}\, J_0^1$. A solution of σ is a smooth map $\varphi : t \in (a, b) \to \varphi(t) \in V_m$ which verifies the following condition: for each $t \in (a, b)$ the 2-jet at $\tau = 0$ of $\varphi(\tau + t)$ belongs to $\sigma(J_0^1(R, V_m))$.

Suppose it is given now a real-valued smooth function F defined on an open set W^s of $J_0^s(R, V_m)$, $s \geq 0$ (here one needs to define $J_0^0(R, V_m) = V_m$ and $\rho_0^1(j_0^1\varphi(t)) = \varphi(0) \in V_m$). It is possible to define the total derivative $\dot{F} = \dfrac{dF}{dt}$ which is a real smooth function on $(\rho_s^{s+1})^{-1} W^s \subset J_0^{s+1}(R, V_m)$. Let $X = j_0^{s+1}\varphi$ be an element of $(\rho_s^{s+1})^{-1} W^s$ and $\varphi(t)$ one representative of X. Consider the composite map $F . j_0^s$ where j_0^s is the function $t \mapsto j_{\tau=0}^s \varphi(t + \tau)$ defined in the neighborhood of zero. By definition

$$\dot{F}(X) = \frac{dF}{dt}(X) = \frac{d}{dt}(F . j_0^s)\Big|_{t=0}$$

It is also useful to see that in local admissible coordinates of $(\rho_s^{s+1})^{-1} W^s$ one has:

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_j} \cdot \dot{q}_j + \frac{\partial F}{\partial \dot{q}_j} \cdot \ddot{q}_j + \cdots + \frac{\partial F}{\partial q_j^{(s)}} \cdot q_j^{(s+1)}.$$

In particular, when $s = 0$,

$$F = F(q_1, \ldots, q_m) \quad \text{then} \quad \frac{dF}{dt} = \frac{\partial F}{\partial q_j} \cdot \dot{q}_j \, .$$

On the manifold $J_0^1(R, V_m)$ it is defined the Lagrangian function $L: J_0^1(R, V_m) \to R$, $L = L_0 + L_1 + T_2$, where

$$L_0(j_0^1 \varphi) = -\pi \, . \, \rho_0^1(j_0^1 \varphi) = -\pi(\varphi(0)),$$
$$L_1(j_0^1 \varphi) = \mathscr{L}(\dot{\varphi}(0)),$$
$$T_2(j_0^1 \varphi) = \tfrac{1}{2} \langle \dot{\varphi}(0), \dot{\varphi}(0) \rangle \, .$$

The Pfaffian locally given in admissible coordinates by

$$\omega_L = \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] dq_j$$

is globally defined on $J_0^2(R, V_m)$ (see [2] for a proof). We know that the dynamical system defined by $\omega_L = 0$ is given locally by the Lagrangian equations ([1], [2]):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, m;$$

and the local representation of $L = L_0 + L_1 + T_2$ is:

$$L_0(j_0^1 \varphi) = L_0(q, \dot{q}) = -\pi(\varphi(0)) = -\pi(q);$$
$$L_1(j_0^1 \varphi) = \mathscr{L}(\dot{\varphi}(0)) = \mathscr{L}\left(\dot{q}_j \frac{\partial}{\partial q_j} \right);$$

and if $\mathscr{L} = b_i dq_i$ one has $L_1(j_0^1 \varphi) = b_j \dot{q}_j$ and finally if $a_{ij}(q) = \left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \right\rangle$

$$T_2(j_0^1 \varphi) = \left\langle \dot{q}_i \frac{\partial}{\partial q_i}, \dot{q}_j \frac{\partial}{\partial q_j} \right\rangle = \dot{q}_i \, \dot{q}_j \left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \right\rangle = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j,$$ (in the last expressions we omit the summation sign). The function T_2 is called the *kinetic energy*, L_1 is the *friction energy* and $V = -\pi$ is the *potential function*. Since T_2 is positive definite in the velocities it is easy to see that the dynamical system $\omega_L = 0$ given locally by the Lagrangian equations defines a second order normal system of ordinary differential equations on V_m.

A non-holonomic constraint on V_m is a smooth non-involutive distribution M of contact elements M_x at each point $x \in V_m$. The contact element M_x is a subspace of the tangent space $T_x(V_m)$, and the dimension of M_x does not depend on the point x. If the dimension is $(m-p)$ we know that each point x has a neighborhood U_x and p smooth Pfaffian forms $\omega^1, \ldots, \omega^p$ linearly independent at each point of U_x. The set of all one-jets $j_0^1 \varphi$ such that $\dot{\varphi}(0) \in M_{\varphi(0)}$ is a regularly embedded submanifold $S(M) \subset J_0^1(R, V_m)$. In fact, if U_x is a coordinate neighborhood, the expressions of the Pfaffian forms ω^μ are:

$$\omega^\mu = A_{\mu j} \, dq_j ; \qquad \mu = 1, 2, \ldots, p;$$

and a jet $j_0^1 \varphi$ of $(\rho_0^1)^{-1} U_x$ with admissible coordinates $(q_i, \dot{q}_i)$ belongs to $S(M)$ if, and only if, $A_{\mu j} \dot{q}_j = 0$; $\mu = 1, 2, \ldots, p$. But the matrix $(A_{\mu j})$ has maximum rank p, then, it is possible, up to a permutation of the coordinates $q_1, \ldots, q_m$, to write

$$\dot{q}_\nu = F^\nu(q_1, \ldots, q_m, \dot{q}_{p+1}, \ldots, \dot{q}_m), \; \nu = 1, 2, \ldots, p;$$

and this proves the following:

Proposition 1. The set $S(M)$ is a regularly embedded submanifold of the manifold $J_0^1(R, V_m)$ with dimension $m + (m-p)$. $S(M)$ is called the *manifold of constraints* and is a vector bundle over V_m.

Given a smooth vector field θ on V_m it is possible to extend it to the manifolds $J_0^1(R, V_m)$ and $J_0^2(R, V_m)$. If the local expression of θ is $\theta = M_i \frac{\partial}{\partial q_i}$, the corresponding local expressions of the extentions θ^1 and θ^2 in admissible coordinates are

$$\theta^1 = M_i \frac{\partial}{\partial q_i} + \dot{M}_i \frac{\partial}{\partial \dot{q}_i} \quad \text{and} \quad \theta^2 = M_i \frac{\partial}{\partial q_i} + \dot{M}_i \frac{\partial}{\partial \dot{q}_i} + \ddot{M}_i \frac{\partial}{\partial \ddot{q}_i} \, .$$

It is a simple matter to define θ^1 and θ^2 globally or to verify that those local formulae are coerent by changing of admissible coordinates.

The Lagrangian multipliers. We reach, locally, for functions:

$$\lambda_1, \ldots, \lambda_p : \Omega \subset J_0^1(R, V_m) \to R$$

such that the system of ordinary and Pfaffian equations given by:

$$(1) \quad \begin{cases} \omega_L - \lambda_\mu \omega^\mu = 0 \\ \omega^\mu(\dot{\varphi}(0)) = 0, \qquad \mu = 1, 2, \ldots, p \end{cases}$$

has existence and uniqueness of solutions. We will prove the following:

Theorem 1. There exists a unique sequence of functions $\lambda_1, \ldots, \lambda_p$ such that for each point of the manifold $S(M)$ we have a unique solution of system (1) and this solution remains on $S(M)$ for all time.

Proof: The mixed system (1) is locally a system of $m + p$ equations in the $m + p$ unknown quantities q_j and $\lambda_\mu, j = 1, 2, \ldots, m, \mu = 1, 2, \ldots, p$:

$$(2) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \lambda_\gamma A_{\gamma j}; \quad j = 1, 2, \ldots, m.$$

$$(3) \qquad A_{\mu j}\dot q_j = 0; \quad \mu = 1, 2, \ldots, p.$$

and the initial condition must satisfy (3).

The *holonomic prolongation* of $S(M)$ is the submanifold $P(S(M))$ of $J_0^2(R, V_m)$ with dimension $3m - 2p$ defined locally by the equations:

$$(3) \qquad A_{\mu j}\dot q_j = 0; \quad \mu = 1, 2, \ldots, p$$

$$(4) \qquad \frac{d}{dt}(A_{\mu j}\dot q_j) = 0 = A_{\mu j}\ddot q_j + \frac{dA_{\mu j}}{dt}\cdot \dot q_j; \quad \mu = 1, 2, \ldots, p.$$

$P(S(M))$ is precisely the manifold $P(S(M)) = J_0^1(R, S(M)) \cap J_0^2(R, V_m)$. If a solution satisfies (3) then it satisfies (4); but, conversely, if it satisfies (4) and for $t = t_0$ satisfies (3) then it satisfies (3) for all time. On the other hand the equation (2) can be written in the form

$$(2)^1 \qquad a_{jk}\ddot q_k + \frac{\partial^2 L}{\partial q_k \partial \dot q_j}\cdot \dot q_k - \frac{\partial L}{\partial q_j} = \lambda_\mu A_{\mu j}; \quad j = 1, 2, \ldots, m.$$

If (B_{rj}) is the inverse of the matrix (a_{jk}) one obtains

$$(5) \qquad \ddot q_r = B_{rj}A_{\mu j}\lambda_\mu + B_{rj}\left[\frac{\partial L}{\partial q_j} - \frac{\partial^2 L}{\partial q_k \partial \dot q_j}\dot q_k\right]$$

and if we put $\ddot q_r$ in (4) we get:

$$(6) \quad (A_{\nu r}B_{rj}A_{\mu j})\lambda_\mu + A_{\nu r}B_{rj}\left[\frac{\partial L}{\partial q_j} - \frac{\partial^2 L}{\partial q_k \partial \dot q_j}\dot q_k\right] + \frac{dA\nu_j}{dt}\cdot \dot q_j = 0, \gamma = 1, 2, \ldots, p$$

Since (B_{rj}) is positive definite it can be proved that the pxp matrix $(A_{\nu r}B_{rj}A_{\mu j})$ is non singular. Then the above system (6) with p equations leaves the unknown quantities $\lambda_1, \ldots, \lambda_p$ uniquely determined. But (6) shows that the λ_μ depend only on L and on the Pfaffians $\omega^1, \ldots, \omega^p$. The uniqueness of the functions $\lambda_1, \ldots, \lambda_p$ is proved. On the other hand, the expression (6) proves the existence of the λ_μ, locally. The geometrical meaning of Theorem 1 is the existence of a global cross section

$$\tau : S(M) \to P(S(M))$$

with respect to ρ_1^2; to each $X_0 \in S(M)$ one can associate the jet $\tau(X_0) =$

$= j_0^2\,\varphi(t)$ where $\varphi(t)$ is the solution considered in the theorem. It is obvious that $\tau(X_0) \subset P(S(M))$, $\rho_1^2\,\tau(X_0) = X_0$ and τ is smooth. Since the equality $\omega_L = \lambda_\mu\,\omega^\mu$ holds on $\tau(S(M)) \subset P(S(M))$, one obtains $\tau^*\,\omega_L = \lambda_\mu\,\omega^\mu$ on $S(M)$ and then, the Pfaffian $\lambda_\mu\,\omega^\mu$ can be globally defined on the manifold $S(M)$; moreover, the functions λ_μ have the same domain on $S(M)$ as the forms ω^μ. If the Pfaffian forms ω^μ are globally defined on V_m the functions λ_μ are globally defined on $S(M)$.

The mixed system (1) is called a *non-holonomic system* if the distribution M is a non-holonomic constraint. If M is involutive, that is, if for it holds the theorem of Frobenius, the system (1) is called *semi-holonomic.* Finally, if the Pfaffian forms ω^μ are identically zero (dimension of $M = m$) the system (1) is called a *holonomic system.*

In the sequel we will suppose that the ω^μ are globally defined on the manifold V_m.

The manifold Z of the zero velocities is the submanifold of $J_0^1(R, V_m)$ of the one-jets of constant functions; the projection ρ_0^1 is a diffeomorphism between Z and V_m. Let us call α_μ the restriction of λ_μ to the manifold Z; the α_μ can be given, locally, by

$$(7) \qquad (A_{\nu r}B_{rj}A_{\mu j})\,\alpha_\mu - A_{\nu r}B_{rj}\frac{\partial \pi}{\partial q_j} = 0; \quad \nu = 1, 2, \ldots, p.$$

It is a simple matter to show that the α_μ can be considered functions on V_m.

Consider now the functions β_μ locally given by

$$(8) \quad (A_{\nu r}B_{rj}A_{\mu j})\,\beta_\mu + A_{\nu r}B_{rj}\left[\frac{\partial L_1}{\partial q_j} - \frac{\partial^2 L_1}{\partial q_k \partial \dot q_j}\dot q_k\right] = 0; \quad \nu = 1, 2, \ldots, p,$$

where $L_1 = b_j\dot q_j$ is the friction energy. The β_μ is precisely the part of the λ_μ which is linear in the velocities:

$$(A_{\nu r}B_{rj}A_{\mu j})\,\beta_\mu + A_{\nu r}B_{rj}\left[\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right]\dot q_k = 0.$$

In a coordinate neighborhood of V_m one has the local Pfaffian forms B_μ given by

$$(A_{\nu r}B_{rj}A_{\mu j})\,B_\mu + A_{\nu r}B_{rj}\left(\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right)dq_k = 0.$$

If $B_\mu = E_{\mu j}\, dq_j$ and $\bar B_\mu = \bar E_{\mu j}\, d\bar q_j$, to prove that the B_μ are globally defined we need to show that

$$E_{\mu j} = \bar E_{\mu k}\frac{\partial \bar q_k}{\partial q_j}.$$

But $\quad (A_{\nu r} B_{rj} A_{\mu j})\, E_{\mu k}\, dq_k + A_{\nu r} B_{rj}\left(\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right) dq_k = 0.$

or

(9) $\quad (A_{\nu r} B_{rj} A_{\mu j})\, E_{\mu k} + A_{\nu r} B_{rj}\left(\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right) = 0.$

In the intersection of two coordinate neigborhoods one has:

$$A_{\nu j} = \bar A_{\nu k}\frac{\partial \bar q_k}{\partial q_j}, \qquad \bar a_{st} = \frac{\partial q_m}{\partial \bar q_s}\frac{\partial q_n}{\partial \bar q_t}\, a_{mn} \quad \text{and} \quad \bar B_{ks} = \frac{\partial \bar q_k}{\partial q_r}\cdot\frac{\partial \bar q_s}{\partial q_j}\, B_{rj};$$

the last one because

$$\left[\frac{\partial \bar q_k}{\partial q_r}\cdot\frac{\partial \bar q_s}{\partial q_j}\cdot B_{rj}\right]\bar a_{st} = \left[\frac{\partial \bar q_k}{\partial q_r}\cdot\frac{\partial \bar q_s}{\partial q_j}\cdot B_{rj}\right]\left[\frac{\partial q_m}{\partial \bar q_s}\cdot\frac{\partial q_n}{\partial \bar q_t}\, a_{mn}\right] = \delta_{kt}$$

Then the equalities (9) give:

$$\bar A_{\nu i}\frac{\partial \bar q_i}{\partial q_r} B_{rj}\, \bar A_{\mu s}\frac{\partial \bar q_s}{\partial q_j} E_{\mu k} + \bar A_{\nu i}\frac{\partial \bar q_i}{\partial q_r} B_{rj}\left(\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right) = 0.$$

and then

$$\bar A_{\nu i}\bar A_{\mu s}\bar B_{is} E_{\mu k} + \bar A_{\nu i}\frac{\partial \bar q_i}{\partial q_r}\cdot\frac{\partial q_r}{\partial \bar q_m}\cdot\frac{\partial q_j}{\partial \bar q_n}\bar B_{mn}\left(\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right) = 0.$$

but $\quad b_k = \bar b_s\frac{\partial \bar q_s}{\partial q_k}, \qquad \frac{\partial b_k}{\partial q_j} = \frac{\partial \bar b_s}{\partial q_j}\frac{\partial \bar q_s}{\partial q_k} + \bar b_s\frac{\partial^2 \bar q_s}{\partial q_k\, \partial q_j}$

and $\quad \left(\frac{\partial b_k}{\partial q_j} - \frac{\partial b_j}{\partial q_k}\right) = \frac{\partial \bar b_s}{\partial q_j}\cdot\frac{\partial \bar q_s}{\partial q_k} - \frac{\partial \bar b_s}{\partial q_k}\cdot\frac{\partial \bar q_s}{\partial q_j}.$

Finally $\bar A_{\nu i}\bar A_{\mu s}\bar B_{is} E_{\mu k} + \bar A_{\nu i}\frac{\partial q_j}{\partial \bar q_n}\bar B_{in}\left(\frac{\partial \bar b_s}{\partial q_j}\cdot\frac{\partial \bar q_s}{\partial q_k} - \frac{\partial \bar b_s}{\partial q_k}\cdot\frac{\partial \bar q_s}{\partial q_j}\right) = \bar A_{\nu i}\bar A_{\mu s}\bar B_{is} E_{\mu k} +$

$$+ \bar A_{\nu i}\bar B_{in}\frac{\partial q_j}{\partial \bar q_n}\left(\frac{\partial \bar b_s}{\partial q_j}\cdot\frac{\partial \bar q_s}{\partial q_k} - \frac{\partial \bar b_s}{\partial q_k}\cdot\frac{\partial \bar q_s}{\partial q_j}\right)$$

and multiplying by $\frac{\partial q_k}{\partial \bar q_r}$ one obtains

$$\bar A_{\nu i}\bar A_{\mu s}\bar B_{is} E_{\mu k}\frac{\partial q_k}{\partial \bar q_r} + \bar A_{\nu i}\bar B_{in}\frac{\partial q_j}{\partial \bar q_n}\cdot\frac{\partial q_k}{\partial \bar q_r}\left(\frac{\partial \bar b_s}{\partial q_j}\cdot\frac{\partial \bar q_s}{\partial q_k} - \frac{\partial \bar b_s}{\partial q_k}\cdot\frac{\partial \bar q_s}{\partial q_j}\right) = 0$$

or $\quad \bar A_{\nu i}\bar A_{\mu s}\bar B_{is} E_{\mu k}\frac{\partial q_k}{\partial \bar q_r} + \bar A_{\nu i}\bar B_{in}\left(\frac{\partial \bar b_r}{\partial q_j}\cdot\frac{\partial q_j}{\partial \bar q_n} - \frac{\partial \bar b_n}{\partial q_k}\cdot\frac{\partial q_k}{\partial \bar q_r}\right) = 0$

which implies

$$\bar A_{\nu i}\bar A_{\mu s}\bar B_{is} E_{\mu k}\frac{\partial q_k}{\partial \bar q_r} + \bar A_{\nu i}\bar B_{in}\left(\frac{\partial \bar b_r}{\partial \bar q_n} - \frac{\partial \bar b_n}{\partial \bar q_r}\right) = 0$$

so $E_{\mu k}\frac{\partial q_k}{\partial \bar q_r} = \bar E_{\mu r}$ and the Pfaffian forms B_μ are globally defined on V_m; also the functions β_μ defined by the forms $B_\mu : \beta_\mu(j_0^1\varphi) = B_\mu(\dot\varphi(0))$ and given locally by equations (8) are globally defined on the manifold $J_0^1(R, V_m)$.

Call $\gamma_\mu : S(M) \to R$ the functions

$$\gamma_\mu = \lambda_\mu - \alpha_\mu - \beta_\mu, \qquad \mu = 1, 2, \ldots, p.$$

Equations (6), (7), (8) and equalities $\lambda_\mu = \alpha_\mu + \beta_\mu + \gamma_\mu$ give the local representation (10) of the functions γ_μ:

(10) $\quad A_{\nu r} B_{rj} A_{\mu j}\gamma_\mu + A_{\nu r} B_{rj}\left[\frac{\partial T_2}{\partial q_j} - \frac{\partial T_2}{\partial q_k\, \partial \dot q_j}\dot q_k\right] + \frac{dA_{\nu j}}{dt}\dot q_j = 0.$

It is easy to see that the functions γ_μ are quadratic in the velocities with coeficients depending only on $q_1, \ldots, q_m$.

Theorem 2 – The Lagrangian multipliers λ_μ ($\mu = 1, 2, \ldots, p$) have a canonical decomposition $\lambda_\mu = \alpha_\mu + \beta_\mu + \gamma_\mu$ where the α_μ are defined on V_m, the β_μ come from Pfaffian forms B_μ on V_m and the γ_μ give rise to tensor fields G_{μ}, of degree two, defined also on V_m.

Proof – We need only to define the tensor fields G_μ, $\mu = 1, 2, \ldots, p$. Let $c_{\mu\nu}$ the inverse of the matrix $A_{\nu r} B_{rj} A_{\mu j}$ then,

$$-\gamma_\mu = c_{\mu\nu} A_{\nu r} B_{rj} \left[\frac{\partial T_2}{\partial q_j} - \frac{\partial T_2}{\partial q_k \partial \dot{q}_j} \dot{q}_k\right] + c_{\mu\nu} \frac{dA_{\nu i}}{dt} \dot{q}_i =$$

$$= c_{\mu\nu} A_{\nu r} B_{rj} \left[\frac{1}{2} \frac{\partial a_{mn}}{\partial q_j} \dot{q}_m \dot{q}_n - \frac{\partial a_{ij}}{\partial q_k} \dot{q}_i \dot{q}_k\right] + c_{\mu\nu} \frac{dA_{\nu i}}{dt} \dot{q}_i =$$

$$= c_{\mu\nu} \left\{A_{\nu r} B_{rj} \left[\frac{1}{2} \frac{\partial a_{ik}}{\partial q_j} - \frac{\partial a_{ij}}{\partial q_k}\right] \dot{q}_i \dot{q}_k + \frac{\partial A_{\nu i}}{\partial q_k} \dot{q}_i \dot{q}_k\right\} =$$

$$= c_{\mu\nu} \left\{A_{\nu r} B_{rj} \left[\frac{1}{2} \frac{\partial a_{ik}}{\partial q_j} - \frac{\partial a_{ij}}{\partial q_k}\right] + \frac{\partial A_{\nu i}}{\partial q_k}\right\} \dot{q}_i \dot{q}_k .$$

Consider the derivatives:

$$-\frac{\partial \gamma_\mu}{\partial \dot{q}_i \partial \dot{q}_k} = c_{\mu\nu} \left\{\left(\frac{\partial A_{\nu i}}{\partial q_k} + \frac{\partial A_{\nu k}}{\partial q_i}\right) - A_{\nu r} B_{rj} \left(\frac{\partial a_{jk}}{\partial q_i} + \frac{\partial a_{ji}}{\partial q_k} - \frac{\partial a_{ik}}{\partial q_j}\right)\right\}.$$

The vector fields θ_ν, orthogonal of ω^ν with respect to the scalar product $\langle , \rangle$ are given locally by

$$\theta_\nu = A_{\nu r} B_{rj} \frac{\partial}{\partial q_j}, \quad \nu = 1, 2, \ldots, p.$$

Now, as we will see with more details, the derivatives $-\frac{\partial \gamma_\mu}{\partial \dot{q}_i \partial \dot{q}_k}$ are the components of the tensor field $c_{\mu\nu} \mathscr{L}_{\theta_\nu} T_2$ where $\mathscr{L}_{\theta_\nu} T_2$ means the Lie derivative of the symetric metric tensor T_2 with respect to the vector field θ_ν. The matrix $(c_{\mu\nu})$ is also globally defined on V_m since $\omega^\mu(\theta_\nu) = A_{\mu k}\, dq_k \left(A_{\nu r} B_{rj} \frac{\partial}{\partial q_j}\right) = A_{\mu k} B_{kr} A_{\nu r}$. Finally

$$G_\mu = c_{\mu\nu} \mathscr{L}_{\theta_\nu} T_2$$

A *first integral* of the mixed system (1) is a smooth function $F : J_0^1(R_1 V_m) \to R$ such that $\frac{dF}{dt}$ is zero on the manifold $\tau(S(M))$. This means that F is constant throught each solution of (1). One classical result is the following:

Proposition 2 (Conservation of energy). The real function $F = T_2 + \pi \cdot \rho_0^1 = T_2 - L_0$ (kinetic plus potential energy) is a first integral of system (1).

Proof: Consider first of all, the local function

$$\bar{H} = -L + \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = -(T_2 + L_1 + L_0) + \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j ;$$

$$\frac{d\bar{H}}{dt} = -\frac{\partial L}{\partial q_j} \dot{q}_j - \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right),$$

then $\frac{d\bar{H}}{dt} = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\right) \cdot \dot{q}_j$. Using system (1) one obtains

$\frac{d\bar{H}}{dt} = (\lambda_\nu A_{\nu j}) \dot{q}_j$ say $\frac{d\bar{H}}{dt}$ is equal to zero throught the solutions of (1). On the other hand

$$L = \tfrac{1}{2} a_{rs} \dot{q}_r \dot{q}_s + b_i \dot{q}_i + L_0 ,$$

then $\frac{\partial L}{\partial \dot{q}_j} = a_{js} \dot{q}_s + b_j$ and $\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = a_{js} \dot{q}_s \dot{q}_j + b_j \dot{q}_j = 2T_2 + L_1$ that is $\bar{H} = -(T_2 + L_1 + L_0) + 2T_2 + L_1$ and finally

$$\bar{H} = T_2 - L_0 = T_2 + \pi \cdot \rho_0^1 .$$

First integrals linear in the velocities.

A *"linear in the velocities"* function f is a smooth real function on $J_0^1(R, V_m)$ induced by a smooth Pfaffian form Ω on $V_m : f(j_0^1 \varphi) = \Omega(\dot{\varphi}(0))$.

The unique vector field θ_f on V_m, orthogonal of Ω with respect to $\langle , \rangle : \Omega(x) = \langle \theta_f , x \rangle$, is called the orthogonal of f. If locally $\theta_f = M_j \frac{\partial}{\partial q_j}$, $\Omega = a_j\, dq_j = a_{ij} M_i\, dq_j$ and $f = a_j \dot{q}_j = a_{ij} M_i \dot{q}_j$.

Remarks: 1. If θ_f is "normal" to the distribution M, $\theta_f = c_\nu \theta_\nu$, where the θ_ν are the same considered in the proof of Theorem 2, $\theta_\nu = A_{\nu r} B_{rj} \frac{\partial}{\partial q_j}$. This way the function f, linear in the velocities, orthogonal of θ_f is then $f = a_{ij} M_i \dot{q}_j$ where $M_i = c_\nu A_{\nu r} B_{ri}$, or $f = a_{ij} B_{ri} c_\nu A_{\nu r} \dot{q}_j = c_\nu A_{\nu j} \dot{q}_j$ which proves that f is identically zero on $S(M)$ then f is a first integral of system (1).

2. If θ is "tangent" to the distribution M, the local flow φ_t of θ leaves M invariant and it is easy to see that θ^1 is tangent to the vector bundle $S(M)$ and θ^2 is tangent to $P(S(M))$.

3. If f is linear in the velocities and θ_f is its orthogonal, since $\theta_f = \bar{\theta}_f + \bar{\bar{\theta}}_f$, $\bar{\theta}_f$ "tangent" to M and $\bar{\bar{\theta}}_f$ is "normal" to M, one obtains $f = \bar{f} + \bar{\bar{f}}$ where, by remark 1., $\bar{\bar{f}}$ is a first integral identically zero on $S(M)$.

Theorem 3. Let f be linear in the velocities and θ_f be the orthogonal of f with respect to the scalar product $\langle , \rangle$. Then f is a first integral of system (1) if, and only if, the equality

$$(\omega_L - \lambda_\nu \omega^\nu)\,\theta_f^2 = \frac{df}{dt}$$

holds on the manifold $P(S(M))$.

Proof: If $(\omega_L - \lambda_\nu \omega^\nu)\,\theta_f^2 = \frac{df}{dt}$ holds on $P(S(M))$ it is quite obvious that f is a first integral of system (1) since $\frac{df}{dt}$ vanishes on the cross-section $\tau(S(M)) \subset P(S(M))$. Conversely, the local condition to be proved is

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right] dq_j \left(M_i \frac{\partial}{\partial q_i} + \dot M_i \frac{\partial}{\partial \dot q_i} + \ddot M_i \frac{\partial}{\partial \ddot q_i}\right) =$$

$$= \frac{df}{dt} = M_j\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right)$$

provided that $\theta_f = M_j \frac{\partial}{\partial q_j}$, $A_{\nu j}\dot q_j = 0$ and the function f be a first integral of system (1). But the functions

$$M_j\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right) \quad \text{and} \quad \frac{df}{dt}$$

vanish on the manifold $\tau(S(M))$. On the other hand $f = a_i \dot q_i$, $\frac{df}{dt} = a_i \ddot q_i + \dot a_i \dot q_i$ and

$$M_j\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right) = M_j\left(a_{ij}\ddot q_i + \frac{\partial^2 L}{\partial q_i \partial \dot q_j}\dot q_i - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right).$$

Since f and θ_f are orthogonal one has the equality between the coeficients of $\ddot q_i$ in the local expressions of $\frac{df}{dt}$ and $(\omega_L - \lambda_\nu \omega^\nu)\,\theta_f^2$. But $\dot a_i \dot q_i$

and $M_j\left[\frac{\partial^2 L}{\partial q_i \partial \dot q_j}\dot q_i - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right]$ do not depend on the coordinates $\ddot q_j$, that is both are constant throught each fiber of $P(S(M))$ over $S(M)$. Since they coincide on $\tau(S(M))$ they must be equal and the equality $(\omega_L - \lambda_\nu \omega^\nu)\,\theta_f^2 = \frac{df}{dt}$ holds on the manifold $P(S(M))$.

Corollary 1. Let the system (1) be holonomic ($\omega^\nu \equiv 0$) and f be linear in the velocities. Then f is a first integral of the Lagrangian system $\omega_L = 0$ if, and only if, the equality

$$\omega_L(\theta_f^2) = \frac{df}{dt}$$

holds on the manifold $J_0^2(R, V_m)$.

Theorem 4. Let f be linear in the velocities and θ_f be the orthogonal of f with respect to the scalar product $\langle , \rangle$. Then f is a first integral of system (1) if, and only if, one has:

i) $\theta_f(\pi) = \alpha_\nu \cdot \omega^\nu(\theta_f)$ on V_m.

ii) $\varphi = \beta_\nu \omega^\nu(\theta_f)$ on $S(M)$ where φ is induced by $\theta_f \lrcorner\, d\mathcal{L}$.

iii) $\mathcal{L}_{\theta_f} T_2 = \omega^\nu(\theta_f) \cdot c_{\mu\nu} \cdot \mathcal{L}_{\theta_\mu} T^2$ restricted to the vector bundle $S(M)$.

Proof: By theorem 3 f is a first integral of system (1) if, and only if the equality $(\omega_L - \lambda_\nu \omega^\nu)\,\theta_f^2 = \frac{df}{dt}$ holds on $P(S(M))$, or locally, the equality

$$M_j\left[\frac{\partial^2 L}{\partial q_k \partial \dot q_j}\dot q_k - \frac{\partial L}{\partial q_j} - \lambda_\nu A_{\nu j}\right] = \dot a_i \dot q_i = \frac{\partial a_i}{\partial q_k}\dot q_i \dot q_k$$

must hold on $S(M)$, say, provided that

$$\theta_f = M_j \frac{\partial}{\partial q_j}, \qquad f = a_i \dot q_i, \qquad a_i = M_j a_{ij} \qquad \text{and} \qquad A_{\nu j}\dot q_j = 0.$$

Recall the expressions of the functions $\lambda_\nu = \alpha_\nu + \beta_\nu + \gamma_\nu$:

$$\alpha_\mu = c_{\mu\nu} A_{\nu r} B_{rj} \frac{\partial \pi}{\partial q_j}$$

$$\beta_\mu = c_{\mu\nu} A_{\nu r} B_{rj} \left(\frac{\partial b_j}{\partial q_k} - \frac{\partial b_k}{\partial q_j}\right)\dot q_k$$

$$\gamma_\mu = c_{\mu\nu}\left\{A_{\nu r} B_{rj}\left(\frac{\partial a_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial a_{ik}}{\partial q_j}\right) - \frac{\partial A_{\nu i}}{\partial q_k}\right\}\dot q_i \dot q_k.$$

One can obtain one first conclusion (doing $\dot q_j = 0,\ j = 1,2,\dots m$):

$$M_j\left[\frac{\partial\pi}{\partial q_j} - A_{\nu j}\, c_{\mu\nu}\, A_{\mu r}\, B_{rk}\frac{\partial\pi}{\partial q_k}\right]=0,$$

or $\theta_f(\pi) = M_j A_{\nu j}\,\alpha_\nu = \omega^\nu(\theta_f).\ \alpha_\nu$ which gives rise to the condition i) $\theta_f(\pi) = \alpha_\nu\cdot\omega^\nu(\theta_f)$. This condition i) implies that

$$M_j\left[\frac{\partial^2(T_2+L_1)}{\partial q_k\,\partial\dot q_j}\dot q_k - \frac{\partial(T_2+L_1)}{\partial q_j} - (\beta_\nu+\gamma_\nu)\,A_{\nu j}\right] = \frac{\partial a_i}{\partial q_k}\dot q_i\,\dot q_k$$

for all $(q_j,\dot q_j)$ such that $A_{\nu j}\dot q_j = 0$. Suppose that $(\dot q_{p+1},\dots,\dot q_m, q_1,\dots,q_m)$ are coordinates of $S(M)$ and $\dot q_\nu = f^\nu_\rho\,\dot q_\rho$ are given by the equations $A_{\nu j}\dot q_j=0,\ \nu=1,2,\dots,p$. By derivative with respect to $\dot q_\rho\ (\rho=p+1,\dots,m)$ and doing after $\dot q_\rho = 0$ (then $\dot q_1=\dot q_2=\dots=\dot q_m=0$) one obtains:

$$\frac{d}{d\dot q_\rho}\left\{M_j\frac{\partial^2 L_1}{\partial q_k\,\partial\dot q_j}\dot q_k - \frac{\partial L_1}{\partial q_j} - \beta_\nu A_{\nu j}\right\}_{\dot q_\rho=0} = 0 \quad\text{then}$$

$$\frac{d}{d\dot q_\rho}M_j\left\{\frac{\partial b_j}{\partial q_k}\dot q_k - \frac{\partial b_k}{\partial q_j}\dot q_k - A_{\nu j}\,c_{\nu\mu}\,A_{\mu r}\,B_{rj}\left(\frac{\partial b_j}{\partial q_k}-\frac{\partial b_k}{\partial q_j}\right)\dot q_k\right\}_{\dot q_\rho=0} = 0.$$

The expression $H_k\dot q_k$ to be derivated, is linear and homogeneous in the coordinates $\dot q_j\ (j=1,2,\dots,m)$. We put $H_k\dot q_k = H_\rho\dot q_\rho + H_\nu\dot q_\nu$, $\nu=1,2,\dots,p$; then,

(11) $$\left.\frac{d(H_k\dot q_k)}{d\dot q_\rho}\right|_{\dot q_\rho=0} = H_\rho + H_\nu f^\nu_\rho = 0;\quad \rho=p+1,\dots,m.$$

Those equalities (11) show that $H_\rho\dot q_\rho + H_\nu f^\nu_\rho\dot q_\rho = H_\rho \dot q_\rho + H_\nu\dot q_\nu = H_k\dot q_k = 0$ and then $M_j\left[\frac{\partial^2 T_2}{\partial q_k\,\partial\dot q_j}\dot q_k - \frac{\partial T_2}{\partial q_j} - \gamma_\nu A_{\nu j}\right] = \frac{\partial a_i}{\partial q_k}\dot q_i\,\dot q_k$ and

$$H_k\dot q_k = \dot q_k\left\{M_j\left(\frac{\partial b_j}{\partial q_k}-\frac{\partial b_k}{\partial q_j}\right) - M_j\,A_{\nu j}\,c_{\nu\mu}\,A_{\mu r}\,B_{rj}\left(\frac{\partial b_j}{\partial q_k}-\frac{\partial b_k}{\partial q_j}\right)\right\}=0.$$

The Pfaffian form $\theta_f \lrcorner\, d\mathscr{L}$ ($\mathscr{L}$ is the Pfaffian form defined by the friction energy L_1) is locally given by

$$\theta_f\lrcorner\, d\mathscr{L} = M_j\left[\frac{\partial b_j}{\partial q_k}-\frac{\partial b_k}{\partial q_j}\right]dq_k$$

and the function (linear in the velocities) induced by $\theta_f\lrcorner\, d\mathscr{L}$ on

$$J^1_0(R,V_m) \quad\text{is}\quad \varphi = M_j\left[\frac{\partial b_j}{\partial q_k}-\frac{\partial b_k}{\partial q_j}\right]\dot q_k.$$

By theorem 2 the functions β_μ are globally defined on $J^1_0(R,V_m)$ and $H_k\dot q_k=0$ means $\varphi-\beta_\mu\cdot\omega^\mu(\theta_f)=0$, provided that $A_{\nu j}\dot q_j=0$. Then one can obtain the second conclusion:

ii) $\varphi = \beta_\mu\,\omega^\mu(\theta_f)$ on the manifold $S(M)$.

The last conclusion comes from (12):

(12) $$\left[M_j\left(\frac{\partial a_{ji}}{\partial q_k} - \frac{1}{2}\frac{\partial a_{ik}}{\partial q_j}\right) - \frac{\partial a_i}{\partial q_k}\right]\dot q_i\,\dot q_k = M_j\,\gamma_\nu\,A_{\nu j} =$$
$$= M_j\,A_{\nu j}\,c_{\mu\nu}\left[A_{\nu r}\,B_{rj}\left(\frac{\partial a_{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial a_{ik}}{\partial q_j}\right)-\frac{\partial A_{\nu i}}{\partial q_k}\right]\dot q_i\,\dot q_k$$

that must hold on $S(M)$, say $H_{ik}\dot q_i\,\dot q_k = 0$ provided that $A_{\nu j}\dot q_j=0$. The symmetric tensor field on V_m with components $(H_{ik}+H_{ki})$ must be zero when computed on pairs of vectors which are "tangent" to the distribution M. Let us show that, in fact, $(H_{ik}+H_{ki})$ are the components of a covariant symmetric tensor field of degree two on the manifold V_m. The sum $(H_{ik}+H_{ki})$ is equal to

$$\left(\frac{\partial a_i}{\partial q_k}+\frac{\partial a_k}{\partial q_i}\right) - M_j\left(\frac{\partial a_{ij}}{\partial q_k}+\frac{\partial a_{kj}}{\partial q_i}-\frac{\partial a_{ik}}{\partial q_j}\right) -$$
$$- M_j\,A_{\mu j}\,c_{\mu\nu}\left[\left(\frac{\partial A_{\nu i}}{\partial q_k}+\frac{\partial A_{\nu k}}{\partial q_i}\right) - A_{\nu r}\,B_{rm}\left(\frac{\partial a_{mi}}{\partial q_k}+\frac{\partial a_{mk}}{\partial q_i}-\frac{\partial a_{ik}}{\partial q_m}\right)\right].$$

Now we will show that the Lie derivative $\mathscr{L}_{\theta_f}T_2$ of the metric tensor T_2 with respect to the vector field θ_f has components A_{ik} given by

$$A_{ik} = \left(\frac{\partial a_i}{\partial q_k}+\frac{\partial a_k}{\partial q_i}\right) - M_j\left(\frac{\partial a_{ij}}{\partial q_k}+\frac{\partial a_{kj}}{\partial q_i}-\frac{\partial a_{ik}}{\partial q_j}\right),$$

then the tensor

$$\mathscr{L}_{\theta_f}T_2 - C_{\mu\nu}\,\mathscr{L}_{\theta_\nu}T_2 .\, w^\mu(\theta_f)$$

must vanish on the vectors which are "tangent" to the distribution M. Since

$$a_i = M_j\, a_{ij}\,, \quad \frac{\partial a_i}{\partial q_r} = M_j \frac{\partial a_{ij}}{\partial q_r} + \frac{\partial M_j}{\partial q_r} a_{ij} \quad \text{then}$$

$$A_{ik} = M_j \cdot \frac{\partial a_{ik}}{\partial q_j} + \frac{\partial M_j}{\partial q_k} \cdot a_{ij} + \frac{\partial M_j}{\partial q_i} a_{kj}$$

and it is easy to see that

$$\mathscr{L}_{\theta_f} T_2 = A_{ik}\, dq_i \otimes dq_k.$$

The last conclusion is then

iii) $\mathscr{L}_{\theta_f} T_2 = \omega^\nu(\theta_f) \cdot c_{\mu\nu} \cdot \mathscr{L}_{\theta_\mu} T_2$ when both tensor fields are restricted to the fiber bundle $S(M)$ over V_m.

Corollary 2. Let the system (1) be holonomic ($\omega^\nu \equiv 0$) and f be linear in the velocities. Then f is a first integral of the Lagrangian system $\omega_L = 0$ if, and only if,

i) $\theta_f(\pi) = 0$.

ii) $\theta_f \lrcorner\, d\mathscr{L} = 0$.

iii) $\mathscr{L}_{\theta_f} T_2 = 0$.

Condition $\theta_f(\pi) = 0$ means that π is a first integral for θ_f, say, π is constant throught the integral curves of the flow θ_f. Condition ii) means that θ_f is a characteristic vector field for the Pfaffian system spanned by $\mathscr{L}$; and condition $\mathscr{L}_{\theta_f} T_2 = 0$ means that θ_f is an infinitesimal isometry of the Riemannian metric $\langle , \rangle$ (θ_f is a symmetry of the Riemannian manifold V_m).

Remark. Given the distribution M, say, the bundle $S(M)$, it induces a normal bundle $M^\perp$ by orthogonality with respect to the scalar product $\langle , \rangle$. It is easy to see that if ω^ν are the Pfaffian forms which define M (at least locally), the vector-fields θ_ν form a local basis for $M^\perp$. Furthermore, any vector-field θ has a canonical decomposition $\theta = \bar{\theta} + \bar{\bar{\theta}}$ where $\bar{\theta}$ is "tangent" to M and $\bar{\bar{\theta}}$ is "normal" to M, say, is "tangent" to the normal bundle $M^\perp$.

Steady Motions

We will call a *motion*, any solution of the system (1). In this paper we reach for a kind of motion which is also an integral curve of a vector-field θ_f, orthogonal of a first integral, of (1) "linear in the velocities". If such a motion there exists it will be called a *steady motion.* As we will see the steady motions come from special vector-fields θ_f which are symmetries of the manifold, in some sense.

The research of steady motions is then equivalent to the research of first integrals, linear in the velocities. But in most of physical examples one can see some steady motions and, using them, one can find the corresponding first integrals.

Steady motions in the holonomic case.

By Corollary 2, f is a first integral linear in the velocities, if, and only if,

$$\theta_f(\pi) = \theta_f \lrcorner\, d\mathscr{L} = \mathscr{L}_{\theta_f} T_2 = 0.$$

Then, in the case that there is no friction energy ($\mathscr{L} \equiv 0$) the orthogonal vector-field θ_f must be an infinitesimal isometry and $\theta_f(\pi)$ must be zero. This way if $\varphi(t)$ is a steady motion it follows that $\pi(\varphi(t))$ does not depend on t and by Proposition 2 the same happens with $T_2(\varphi(t)), \dot{\varphi}(t))$.

Example 1. *The spherical pendulum.*

This is the situation of a material point which moves on the manifold $V_m = S^2$ Let θ, $0 < \theta < \pi/2$, the latitude and φ the longitude of a point on S^2. The kinetic energy is

$$T_2 = \tfrac{1}{2}[\dot{\theta}^2 + (\cos^2\theta)\,\dot{\varphi}^2],$$

there is no friction and the potential energy is $\pi = -\sin\theta . g$

The Lagrangian equations in that local chart are:

$$\begin{cases} \ddot{\theta} + \sin\theta \cdot \cos\theta \cdot \dot{\varphi}^2 - g\cos\theta = 0 \\ \cos\theta\ddot{\varphi} - 2\sin\theta \cdot \dot{\varphi} \cdot \dot{\theta} = 0 \end{cases}$$

The linear in the velocities function $f = (\cos^2\theta) \cdot \dot{\varphi}$ is a first integral for the considered Lagrangian system since

$$\dot{f} = (\cos^2\theta)\,\ddot{\varphi} - 2\cos\theta\sin\theta\dot{\theta}\dot{\varphi} = 0.$$

The vector-field θ_f orthogonal of f with respect to T_2 is

$$\theta_f = 2 \cdot \frac{\partial}{\partial\varphi}.$$

The steady motions, integral curves of $\theta_f = 2\dfrac{\partial}{\partial\varphi}$, are, then, horizontal circles on S^2. More precisely, with the initial conditions $\theta = \theta_0$; $\varphi = \varphi_0$; $\dot{\theta}_0 = 0$; $\dot{\varphi}_0 = \sqrt{\dfrac{g}{\sin\theta_0}}$, the motion of equations

$$\theta = \theta_0; \quad \varphi = \varphi_0 + t \cdot \sqrt{g/\sin\theta_0}$$

is a steady motion of the given holonomic Lagrangian system.

Steady motions in the non-holonomic case.

If a motion $\varphi(t)$ must come from a vector-field θ_f, orthogonal of a first integral f, linear in the velocities, it is clear that $\dot{\varphi}(t)$ is "tangent" to M, that is, $\omega^v(\dot{\varphi}(t)) = 0$ for all t. This shows that the decomposition $\theta_f = \bar{\theta} + \bar{\bar{\theta}}$ (with $\bar{\theta}$ "tangent" to M and $\bar{\bar{\theta}}$ "normal" to M) induces a decomposition $f = \bar{f} + \bar{\bar{f}}$ where $\bar{\bar{f}}$ is identically zero on tha bundle $S(M)$; $\bar{f}$ is also a first integral of system (1) and coincides with f on $S(M)$. The motion $\varphi(t)$ is then a solution of $\bar{\theta} = \theta_{\bar{f}}$. By Theorem 4 appied to $\bar{f}$ one can say that:

a) $\theta_{\bar{f}}(\pi) = 0$;

b) the function φ induced by $\theta_{\bar{f}} \lrcorner\, d\mathscr{L}$ is zero on $S(M)$;

c) $\mathscr{L}_{\theta_{\bar{f}}} T_2$ is zero when restricted to the bundle $S(M)$.

These considerations show that if $\varphi(t)$ is a motion, the following conditions are equivalent:

I) $\varphi(t)$ is a steady motion;

II) $\varphi(t)$ is an integral curve of a "horizontal" (tangent to M) vector-field $\theta_{\bar{f}}$, orthogonal of a first integral $\bar{f}$ of system (1), linear in the velocities;

III) $\varphi(t)$ is an integral curve of a "horizontal" (tangent to M) vector-field $\theta_{\bar{f}}$ satisfying conditions a), b) and c).

Example 2. A homogeneous vertical material disc which rolls without sliping on a horizontal plane.

The manifold is, in this case, the product $R^2 \times T^2$. Let x, y, φ, θ be local coordinates where (x, y) are Cartesian retangular coordinates, φ is the rotation of the disc and θ is the angle between the vertical plane of the disc with another vertical plane. The non-holonomic Lagrangian system is:

$$\ddot{x} = \lambda_1$$
$$\ddot{y} = \lambda_2$$
$$\ddot{\varphi} = -2(\lambda_1 \,.\, \cos\theta + \lambda_2 \,.\, \sin\theta)$$
$$\ddot{\theta} = 0$$
$$\dot{x} - \cos\theta \cdot \dot{\varphi} = 0$$
$$\dot{y} - \sin\theta \cdot \dot{\varphi} = 0$$

The kinetic energy is $T_2 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{8}(2\dot{\varphi}^2 + \dot{\theta}^2)$, there is no friction and the potential energy is zero.

It is easy to see that $f_1 = \dot{\varphi}$ and $f_2 = \dot{\theta}$ are linear in the velocities first integrals of the system. A simple computation shows that $\theta_{f_1} = 4\dfrac{\partial}{\partial\varphi}$ and $\theta_{f_2} = 8\dfrac{\partial}{\partial\theta}$; θ_{f_2} is "tangent" to the distribution M, in this case given by the Pfaffian forms

$$\omega^1 = dx - \cos\theta \cdot d\varphi \quad \text{and} \quad \omega^2 = dy - \sin\theta \cdot d\varphi.$$

The vector field θ_{f_1} is not "tangent" to M. The steady motions defined by θ_{f_1} are also solutions of the projection on M of it:

$$\bar{\theta}_{f_1} = \tfrac{4}{3}\left(\cos\theta \cdot \frac{\partial}{\partial x} + \sin\theta \cdot \frac{\partial}{\partial y} + \frac{\partial}{\partial\varphi}\right)$$

One can prove that in this example all motions are steady motions. The normal bundle is spanned by $\left(2\dfrac{\partial}{\partial x} - 4\cos\theta \cdot \dfrac{\partial}{\partial\varphi}\right)$ and $\left(2\dfrac{\partial}{\partial y} - 4\sin\theta\dfrac{\partial}{\partial\varphi}\right)$, and the bundle $S(M)$ has generators $\dfrac{\partial}{\partial\theta}$ and $\cos\theta\dfrac{\partial}{\partial x} + \sin\theta\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial\varphi}$.

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University of São Paulo and
University of Campinas.