Composition of contractions

by
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A map $f: X \to X$ of a metric space is a contraction if for some λ, $0 \le \lambda \le 1$, $d(fx, fy) \le \lambda\, d(x, y)$ for all $x, y \in X$. The least such λ is the Lipschitz constant, $L(f)$. If X is complete, a contraction f has a unique fixed point which we call $F(f)$.

Now suppose m contractions $f_1, \ldots, f_m : X \to X$ are given, where X is complete. Then each composite "word" $w = f_{i_1} \circ \ldots \circ f_{i_r}$ has a unique fixed point $F(w)$. Here we are concerned with the closure $\bar{F}$ of the set $F = F(f_1, \ldots, f_m)$ of all such fixed points, $F(w)$. This paper can be regarded as a step toward studying generic properties of the action of free (non-abelian) groups on manifolds. See S. Smale [2]. Conversations with R. Thom and S. Smale were very helpful in writing this paper.

We would also like to thank P. Fernandez who pointed out that we had overlooked compactness and, in particular, our most general result:

Theorem A'. For any finite set of contractions, $\bar{F}$ is compact.

Note, however, that this is included in theorem *D*, below.

Theorem A. If the Lipschitz constants satisfy $L(f_1) + \ldots + L(f_m) < 1$, then $\bar{F}(f_1, \ldots, f_m)$ is zero dimensional.

In general $\bar{F}$ can have dimension > 0. Our most general result in this conection is

Theorem B. If $f, g : \mathbb{R} \to \mathbb{R}$ are two 1-1 contractions with distinct fixed points and if

$$(*) \quad L(f^{-1})^{-1} + L(g^{-1})^{-1} \ge 1$$

then $\bar{F}(f, g)$ is a closed line interval.

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Note that (*) reduces to $L(f) + L(g) \geq 1$ in case f and g are affine. We know of no counterpart to theorem B for $R^n, n \geq 2$ except consequences of theorem B in case f and g (and possibly other maps) have something like an "eigen line" in common. The proof of theorem B uses the elementary:

Principle C. If $A \subset X$ is compact, invariant under each f_i and if $f_1(A) \cup \ldots \cup f_m(A) \supset A$, then $A = \overline{F}(f_1, \ldots, f_m)$.

The difficulty with analogues to theorem B for R^n, $n \geq 2$, seems to be that this principle does not apply.

Question: What is the structure of $\overline{F}(f, g)$ for $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, affine contractions satisfiyng (*).

We close with the remark that the space W of all words $w = f_{i_1} \circ \ldots \circ f_{i_r}$ has a natural topology with Cantor set $\hat{W}$ as compatification. In forming W we identify u and v if $u^n = v^r$ for some n, r. In this case, of course, $F(u) = F(v)$.

Theorem D. If $f_i : X \to X$, $i = 1, \ldots, m$ are contractions, then the function $\Phi : \hat{W} \to \overline{F}$ induced by $w \to F(w)$ is continuous. If in addition

a) the f_i are all 1-1;
b) the fixed points $F(f_1), \ldots, F(f_n)$ are distinct;
and
c) $L(f_1) + \ldots + L(f_m) < 1$,
then Φ is a homeomorphism.

Proof of Theorem A: Lef $L(f_i) = \lambda_i$, $i = 1, \ldots, m$.

Step 1. There is a closed and bounded set $A \subset X$ such that $f_i(A) \subset A$ for $i = 1, \ldots, m$. Hence $\overline{F} \subset A$.

Proof: We may take A to be the closed ε-neighborhood of $F(f_1)$ for $\varepsilon > 0$ and so large that

$$\lambda_i(\varepsilon + d_i) + d_i < \varepsilon, \quad i = 1, \ldots, m,$$

where $d_i = d[F(f_1), F(f_i)]$. For then $f_i(A) \subset A$, for each i so that in particular $w(A) \subset A$ for each word w. Thus $w|A$ has a fixed point which must be $F(w)$ so that $F \subset A$. As A is closed, $\overline{F} \subset A$.

Next, let W_n consist of all words w of length n.

Step 2. For each $\varepsilon > 0$, there is an integer n such that

$$\sum_{w \in W_r} \text{diam } w(A) < \varepsilon.$$

Proof. For $w = f_{i_1} \circ \ldots \circ f_{i_n}$, diam $w(A) \leq \lambda_{i_1} \cdot \lambda_{i_2} \ldots \lambda_{i_n} \cdot$ (diam A) so that

$$\sum_{w \in W_n} \text{diam } w(A) \leq (\lambda_1 + \ldots + \lambda_m)^n \text{ diam } A.$$

Step 3. $\overline{F} \subset \bigcup_{w \in W_n} \overline{w(A)} = A_n$ for each n.

Proof. For if w' is a word of length $\geq n$, then $w' = w \circ w''$ where w has length n. Hence $F(w') \in w'(A) \subset \overline{w(A)} \subset A_n$. For w any word $F(w) = F(w \circ \ldots \circ w)$ (n-times) $\subset A_n$.

Theorem A now follows, as no component of $\overline{F}$ could have positive diameter.

Note also that step 3 plus all but the last line of step 2 shows that $\overline{F}$ is totally bounded and hence compact. This gives a more direct proof of Theorem A' than is provided by Theorem D.

Remark. We have shown that the Hausdorf p-measure [1; p. 102] of $\overline{F}$ is 0 for $p = 1$.

Proof of principle C. Note that the inclusion $\overline{F} \subset A$ is proved just as above. Next, it follows by induction on n that $\bigcup_{w \in W_n} w(A) = A$. But diam $w(A) \leq \lambda^n A$ for $w \in W_n$ where $\lambda_i \leq \lambda < 1$ for $i = 1, \ldots, m$. Therefore, as $F(w) \subset w(A)$, $F(f_1, \ldots, f_m)$ is dense in A so that $\overline{F}(f_1, \ldots, f_m) = A$.

Proof of Theorem B. We find an interval I for which $f(I) \cup g(I) = I$ and apply principle C.

Case 1. f and g preserve orientation. Then let $I = [F(f), F(g)]$. Then $f(I) \subset I$ as f preserves orientation; similarly for g. Now $f(I) \cup g(I)$ contains I's end points and as the sum the lengths of $f(I)$ and $g(I) \geq$

$$[L(f^{-1})^{-1} + L(g^{-1})^{-1}] \text{ length I}, f(I) \cap g(I) \neq \emptyset.$$

Hence $f(I) \cup g(I) = I$.

Case. 2. f reverses orientation, g preserves orientation and $F(f) = a < b = F(g)$. Then let $c = f(b)$ and note $c \geq a$. Let $I = [c, b]$. Again $f(I) \cup g(I) \subset I$, $c \in f(I)$, $b \in g(I)$ and as the lengths of $f(I)$ and $g(I)$ together exceed that of I, $f(I) \cup g(I) = I$.

Case 3. Both f and g reverse orientation.

Let $a = F(f)$, $b = F(g)$ and assume $a < b$. Now let $c = F(f \circ g)$ and $d = F(g \circ f)$.

Step 1. $c < a$ and $b < d$.

Proof. First assume $c \in [a, b]$. Then $g(c) \geq b > a$ as g reverses orientation, so that $c = fg(c) < a \leq c$ which is absurd. Next assume $c > b$. Then $gc < a$ as otherwise $fgc < a$. Hence

$$d(gc, a) = a - gc < b - gc = d(b, gc) < d(b, c)$$

so that $d(fgc, fa) < d(b, c) < d(c, a)$. This last is absurd as the first and third terms are identical. Thus $c \leq a$. But $c = F(fg) = a$ is impossible as $g(a) \neq a$ and f is 1-1. Therefore $c < a$. Similarly, $b < d$. (Note: we have not yet used the special assumption (*)).

Step 2. Let $I = [c, d]$. Then $f(I) \cup g(I) \supset I$.

Proof. First, as $gf(d) = d, fgf(d) = f(d)$ so that $f(d) = c$. Similarly $g(c) = d$. Thus $f(I) \cup g(I)$ contains I's end points and as above,

$$f(I) \cup g(I) \supset I.$$

The space $\hat{W}$ and proof of theorem D. Let $W = \cup_{n=1}^{\infty} W_n$ be the set of all (finite) words and for $w = f_{i_1} \circ \ldots \circ f_{i_n} \in W$, let $\hat{w}$ be the infinite, periodic word $i_1, \ldots, i_n, i_1, \ldots, i_n, i_1, \ldots$. Let $\hat{W}$ consist of all infinite "words" or sequences $i_1, i_2, \ldots$ of the integers $\{1, \ldots, m\}$. Then $w \mapsto \hat{w}$ sends W into $\hat{W}$, identifying $\hat{w}$ and $\hat{u}$ if and only if $w^n = u^r$ for some n and r. A metric for $\hat{W}$ is

$$d(\hat{w}, \hat{u}) = \sum_{i=1}^{\infty} \sigma(\hat{w}_i, \hat{u}_i) 2^{-i}$$

Where

$$\sigma(i, j) = \begin{cases} 1 & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j. \end{cases}$$

$\hat{W}$ is the "one-sided symbol space on m symbols" and is a Cantor set.

For $w \in W$, define $\phi(\hat{w}) = F(w)$. This is well defined by our remark above. As the periodic words are dense in $\hat{W}$, we need only to see that ϕ is uniformly continuous to know that it extends uniquely to $\phi : \hat{W} \to F$.

To this end, suppose, $w, v \in W$, where w has length n. Then $F(w)$ and $F(w \circ v)$ lie together in $w(A)$, a set of diameter $\leq \lambda^n$ diam A, where A, λ are as in the proof of theorem A. Hence $d(F(w), F(w \circ v)) \leq \lambda^n$ diam A, which shows ϕ is uniformly continuous. This proves the first assertion of Theorem D.

Now, assume the additional hypothesis the second part of Theorem D and let w and u be distinct words.

Case 1. w and u are of the same finite length n. We claim $F(w) \neq F(u)$. For $n = 1$, this is part of hypothesis. For the inductive step, consider

case 1a. $w = v \circ w'$, $u = v \circ u'$ where by induction, $F(u') \neq F(w')$. Then for sufficiently large n, $w' \circ (v \circ w')^n(A)$ and $u' \circ (v \circ u')^n(A)$ are disjoint as they contain $F(w')$ and $F(u')$ respectively. Composing with v, we know $(v \circ u')^{n+1}(A) \cap (v \circ w')^{n+1}(A) = \varnothing$ as v is 1-1. Thus as these sets contain $F(u)$, and $F(w)$ respectively, $F(u) \neq F(w)$.

Case 1b. $w = w' \circ v$, $u = u' \circ v$ where by induction, $F(w') \neq F(u')$. As above, $(w' \circ v)^{n+1}(A) \cap (u' \circ v)^{n+1}(A) = \varnothing$ for these sets contain the distinct points $F(w')$ and $F(u')$. Hence $F(w) \neq F(u)$.

Case 2. w and u are distinct infinite words. Let w_i and u_i denote the finite words consisting of the first i terms of w and u respectively. Then for some n, $\phi(\hat{w}_n) \neq \phi(\hat{u}_n)$, but for i sufficiently large, $w_{n+i}(A) \cap \cap u_{n+i}(A) = \varnothing$ as above. As there sets contain $\phi(w)$ and $\phi(u)$ respectively, $\phi(w) \neq \phi(u)$.

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