Compact riemann surfaces with prescribed ramifications and Puiseux series

by

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It is well known that $\mathbb{C}(z)$, the field of rational functions over $\mathbb{C}$, coincides with the field $M(S_0)$ of meromorphic functions on the Riemann sphere S_0, when any $a(z) \in \mathbb{C}(z)$ is identified with the mapping $a : S_0 \to S_0$ defined by $\zeta \to a(\zeta)$ (with the usual convention with respect to ∞).

Let K be a field of algebraic functions, i.e. a finite field extension of $\mathbb{C}(z)$. Then $K = \mathbb{C}(z, w)$, where w is a root of some irreducible polynomial $F(W) = W^n + a_1(z) \cdot W^{n-1} + \ldots + a_n(z) \in \mathbb{C}(z)[W]$. The elements of K cannot be interpreted as functions on S_0, but as functions on the Riemann surface S_K associated with the field K. S_K can be defined in an analytical way (see for example [5] or [8]) or by means of valuation theory (cf. [1]). In fact, let S_K be the set of all normalized discrete valuations of K over $\mathbb{C}$, i.e. surjective mappings $v : K \to \mathbb{Z} \cup \{\infty\}$ such that $v0 = \infty$, $v(f \cdot g) = vf + vg$, $v(f + g) \geq \min\{vf, vg\}$ for all $f, g \in K$, and $v\zeta = 0$ for all non-zero $\zeta \in \mathbb{C}$. Any $f \in K$ is identified with the mapping $S_K \to S_0$ which assigns to each $v \in S_K$ the unique $\zeta_{f,v} \in S_0$ such that $v(f - \zeta_{f,v}) > 0$ (where $v(f - \infty) > 0$ means $vf < 0$), and S_K is endowed with the weakest topology with respect to which all mappings $f \in K$ are continuous; it is compact. Any $t_v \in K$ such that $vt_v = 1$ is a local chart for the point $v \in S_k$ (i.e. induces a homeomorphism from some neighborhood of v onto a neighborhood of 0); thus the notion of "meromorphic function" can be carried over to S_K. The field K coincides with the field $M(S_K)$ of meromorphic functions on S_K, and the containment $\mathbb{C}(z) \subseteq K$ implies the imbedding of $M(S_0)$ in $M(S_K)$ by $a(z) \to a \circ z$.

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Considering the projection $z: S_K \to S_0$, we say that the point $v \in S_K$ lies over $\zeta \in S_0$ if $z(v) = \zeta$, and the positive integer $e(v) = v(z-\zeta)$[2] is called the *ramification index* of v (with respect to z). For any $\zeta \in S_0$ there exist only finitely many points $v_1, \ldots, v_r \in S_K$ which lie over ζ, and we have $e(v_1) + \ldots + e(v_r) = n$. In particular, $r = n$ if and only if $e(v_1) = \ldots = e(v_r) = 1$. These equations hold for almost all $\zeta \in S_0$; in fact, any *branch* point (i.e. $v \in S_K$ such that $e(v) > 1$) lies over a pole of $a_1(z)$ or $a_2(z)$ or ... or $a_n(z)$, or over a root of the discriminant $d(z) \varepsilon \mathbb{D}(z)$ of $F(W)$, or over ∞. Therefore the sum $\sum_{v \varepsilon S_K} (e(v) - 1)$ *is finite*; it equals $2 \cdot (g_K + n - 1)$, where g_K is the genus of S_K (see [5] or [8]).

It is natural to ask whether there exists a field K of algebraic functions such that S_K has prescribed ramifications. Using results due to HASS [6] and KRULL [7][3], one can prove the following theorem.

Theorem 1. *Let H be a finite subset of S_0 and, for any $\eta \in H$, let r_η, $e(\eta, 1), \ldots, e(\eta, r_\eta)$ be positive integers such that $e(\eta, 1) + \ldots + e(\eta, r_\eta) = n$. Then there is a field extension K of $\mathbb{C}(z)$ of degree n such that, for any $\eta \in H$, there are exactly r_η points $v_{\eta,i}$ lying over η, and $e(v_{\eta,i}) = e(\eta, i)$ $(i = 1, \ldots, r_\eta)$.*

Note that S_K may have (finitely many) branch points outside the set $H_K = \{v_{\eta,i} \mid i = 1, \ldots, r_\eta\,;\ \eta \in H\}$; hence for the genus of S_K we get only the inequality $g_K \geq 1 - n + \frac{1}{2} \cdot \sum_{\eta \in H} \sum_{i=1}^{r_\eta} (e(\eta, i) - 1)$. For example, setting $r_\eta = 1$ and $e(\eta, 1) = 2$ for any $\eta \in H$, we obtain a quadratic field extension K of $\mathbb{C}(z)$ such that S_K has at least $\# H$ branch points and $g_K \geq \frac{1}{2} \# H - 1$.

Under certain hypotheses, it is possible to restrict the ramification indices of all points outside H_K, as will be shown in the following corollaries. We say that a finite r-tuple $E = (e(1), \ldots, e(r))$ of positive integers such that $e(1) + \ldots + e(r) = n$ *factors* according to the factorization $n = n' \cdot n''$ if there exist positive integers r', r_i'', $e'(i)$, $e''(i, j)$ such that $r_1'' + \ldots + r_{r'}'' = r$, $e'(1) + \ldots + e'(r') = n'$, $e''(i, 1) + \ldots + e''(i, r_i'') = n''$, and $e'(i) \cdot e''(i, j) = e(r_1'' + \ldots + r_{i-1}'' + j)$ for all $j = 1, \ldots, r_i''$ and $i = 1, \ldots, r'$. Factorization of E according to a factorization $n = n_1 \cdot \ldots \cdot n_m$ with $m > 2$ is defined similarly.

Corollary 1. *With the same hypotheses as in theorem 1, assume that the r_η-tuple $E_\eta = (e(\eta, 1), \ldots, e(\eta, r_\eta))$ factors according to the factorization $n = n_1 \cdot \ldots \cdot n_m$, for any $\eta \in H$. Then there exists a field K*

[2] Here and in the following, substitute z^{-1} for $z - \infty$.

[3] For a résumé without proofs see [3].

with the properties indicated in theorem 1 such that $e(v) \leq \max\{n_1, \ldots, n_m\}$ for all $v \in S_K \setminus H_K$.

In fact, we can construct a chain of fields $\mathbb{C}(z) = K_0 \subset K_1 \subset \subset \ldots \subset K_m = K$ in which K_j is obtained from K_{j-1} in the following manner (using a slight generalization of theorem 1, with K_{j-1} instead of $\mathbb{C}(z)$): For any $\mu \in S_{K_{j-1}}$ lying over some $\eta \in H$ one prescribes an r_μ-tuple $E_\mu = (e(\mu, 1), \ldots, e(\mu, r_\mu))$ of positive integers such that $e(\mu, 1) + \ldots + e(\mu, r_\mu) = n_j$, according to the factorization of E_η, and for any branch point $\mu' \in S_{K_{j-1}}$ lying over some $\eta' \in S_0 \backslash H$ one prescribes the n_j-tuple $(1, \ldots, 1)$.

In the case $n = n_1^m$ any r-tuple $E = (e(1), \ldots, e(r))$ such that $e(1) \geq \ldots \geq e(r)$, $e(1) + \ldots + e(r) = n$, and all $e(i)$ are non-negative powers of n_1, factors according to the factorization $n = n_1 \cdot \ldots \cdot n_1$ (m times), as is checked easily. Therefore corollary 1 yields:

Corollary 2. *With the same hypotheses as in theorem 1, assume that n and all $e(\eta, i)$ $(i = 1, \ldots, r_\eta\,;\ \eta \in H)$ are non-negative powers of some positive integer n_1. Then there exists a field K with the properties indicated in theorem 1 such that $e(v) \leq n_1$ for all $v \in S_K \setminus H_K$.*

For example, there exists a field extension K of $\mathbb{C}(z)$ of degree 16 with the following properties: There are exactly 2 (resp. 1, resp. 11) points of S_K lying over the point 0 (resp. 1, resp. ∞) of S_0, with ramification indices 8, 8 (resp. 16, resp. 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1), whereas all other branch points of S_K have ramification index 2.

We are going to show that we may prescribe, in addition to the ramification indices $e(\eta, 1), \ldots, e(\eta, r_\eta)$, the initial parts of the *Puiseux series* $P_{\eta,i} = \sum_{q=-s}^{\infty} \omega_{\eta,i,q} \cdot (z-\eta)^{q/e(\eta,i)}$ $(i = 1, \ldots, r_\eta)$ for a generator w of K over $\mathbb{C}(z)$. To make this statement more precise, we recall that for any $\zeta \in S_0$ the field $M_\zeta(S_0)$ of germs of meromorphic functions at ζ contains $M(S_0)$ and is contained in the quotient field $\mathbb{C}((z-\zeta))$ of the ring $\mathbb{C}[[z-\zeta]]$ of formal power series in $z-\zeta$, with coefficients in $\mathbb{C}$. Similarly, for any finite extension K of $\mathbb{C}(z)$ and any $v \in S_K$, the field $M_v(S_K)$ of germs of meromorphic functions at v contains $M(S_K)$ and is contained in $\mathbb{C}((t\{v\}))$, where $t\{v\}$ is an $e(v)$-th root of $z - z(v)$. The containment $\mathbb{C}((z - z(v))) \subset \mathbb{C}((t\{v\}))$ implies the imbedding of $M_{z(v)}(S_0)$ in $M_v(S_K)$ by $f \to f \circ z$, and we have $[M_v(S_K) : M_{z(v)}(S_0)] = e(v)$. In particular, any element $c \in K = M(S_K)$ may be represented by a Laurent series in $t\{v\}$, say $c = \sum_{q=-s}^{\infty} \gamma_{v;q} \cdot (t\{v\})^q$, which is sometimes written as a Puiseux series $\sum_{q=-s}^{\infty} \gamma_{v,q} \cdot (z - z(v))^{q/e(v)}$.

Using a theorem on valuations ([2], Satz 7, or [4], theorem (25.7))[3], which strengthens the above mentioned results obtained by HASSE and KRULL, we get:

Theorem 2. *With the same hypotheses as in theorem* 1, *let* $\Omega_{\eta,i} = (\omega_{\eta,i,q})_{q\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ *such that* $\{q \in \mathbb{Z} \setminus \mathbb{N} \mid \omega_{\eta,i,q} \neq 0\}$ *is finite* $(i = 1,\ldots,r_\eta;\ \eta \in H)$. *Then for any* $q_0 \in \mathbb{Z}$ *there exists an algebraic function* w *such that* $K = \mathbb{C}(z,w)$ *has the properties indicated in theorem* 1 *and*

$$w - \sum_{q\in\mathbb{Z}} \omega_{\eta,i,q} \cdot t_{\eta,i}^{\ q} \in t_{\eta,i}^{q_0+1} \cdot \mathbb{C}[[t_{\eta,i}]]$$

for any $\eta \in H$ *and* $i \in \{1, \ldots, r_\eta\}$, *where* $t_{\eta,i} = t\{v_{\eta,i}\}$.

In other words, the coefficients of the Laurent series in $t\{v_{\eta,i}\}$ of some generator w of K over $\mathbb{C}(z)$ coincide with those of the prescribed series $\sum_{q\in\mathbb{Z}} \omega_{\eta,i,q} \cdot X^q$ up to the q_0-th coefficient, for any $\eta \in H$ and $i = 1,\ldots,r_\eta$.

In the special case $n = 1$, theorem 2 is well known. It affirms the existence of a rational function $w \in \mathbb{C}(z)$ which has prescribed Laurent series at any $\eta \in H$, up to the q_0-th coefficient.

Most part of the results presented in this paper can be easily generalized to algebraic functions over an arbitrary field K_0 (instead of $\mathbb{C}$).

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