A note on random sums

by

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In [1] the following theorem was proved.

Theorem 1. Suppose that $X_1, X_2, \dots$ is a sequence of independent and identically distributed random variables with finite mean $m \neq 0$.

Suppose also that τ_n is a sequence of positive integer valued random variables for which

$$\text{(I)} \qquad \lim_{n\to\infty} P\left[\frac{\tau_n}{a_n} \leq x\right] = G(x), \qquad G(+0) = 0$$

where $0 < a_n \to +\infty$ and G is a distribution function then

$$\lim_{n\to\infty} P\left[\frac{\sum_{i=1}^{\tau_n} X_i}{ma_n} \leq x\right] = G(x)$$

The purpose of this note is to give a short proof of a generalization of this result.

Note that the condition (I) implies $\tau_n \xrightarrow{P} +\infty$. In what follows we use the notation of [2], Chapter 1. Since there exist applications to sequences in $C[0,1]$ and the proof is essentially the same as in the case of real valued random variables we take random elements taking values in a separable normed space E.

Theorem 2. Let $\{Y_n\}_{n=1,2,\dots}$ be a sequence of random variables taking values in E and suppose $Y_n \xrightarrow{a\cdot s} a$ (a point of E) and $\frac{\tau_n}{a_n} \to \gamma$ with $P[\gamma > 0] = 1$, $0 < a_n \to +\infty$ then

$$Y_{\tau_n} \frac{\tau_n}{a_n} \to a\gamma$$

Proof: By using theorems 4.4 and 5.1 of [2] is enough to prove that $Y_{\tau_n} \xrightarrow{P} a$. Given $\varepsilon > 0$ and $\delta > 0$ there exist A and n_0 such that $P(A) < \varepsilon$ and for all $\omega \in A^c$ and $n \geq n_0$ $\| Y_n(\omega) - a \| < \delta$ (See theorem A, Sec. 21 of [3]; the proof is essentially valid in general).

$$P[\| Y_{\tau_n} - a \| \geq \delta] \leq \varepsilon + P[A^c, \| Y_{\tau_n} - a \| \geq \delta]$$
$$\leq \varepsilon + P[\tau_n < n_0]$$

Therefore

$$\lim_n \sup P[\| Y_{\tau_n} - a \| \geq \delta] \leq \varepsilon$$

for all $\varepsilon > 0$ and $\delta > 0$. This clearly implies $Y_{\tau_n} \xrightarrow{P} a$.

Note: The conclusion of Theorem 2 can not be strengthen to $\xrightarrow{a \cdot s}$. For example if the probability space consist of the interval $[0, 1)$ with the Borel σ-field and the Lebesque measure, $Y_1 = 1$, $Y_k = -1$ for all $k \geq 2$,

$$\tau_j(\omega) = \begin{cases} 2^i & \omega \in [0, 1) - \left[\dfrac{j-2^i}{2^i}, \dfrac{j-2^i+1}{2^i}\right) \\ 1 & \omega \in \left[\dfrac{j-2^i}{2^i}, \dfrac{j-2^i+1}{2^i}\right) \end{cases}$$

for $2^i \leq j < 2^{i+1}$ $\quad i = 0, 1, \ldots \quad j = 1, 2, \ldots$

and $\quad a_j = 2^i \quad$ iff $\quad 2^i \leq j < 2^{i+1}$

the hypothesis of theorem 2 are satisfied but $\{Y_{\tau_n}\}_{n=1,2,\cdots}$ are precisely the Rademacher functions. Also by using these functions is not dificult to give an example to show that the condition $Y_n \xrightarrow{a \cdot s} a$ in Theorem 2 can not be changed to $Y_n \xrightarrow{P} a$.

References

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