On the Method of Moving Planes and the Sliding Method

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—dedicated to Shmuel Agmon

Abstract. The method of moving planes and the sliding method are used in proving monotonicity or symmetry in, say, the x_1 direction for solutions of nonlinear elliptic equations $F(x, u, Du, D^2u) = 0$ in a bounded domain Ω in $\mathbb{R}^n$ which is convex in the x_1 direction. Here we present a much simplified approach to these methods; at the same time it yields improved results. For example, for the Dirichlet problem, no regularity of the boundary is assumed. The new approach relies on improved forms of the Maximum Principle in "narrow domains". Several results are also presented in cylindrical domains — under more general boundary conditions.

1. The Methods for Simple Equations in General Domains

1.1. Symmetry in general domains

The moving plane and the sliding methods are techniques that have been used in recent years to establish some qualitative properties of positive solutions of nonlinear elliptic equations like symmetry, monotonicity etc... For instance, they are used to prove monotonicity in, say, the x_1 direction of scalar solutions of nonlinear second order elliptic equations in domains Ω in $\mathbb{R}^n$ (and even parabolic equations). The essential ingredient in their use is the maximum principle. Both methods compare values of the solution of the equation at two different points. In the first method one point is the reflection of the other in a hyperplane $x_1 = \lambda$, and then, the plane is moved up to a critical position. In the other, the second point is obtained from the first by sliding the domain in the x_1-direction. Again, the domain is slid up to a critical position.

The purpose of this work is to present a considerably simpler approach to these methods for nonlinear elliptic equations of the type

$$F(x, u, Du, D^2u) = 0 .$$

It also yields extensions of the previous results.

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The moving plane method goes back to A. D. Alexandroff (in his study of surfaces of constant mean curvature) and then J. Serrin [S]. In [GNN1], it was used to prove monotonicity of positive solutions vanishing on $\partial\Omega$ and, as a corollary, symmetry; [GNN2] extended these techniques to equations in all of $\mathbb{R}^n$. Since then, further extensions and generalizations have been made. We mention a few. In [BN1-2] (where other references may be found, as well as in [GNN1]) solutions were no longer required to be constant on the boundary; also, more general equations were treated. Subsequently [L1-2] extended and simplified some of the results there. In [BN1-2] the sliding method was introduced — first for infinite cylinders; L. Caffarelli then suggested its use for finite cylinders. (Both methods will be described in full detail.)

In all of these papers the maximum principle plays, as we said, the crucial role, but the papers had to rely on *many* forms of the maximum principle. These included the Hopf lemma at the boundary and its refinement at corners — the method of moving planes indeed always forces one to deal with domains having corners (see [S] and [GNN1].) Another basic form of the maximum principle which is sometimes used in order to get the methods started is the maximum principle for narrow domains. In addition, [BN1] introduced the use of maximum principles for degenerate parabolic equations in these problems. The various forms of the maximum principle may be found in [GNN1] and [BN1].

However, up to now, because of the difficulties at corners, certain simple domains could not be treated. For example, a simple and basic result of [GNN1] is the following.

Theorem 1.1. *In the ball* $\Omega : |x| < R$ *in* $\mathbb{R}^n$, *let* u *be a positive solution belonging to* $C^2(\overline{\Omega})$ *of*

$$(1.1) \qquad \Delta u + f(u) = 0 \quad with \quad u = 0 \ on \ \partial\Omega .$$

Here $f \in C^1$. *Then* u *is radially symmetric and the radial derivative satisfies* $u_r < 0$ *for* $0 < r < R$.

A more general form of this result is the following, also proved in [GNN1].

Theorem 1.2. *Consider a domain* Ω *in* $\mathbb{R}^n$ *of class* C^2 *which is convex in the* x_1 *direction and symmetric with respect to the plane* $x_1 = 0$. *Let* u *be a positive solution belonging to* $C^2(\overline{\Omega})$ *of* (1.1) *with* $f \in C^1$. *Then* u *is symmetric with respect to* x_1, *and* $u_{x_1} < 0$ *for* $0 < x_1$ *in* Ω.

At the time of this result, the question came up of proving the corresponding symmetry (in each argument) for Ω a cube. The method failed — because of the corners — and the problem even in the simple case of the cube has remained open until now.

Recently we found a considerable simplification of the method and the purpose of this paper is to describe this argument and use it to rederive and improve various results of [GNN1], [BN1] and [L1], and to treat more general domains, including the cube. For instance, in the case of (1.1), here is the general result of symmetry in *any* domain and with more general regularity assumptions on the function.

Theorem 1.3. *Let* Ω *be an arbitrary bounded domain in* $\mathbb{R}^n$ *which is convex in the* x_1 *direction and symmetric with respect to the plane* $x_1 = 0$. *Let* u *be a positive solution of* (1.1) *belonging to* $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$. *We assume that* f *is Lipschitz continuous. Then* u *is symmetric with respect to* x_1, *and* $u_{x_1} < 0$ *for* $0 < x_1$ *in* Ω.

Using the new method, we give a short proof of this result in section 1.3. Much more general equations – in fact fully nonlinear equations– are considered later on, in section 2.

The new idea enables us to avoid careful study of the boundary: we no longer (rather, hardly) need the Hopf lemma or its refinements at corners. Also, the previous application of these methods involved careful examination of various cases of boundary points; we now avoid this. Instead we use an improved form of the maximum principle in "narrow domains".

1.2. Maximum principle in "narrow domains".

In our treatment it is essential to find conditions ensuring that the maximum

principle holds. In particular, to consider such general domains as in Theorem 1.3, we need generalized forms of the Maximum principle. The crucial observations enabling us to treat such general domains were furnished by S. R. S. Varadhan — to whom we are extremely grateful.

Consider a second order elliptic operator in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$(1.2) \qquad L = M + c = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

with L^∞ coefficients and which is uniformly elliptic:

$$(1.3) \qquad c_0|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C_0|\xi|^2 , \quad c_0, C_0 > 0 , \ \forall \xi \in \mathbb{R}^n ,$$

and satisfying

$$(1.4) \qquad \sqrt{\Sigma b_i^2} , \ |c| \leq b .$$

The functions on which L will be applied will always be assumed to belong to $W_{loc}^{2,n}(\Omega)$.

Definition. We say that the maximum principle holds for L in Ω if

$$(1.5) \qquad Lz \geq 0 \quad \text{in } \Omega$$

and

$$(1.6) \qquad \overline{\lim_{x \to \partial\Omega}} z(x) \leq 0$$

implies $z \leq 0$ in Ω.

The following are three well known sufficient conditions:

i) $c \leq 0$ (see [Bo], [L], [BN1] and Theorem 9.6 in [GT]).

ii) There exists a continuous positive function g in $W^{2,\infty}(\Omega) \cap C(\overline{\Omega})$ satisfying $Lg \leq 0$. Indeed z/g satisfies a new elliptic inequality like (1.5) but with a new coefficient c which is nonpositive.

iii) Ω lies in a narrow band $\alpha < x_1 < \alpha + \epsilon$, with ϵ small. In this case, one constructs a function $g(x_1)$ satisfying the conditions of ii).

Varadhan's first observation is the following:

Proposition 1.1. *Assume* diam $\Omega \leq d$. *There exists* $\delta > 0$ *depending only on* n, d, c_0 *and* b, *such that the maximum principle holds for* L *in* Ω *provided*

$$(1.7) \qquad \text{meas } \Omega = |\Omega| < \delta .$$

In the proof only $c \leq b$ is required, rather than $|c| < b$. We have just found out that a similar observation had already been made by A. Bakelman [B]. We also refer the reader to [BNV] for various generalizations and applications of this type of results.

The proposition follows easily from the theorem of Alexandroff, Bakelman and Pucci which we use in the following form: If $c \leq 0$ and z satisfies $Lz \geq f$ and (1.6), then

$$(1.8) \qquad \sup_\Omega z \leq \overline{C} \|f\|_{L^n}$$

where $\overline{C}$ depends only on n, c_0, b and d.

This result is proved in a more general form in Theorem 9.1 in [GT]. It is assumed there that $z \in C(\overline{\Omega})$ but, as they remark, the argument works under condition (1.6).

Proof of Proposition 1.1. Consider z satisfying (1.5), (1.6). We write (1.5) in the form

$$(M - c^-)z \geq -c^+ z^+$$

where $c = c^+ - c^-$, $c^+ = \max\{c, 0\}$, and simply apply (1.8). It yields

$$\sup_\Omega z^+ \leq \overline{C} \, b |\Omega|^{1/n} \sup_\Omega z^+ ,$$

and it follows that $\sup z \leq 0$ in case $\overline{C} \, b \, \delta^{1/n} < 1$. $\square$

To illustrate the new ideas, including how to use this proposition, we give now our short proof of Theorem 1.3, that is, of monotonicity (and symmetry) for (1.1) in *any* bounded domain Ω which is convex in the x_1 direction (and symmetric about the plane $x_1 = 0$) — for example, a cube.

1.3. Proof of Theorem 1.3

With $x = (x_1, y) \in \Omega$ we will prove

$$u_1 > 0 \quad \text{if } x_1 < 0$$

and

$$(1.9) \qquad u(x_1, y) < u(x_1', y) \quad \text{if} \quad x_1 < x_1' , \ x_1 + x_1' < 0 .$$

Indeed, using (1.9), and letting $x_1' \nearrow -x_1$, we find by continuity,

$$(1.10) \qquad u(x_1, y) \leq u(-x_1, y) \quad \text{if } x_1 < 0 .$$

Since we may replace x_1 by $-x_1$ and use $u(-x_1, y)$, we see that equality must hold in (1.10), i.e., we obtain the symmetry of u in x_1. As we will see, that $u_1 > 0$ if $x_1 < 0$, follows easily from (1.9). In the following, we let $-a = \inf_{x \in \Omega} x_1$.

We now start the method of moving planes: for $-a < \lambda < 0$, let T_λ be the plane $x_1 = \lambda$, and

$$\sum(\lambda) = \{x \in \Omega \; ; \; x_1 < \lambda\} .$$

In $\Sigma(\lambda)$ set

$$v(x_1, y) = u(2\lambda - x_1, y) , \quad w(x, \lambda) = v(x) - u(x) .$$

Since f is Lipschitz we see that w satisfies an equation

$$(1.11) \qquad \begin{aligned} \Delta w + c(x, \lambda) w &= 0 \quad \text{in } \Sigma(\lambda) , \\ w &\overset{\geq}{\not\equiv} 0 \quad \text{on } \partial\Sigma(\lambda), \end{aligned}$$

for some bounded function $c(x, \lambda)$, $x \in \Omega$, with $|c| \leq b$, $\forall x \in \Sigma(\lambda)$, $\forall \lambda$. The boundary inequality in (1.11) holds because $w = 0$ on T_λ and $u = 0$ on $\partial\Omega$. To prove (1.9) we wish to get

$$(1.12) \qquad w(x, \lambda) > 0 \quad \text{for } x \in \Sigma(\lambda) .$$

It then follows from the Hopf lemma that on $T_\lambda \cap \Omega$, where $w = 0$, we have $0 > w_{x_1} = -2u_1$. Thus once we have proved (1.12) our proof of the theorem will be complete.

Now for $0 < \lambda + a$ small, the domain $\Sigma(\lambda)$ is narrow in the x_1-direction and it follows from Proposition 1.1, (or the corollary on page 213 of [GNN1] and remarks on page 24 of [BN1]) that the maximum principle holds, so that we may infer $w > 0$ in $\Sigma(\lambda)$. In fact we are in case iii) mentioned earlier. Let $(-a, \mu)$ be the largest open interval of values of λ such that (1.12) holds. We want to show that $\mu = 0$. We suppose $\mu < 0$ and argue by contradiction. By continuity, then, $w(x, \mu) \geq 0$ in $\Sigma(\mu)$. Since $w \not\equiv 0$ on $\partial\Sigma(\mu)$ it follows by the usual maximum principle that $w > 0$ in $\Sigma(\mu)$. We will show that for all positive small ϵ, $w(x, \mu + \epsilon) > 0$ in $\Sigma(\mu + \epsilon)$.

Here is the new idea. Fix $\delta > 0$ as in (1.7) (Proposition 1.1.) Let K be a closed set in $\Sigma(\mu)$ such that $|\Sigma(\mu) \setminus K| \leq \delta/2$. Clearly, by compactness,

$w(x, \mu) > 0$ for $x \in K$. Hence, by continuity it follows that $\forall \epsilon$ with $0 < \epsilon \leq \epsilon_0$ small, the following hold: $|\Sigma(\mu + \epsilon_0) \setminus K| \leq \delta$ and

$$w(x, \mu + \epsilon) > 0 \quad \text{on } \cdot K .$$

In the remaining part $\tilde\Sigma$ of $\Sigma(\mu + \epsilon)$, $w = w(x, \mu + \epsilon)$ satisfies

$$(1.13) \qquad \begin{cases} \Delta w + cw = 0 , & \text{in } \tilde\Sigma = \Sigma(\mu + \epsilon) \setminus K, \\ w \overset{\geq}{\not\equiv} 0 & \text{on } \partial\tilde\Sigma . \end{cases}$$

here $c = c(x, \mu + \epsilon)$. Indeed, the boundary inequality in (1.13) follows from $w \geq 0$ on $\partial\Sigma(\mu + \epsilon)$ by construction (since $u = 0$ on $\partial\Omega$, and $w = 0$ on $T_{\mu+\epsilon}$) and from the fact that $w > 0$ on ∂K.

Applying Proposition 1.1 in $\tilde\Sigma$ we infer that $w > 0$ in $\tilde\Sigma$. Hence $w(x, \mu + \epsilon) > 0$ in $\Sigma(\mu + \epsilon)$. But this contradicts the maximality of the interval $(-a, \mu)$.

Finally we show that $u_1 > 0$ if $x_1 < 0$. Since $w(x, \lambda) > 0$ in $\Sigma(\lambda)$ and $w(\lambda, y) = 0$, we may apply the Hopf lemma at the planar boundary, $x_1 = \lambda$, of $\Sigma(\lambda)$ and conclude that $w_{x_1}(\lambda, y) = -2u_{x_1}(\lambda, y) < 0$ — for every $\lambda < 0$. $\square$

1.4. The sliding method in general domains

In the sliding method introduced in [BN2] one compares translations of the function rather than reflections. In this context too, the approach based on the maximum principle for "narrow domains" yields not only a simpler but a more general result. To illustrate the main idea we now state and prove a monotonicity result for equation (1.1), but with other boundary conditions, in an arbitrary domain and with general regularity assumptions on the solution. In section 2 we treat fully nonlinear equations.

Theorem 1.4. *Let Ω be an arbitrary bounded domain of $\mathbb{R}^n$ which is convex in the x_1-direction. Let $u \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be a solution of*

$$(1.14) \qquad \Delta u + f(u) = 0 \quad \text{in } \Omega$$

$$(1.15) \qquad u = \varphi \quad \text{on } \partial\Omega .$$

The function f is supposed to be Lipschitz continuous. Here we assume that for any three points $x' = (x_1', y)$, $x = (x_1, y)$, $x'' = (x_1'', y)$ lying on a segment

parallel to the x_1-axis, $x_1' < x_1 < x_1''$, with $x', x'' \in \partial\Omega$, the following hold

(1.16) $$\varphi(x') < u(x) < \varphi(x'') \quad \text{if} \ x \in \Omega$$

and

(1.17) $$\varphi(x') \leq \varphi(x) \leq \varphi(x'') \quad \text{if} \ x \in \partial\Omega \ .$$

Then, u is monotone with respect to x_1 in Ω:

$$u(x_1 + \tau, y) > u(x_1, y) \ \text{ for } \ (x_1, y) \ , \ (x_1 + \tau; y) \in \Omega \text{ and } \ \tau > 0 \ .$$

Furthermore, if f is differentiable, then $u_{x_1} > 0$ in Ω. Finally, u is the unique solution of (1.14), (1.15), in $W_{loc}^{2,n}(\Omega) \cap C(\overline{\Omega})$ satisfying (1.16).

Condition (1.17) requires monotonicity of φ on any segment parallel to the x_1-axis lying on $\partial\Omega$. It is obviously a necessary condition for the result to hold.

Proof. Theorem 1.4 is proved by using the sliding technique. For $\tau \geq 0$, we let $u^\tau(x_1, y) = u(x_1 + \tau, y)$. The function u^τ is defined on the set $\Omega^\tau = \Omega - \tau e_1$ obtained from Ω by sliding it to the left a distance τ parallel to the x_1-axis. The main part of the proof consists in showing that

(1.18) $$u^\tau > u \quad \text{in} \ \Omega^\tau \cap \Omega \text{ for any } \ \tau > 0 \ .$$

Indeed, (1.18) means precisely that u is monotone increasing in the x_1 direction.

Set $w^\tau(x) = u^\tau(x) - u(x)$ i.e. $w^\tau(x_1, y) = u(x_1 + \tau, y) - u(x_1, y)$; w^τ is defined in $D^\tau = \Omega \cap \Omega^\tau$. As before, since u^τ satisfies the same equation (1.14) in Ω^τ as does u in Ω, we see that w^τ satisfies an equation

(1.19) $$\begin{cases} \Delta w^\tau + c^\tau(x) w^\tau = 0 & \text{in} \ D^\tau \\ w^\tau \geq 0 & \text{on} \ \partial D^\tau \end{cases}$$

where c^τ is some L^∞ function satisfying $|c^\tau(x)| \leq b, \ \forall x \in D^\tau, \ \forall \tau$. The inequality on the boundary $\partial D^\tau \subset \partial\Omega \cup \partial\Omega^\tau$ follows from the assumptions (1.16)–(1.17).

Let $\tau_0 = \sup\{\tau > 0 \ ; \ D^\tau \neq \varnothing\}$. For $0 < \tau_0 - \tau$ small, $|D^\tau|$ is small, that is D^τ is a "narrow domain". Therefore, from (1.19) it follows that for $0 < \tau_0 - \tau$ small, $w^\tau > 0$ in D^τ.

Next, let us start sliding Ω^τ back to the right, that is we decrease τ from τ_0 to a critical position $\tau \in [0, \tau_0)$: let (τ, τ_0) be a maximal interval, with $\tau \geq 0$, such that for all τ' in $\tau < \tau' \leq \tau_0$, $w^{\tau'} \geq 0$ in $D^{\tau'}$. We want to prove that $\tau = 0$. We argue by contradiction, assuming $\tau > 0$.

By continuity, we have $w^\tau \geq 0$ in D^τ. Furthermore, we know by (1.16) that for any $x \in \Omega \cap \partial D^\tau$, $w^\tau(x) > 0$. It follows easily that $w^\tau \not\equiv 0$ in every component of the open set D^τ. By the strong Maximum Principle (see [GNN1]) it follows from (1.19) that $w^\tau > 0$ in D^τ. Indeed if $x = (x_1, y)$ is any interior point of D^τ, then the half line $\{x_1 + t, y; \ t \geq 0\}$ hits ∂D^τ at a point $\bar{x} \in \partial\Omega^\tau \cap \partial\Omega$. This point $\bar{x}$ is on the boundary of that component of D^τ to which x belongs, and $u(\bar{x}) > 0$.

Now choose $\delta > 0$ as in Proposition 1.1 and carve out of D^τ a closed set $K \subset D^\tau$ such that $|D^\tau \setminus K| < \delta/2$. We know that $w^\tau > 0$ on K. Hence, for small $\epsilon > 0$, $w^{\tau-\epsilon}$ is also positive on K. Moreover, for $\epsilon > 0$ small, $|D^{\tau-\epsilon} \setminus K| < \delta$. Since $\partial(D^{\tau-\epsilon} \setminus K) \subset \partial D^{\tau-\epsilon} \cup K$, we see that $w^{\tau-\epsilon} \geq 0$ on $\partial(D^{\tau-\epsilon} \setminus K)$. Thus, $w^{\tau-\epsilon}$ satisfies

(1.20) $$\begin{cases} \Delta w^{\tau-\epsilon} + c^{\tau-\epsilon}(x) w^{\tau-\epsilon} = 0 & \text{in} \ D^{\tau-\epsilon} \setminus K \\ w^{\tau-\epsilon} \geq 0 & \text{on} \ \partial(D^{\tau-\epsilon} \setminus K) \ . \end{cases}$$

It then follows from Proposition 1.1 that $w^{\tau-\epsilon} \geq 0$ in $D^{\tau-\epsilon} \setminus K$ and hence in all of $D^{\tau-\epsilon}$. We have reached a contradiction and thus proved that u is monotone:

$$u^\tau > u \quad \text{in} \ D^\tau \ , \ \forall \tau > 0 \ .$$

(Again this follows from equation (1.19) since we know $w^\tau \geq 0$ and $w^\tau \not\equiv 0$ which implies $w^\tau > 0$.)

If, furthermore, f is differentiable, u_{x_1} satisfies a linear equation in Ω, by differentiation of (1.14):

$$\Delta u_{x_1} + f'(u) u_{x_1} = 0 \quad \text{in} \ \Omega$$

Since we already know that $u_{x_1} \geq 0$, $u_{x_1} \not\equiv 0$, we infer from this equation that $u_{x_1} > 0$ in Ω.

To prove the last assertion of the theorem suppose v is another solution. We argue exactly as before but instead of $w^\tau = u^\tau - u$ we now take $w^\tau = v^\tau - u$. The same proof shows that $v^\tau \geq u \ \forall \tau \geq 0$. Hence $v \geq u$, and by symmetry we have $v = u$. $\square$

1.5. Fully nonlinear equations

In the remainder of the paper we will primarily consider real functions u in a bounded domain Ω in $\mathbb{R}^n$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, which are solutions of a fully

nonlinear elliptic equation

$$(1.21) \qquad F(x, u, Du, D^2 u) = 0 \ .$$

Ω is always assumed to be convex in the x_1-direction and our aim is to prove monotonicity of u in x_1 in all or part of Ω.

The function $F(x, u, p_i, p_{jk})$ (here $i, j, k = 1, \ldots, n$) is always assumed to be continuous in all arguments for $x \in \overline{\Omega}$, $(u, p_1, \ldots, p_n)$ lying in a convex region in $\mathbb{R}^{n+1}$, and for $\{p_{jk}\}$ belonging to a convex region in the space of symmetric matrices. In addition, we assume that the derivatives

$$\frac{\partial F}{\partial p_{jk}}, \qquad j, k = 1, \ldots, n$$

are also continuous there and that the equation is uniformly elliptic, i.e., for some constants $c_0, C_0 > 0$,

$$(1.22) \qquad c_0 |\xi|^2 \le \sum \frac{\partial F}{\partial p_{jk}} \xi_j \xi_k \le C_0 |\xi|^2 \ ,$$

Assume furthermore that, here $p = (p_1, \ldots, p_n)$,

$$(1.23) \qquad F \text{ is Lipschitz-continuous in } (p, u) \text{ with Lipschitz constant } b \ge 0.$$

We will concentrate more on the sliding method since we will use some of the results in [BN4] (see also [BN3]). In Theorems 4.2 and 4.3 of [BN1] we used the sliding method to prove monotonicity. We have discovered however that the proofs are not quite correct: Proposition 1.1 of [BN1] does not apply as stated. In section 4 we will present corrected and improved results.

The sliding method is used here to prove x_1-monotonicity of u in all of Ω, and uniqueness of u, and when applying it we assume, as in Theorem 4.1 in [BN1]:

$$(1.24) \qquad F(x, u, p_i, p_{jk}) \text{ is nondecreasing in } x_1 \text{ for } p_1 \ge 0 \ .$$

In using the method of moving planes, however, we will usually prove x_1-monotonicity of u only in the region in Ω where $x_1 < 0$, and in place of (1.24) we assume as in [BN1] and [L1]: for $x_1 < x_1'$, $x_1 + x_1' < 0$ and $p_1 \ge 0$,

$$(1.25) \qquad \begin{aligned} &F(x_1, y, u, p_i, p_{jk}) \\ &\le F(x_1', y, u, -p_1, p_2, \ldots, p_n, p_{11}, -p_{12}, \ldots, -p_{1n}, p_{22}, \ldots, p_{nn}). \end{aligned}$$

We will first show in section 2 that the arguments used in proving Theorems 1.3 and 1.4 carry over directly to the Dirichlet problem in any bounded domain Ω which is convex in the x_1-direction in case (1.24) or (1.25) hold for all p_1, not merely for $p_1 \ge 0$.

As in [BN1], the fact that the condition (1.24) or (1.25) is assumed only for $p_1 \ge 0$ leads us to the use of the maximum principle for degenerate *parabolic* equations. In this connection we were led to a related question for *families* of elliptic operators. S.R.S. Varadhan gave an elegant affirmative answer to this question and we will present this result in section 3. It deals with the construction of some auxiliary function; earlier, we had constructed such functions, but only for fairly regular domains.

1.6. Combining sliding and moving planes methods ?

It is natural to ask whether the moving planes and sliding methods can be subsumed in a more general one which uses some combination of conditions (1.24), (1.25). (In Theorem 2.5 we use the method of moving planes to prove monotonicity in all of Ω under suitable conditions.)

Even in one dimension (our results also hold for $n = 1$), for a simple ordinary differential equation,

$$(1.26) \qquad \ddot{u} + b(x)\dot{u} + f(u) = 0 \quad \text{on} \quad (-a, a)$$

the methods lead to different results:

Theorem 1.5. *Let u be a C^2 solution of* (1.26) *in* $(-a, a)$ *which is continuous on* $[-a, a]$, *and satisfies*

$$(1.27) \qquad u(-a) < u(x) < u(a) \qquad \text{for} \quad -a < x < a \ .$$

Suppose b is continuous and f Lipschitz continuous. Then u is strictly increasing in x under any one of the following conditions:

(i) *$b \le 0$ on $(-a, a)$*

(ii) *$b \ge 0$ on $(-a, a)$*

(iii) *b is nondecreasing.*

The case (i) is contained in Theorem 3.2 of [BN1] (or Theorem 2.5 here), as is the case (ii) if x_1 is replaced by $-x_1$ and u by $-u$, while case (iii) is contained

in Theorem 4.1 of [BN1], or in Theorem 2.1 here. In case (iii), any solution satisfying (1.27) is also unique.

We conjectured that Theorem 1.3 holds if in place of (i), (ii) or (iii) one assumes the following single condition. If $b(x) > 0$ for some x in $(-a, a)$ then $b(x') \geq 0$ if $x' \geq x$ in $(-a, a)$. Alan Weinstein then found a direct proof. For our fully nonlinear equations, though, we do not know how to formulate a corresponding conjecture.

1.7. Organization of the paper

In Section 2 we state our monotonicity results for fully nonlinear elliptic equations. In addition to the Dirichlet problem in a general domain, we consider also a cylinder with generators parallel to the x_1-axis. On the curved side of the cylinder we consider more general conditions than Dirichlet — for application in [BN3-4]. In that same section, as mentioned, we also show that the proofs of section 1 carry over easily to the fully nonlinear equations under slightly stronger hypotheses (if conditions (1.24) or (1.25) hold for all p_1). Proofs of the general results are given in sections 5 and 6. In Section 3 we first present a preliminary useful result due to Varadhan concerning the construction of some auxiliary function. This is related to the maximum principle for a *family* of operators. In section 4, we derive some auxiliary results (some from [BN1]) for parabolic inequalities. In the last section, as an application of Theorem 2.2, we present an existence and uniqueness theorem for a semilinear equation in a finite cylinder. It will be used in [BN4].

For the reader who just wishes to understand the method for the Dirichlet problem we suggest he skip Theorems 2.4, 2.5 and Lemma 3.1 since their proofs are more intricate.

We wish to express our thanks to L. Caffarelli, as well as to S. R. S. Varadhan, for several very useful discussions.

2. Principal Results and Proofs under Relaxed Conditions

Throughout the paper, Ω is always assumed to be a bounded domain which is convex in the x_1 direction. u is assumed to be in $C^2(\Omega) \cap C(\overline{\Omega})$ and to satisfy

a nonlinear elliptic equation:

$$(2.1) \qquad F(x, u, Du, D^2 u) = 0 \quad \text{in } \Omega \, ,$$

with F satisfying conditions (1.22), (1.23). We begin with the sliding method for proving monotonicity in x_1. The first result is the general form of Theorem 1.4. It refers to a solution of the Dirichlet problem: u satisfies (2.1) and

$$(2.2) \qquad u = \phi \quad \text{on } \partial\Omega \, , \quad \text{with } \phi \in C(\partial\Omega) \, .$$

Theorem 2.1. *Assume F satisfies* (1.24) *and assume*

$$(2.3) \qquad
\begin{aligned}
&\text{If } (x_1', y) \, , \, (x_1'', y) \text{ lie on } \partial\Omega, \, x_1' < x_1'', \text{ then } \phi(x_1', y) \leq \phi(x_1'', y) \\
&\text{and if in addition the interior of the segment joining} \\
&(x_1', y) \text{ to } (x_1'', y) \text{ belongs to } \Omega \text{ then for } x \text{ on it,} \\
&\phi(x_1', y) < u(x) < \phi(x_1'', y) \, .
\end{aligned}$$

Then u is strictly increasing in x_1. Furthermore it is the unique solution satisfying (2.2) *and* (2.3)*. Finally, if F (and hence u) is smooth, then $u_1 > 0$ in Ω.*

Remarks. This is an improved form of Theorem 4.3 of [BN1] in that we no longer require Hypothesis 3 there. Moreover, in the (incorrect) proof of Theorem 4.3 in [BN1] we relied on Proposition 1.1 of that paper. But the proposition cannot really be applied in the region U of page 265 of [BN1]. In case $0 < \alpha + \lambda$ small, the region is narrow in the x_1-direction but not in the sense of Proposition 1.1 there. Rather, it is narrow in the sense that it is convex in the x_1-direction, and the length of any interval in U parallel to the x_1 axis is at most $a + \lambda$, which is small.

In place of Proposition 1.1 of [BN1] we will, at a certain point, make use of an auxiliary function with special properties — as described in Theorem 3.1 below.

Next, we present the general form of Theorem 1.3. In case $\partial\Omega$ is smooth and $u \in C^{2,\alpha}(\overline{\Omega})$, $\alpha > 0$ it is due to Li, Cong Ming (Theorem 3.1 in [L1]). It is an improvement of Theorem 2.1′ in [GNN1]. Li's proof is fairly intricate. It involves considerable use of the condition (1.25) and careful treatment of corners while ours does not. We suppose $-a = \min\{x_1; \, x \in \overline{\Omega}\}$ and assume

$$(2.4) \qquad (x_1, y) \in \Omega \, , \quad x_1 < 0 \Rightarrow (x_1', y) \in \Omega \quad \text{for } x_1 < x_1' < -x_1 \, .$$

Theorem 2.2. *In Ω consider a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (2.1). F is assumed to satisfy (1.25). Assume also:*

(2.5)   *If $(x_1, y) \in \partial\Omega$, $(x_1', y) \in \Omega$, $x_1 < x_1'$, then $u(x_1, y) < u(x_1', y)$.*

(2.6) $\begin{cases} \text{If } (x_1, y), (x_1', y) \in \partial\Omega, x_1 < x_1', x_1 + x_1' < 0 \text{ then} \\ u(x_1, y) \leq u(x_1', y) \end{cases}$

Then, for $(x_1, y), (x_1', y) \in \Omega$, $x_1 < x_1'$, $x_1 + x_1' < 0$,

(2.7) $u(x_1, y) < u(x_1', y)$ *and $u_1(x_1, y) > 0$.*

Furthermore if $u_1 = 0$ at some point $(0, y) \in \Omega$ then u is symmetric in x_1.

The conclusion (2.7) of Theorem 2.2 asserts x_1-monotonicity in only part of the domain Ω. In Theorems 3.2 and 3.3 of [BN1] we also showed how to use the method of moving planes to obtain full monotonicity in this situation. Here is an improvement:

Theorem 2.3. *Theorem 2.1 holds if condition (1.24) is replaced by (1.25) which is assumed to hold whenever $x_1 < x_1'$ and $p_1 \geq 0$. In addition, $u_1 > 0$ in Ω.*

Before stating other results it is worthwhile pointing out that the proofs of Theorems 1.3 and 1.4 carry over immediately to Theorems 2.2 and 2.1, respectively, if we assume that (1.24) and (1.25) hold *for all p_1*, not merely for $p_1 \geq 0$.

Proof of Theorem 2.2 in this case. We follow the proof of Theorem 1.3. With $\Sigma(\lambda)$, v and $w(x, \lambda)$ as defined there, we have to prove that (1.12) holds, i.e.

(2.8) $w(x, \lambda) > 0$ for $x \in \Sigma(\lambda)$, $\lambda < 0$.

We no longer have the equation (1.13) for w, but instead w satisfies an elliptic inequality which we now derive. In $\Sigma(\lambda)$, $v(x_1, y) = u(2\lambda - x_1, y)$ satisfies

(2.9) $F(2\lambda - x_1, y, v, -v_1, v_2, \ldots, v_n, v_{11}, -v_{12}, \ldots, -v_{1n}, v_{22}, \ldots) = 0$.

It follows from (1.25) — assumed to hold for all p_1 — that

(2.10) $F(x_1, y, v, v_i, v_{jk}) \leq 0$.

Subtracting the equation (2.1) and using the integral form of the theorem of the mean we find that $w(x, \lambda) = v - u$ satisfies an elliptic inequality similar to (1.5).

(2.11) $Lw = a_{ij}(x, \lambda)w_{x_i x_j} + b_i(x, \lambda)w_{x_i} + c(x, \lambda)w \leq 0$,

with coefficients satisfying conditions (1.3), (1.4). Furthermore

(2.12) $w(x, \lambda) \gneq 0$ on $\partial\Sigma(\lambda)$.

The proof of Theorem 1.3 now just carries over, with the equation in (1.13) replaced by the inequality (2.11). The last assertion in Theorem 2.2 follows from the Hopf lemma. For if $u_1(0, y) = 0$ then in $\Sigma(0)$, $w(x, 0)$ satisfies (2.11), it has a minimum at the boundary point $(0, y)$ and there, $w_{x_1} = -2u_1 = 0$. Since $w(x, 0) \geq 0$, it follows by the Hopf lemma that $w(x, 0) \equiv 0$ — which implies the symmetry of u. $\square$

Proof of Theorem 2.1 in case (1.24) holds for all p_1. Again we just follow the proof of Theorem 1.4. With D^τ, u^τ and w^τ as in that proof we find that in D^τ, w^τ satisfies, in place of the equation (1.19), an elliptic inequality of the form (2.11). Namely $v = u^\tau$ satisfies

$$F(x_1 + \tau, y, v, v_i, v_{jk}) = 0 .$$

Condition (1.24) for all p_1 then yields

$$F(x_1, y, v, v_i, v_{jk}) \leq 0 .$$

Arguing as in the preceding proof we see that in D^τ, w^τ satisfies (2.11). The remainder of the proof of Theorem 1.4 is then easily adapted.

We now continue with general results.

We take Ω to be a finite cylinder and consider more general boundary conditions.

(2.13) $\Omega = S_a = \{x = (x_1, y) \in \mathbb{R}^n ; |x_1| < a , y = (x_2, \ldots, x_n) \in \omega\}$;

here ω is a bounded domain in $\mathbb{R}^{n-1}$, with smooth boundary; we denote by ν the exterior unit normal to Ω (and to ω) at a point $x \in \partial\Omega$ with $-a < x_1 < a$. We consider a function $u \in C^2(\Omega) \cap C^1((-a, a) \times \overline{\omega}) \cap C(\overline{\Omega})$ which satisfies (2.1) in Ω and nonlinear boundary conditions:

(2.14) $\sigma(x_1, y, u, \nabla u) = 0$ for $-a < x_1 < a , y \in \partial\omega$,

(2.15) $u(-a, y) = \psi_1(y) , \quad u(a, y) = \psi_2(y) , \quad \forall y \in \omega$.

We assume that

$$(2.16) \quad \begin{cases} \psi_1(y) \leq u(x_1, y) \leq \psi_2(y) & \text{for } -a < x_1 < a \,, \, y \in \omega \\ \text{and } \forall \, x_1 \text{ in } (-a, a) \,, \, \exists \, y \in \omega & \text{such that } \psi_1(y) < u(x_1, y) \,, \\ \text{or } \forall \, x_1 \text{ in } (-a, a) \,, \, \exists \, y \in \omega & \text{such that } u(x_1, y) < \psi_2(y) \,. \end{cases}$$

We suppose that $\sigma(x, u, p)$, as well as σ_u, $\nabla_p \sigma$, are continuous in all arguments for (u, p) contained in a convex region in $\mathbb{R}^{n+1}$ and bounded in absolute value by b. In addition we assume that for $-a < x_1 < a$, $y \in \partial\omega$,

$$(2.17) \qquad \sigma_u \geq 0 \,, \quad \nu \cdot \nabla_p \sigma \geq 0 \text{ and } \sigma_u + \nu \cdot \nabla_p \sigma > 0$$

and (analogous to (16) in [L1] where, however, σ is assumed to be independent of p_1):

$$(2.18) \qquad \sigma(x_1, y, u, p) \text{ is nonincreasing in } x_1 \text{ for } p_1 \geq 0 \,.$$

Theorem 2.4. (i) *Assume the conditions above and assume F satisfies (1.24) then, u is strictly increasing in x_1. In addition, if F is also smooth then $u_{x_1} > 0$ in Ω.* (ii) *The solution u is unique, i.e. if $\underline{u}$ is another solution satisfying all the conditions above (in particular (2.16)) then $\underline{u} = u$.*

Remark. In case F is semilinear,

$$F = a_{ij}(x)u_{x_i x_j} + f(x, u, \nabla u),$$

we need only suppose about u, that

$$u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^1\{(-a, a) \times \overline{\omega}\} \cap C(\overline{\Omega}).$$

This may be seen from the proof.

The method of moving planes yields a similar but slightly different result in the finite cylinder $\Omega = S_a$. Let u be a solution in Ω of (2.1) with the same regularity properties as in Theorem 2.4, satisfying the nonlinear boundary conditions (2.14), (2.15). Concerning F and σ, we assume they satisfy the same properties as in Theorem 2.4, except that in place of conditions (1.24) we assume (1.25) and in place of (2.18) we assume:

$$(2.19) \qquad \begin{array}{l} \text{for } x_1 < x_1' \,, \quad x_1 + x_1' < 0 \quad \text{and} \quad p_1 \geq 0 \,, \\ \sigma(x_1, y, u, p_1, \dots, p_n) \geq \sigma(x_1', y, u, -p_1, p_2, \dots, p_n) \,. \end{array}$$

This condition is assumed in [BN1] and [L1] (actually Li assumes σ independent of p_1).

Theorem 2.5. *In addition to all the hypotheses above, in particular that F satisfies (1.25), assume*

$$(2.20) \qquad \begin{array}{l} u(-a, y) \leq u(x_1, y) \text{ for } -a < x_1 < a \,; \\ \forall \, x_1 \text{ in } (-a, a), \text{ strict inequality holds for some } y \in \omega \,. \end{array}$$

Then, for $-a < x_1 < x_1'$, $x_1 + x_1' < 0$, $y \in \omega$,

$$(2.21) \qquad u(x_1, y) < u(x_1', y) \text{ and } u_1(x_1, y) > 0 \,.$$

Furthermore if $u_1(0, y) = 0$ for some $y \in \omega$ then u is a symmetric function of x_1.

The following is an immediate consequence:

Corollary 2.1. *Assume all the conditions in Theorem 2.5 and assume in addition:*

$$(2.20)' \qquad \begin{array}{l} u(a, y) \leq u(x_1, y) \quad \text{for } -a < x_1 < a \,, \, y \in \omega \\ \text{with strict inequality holding for some } y_1 \text{ for each } x_1 \,, \end{array}$$

as well as (2.19) with the opposite inequalities, in case $x_1 < x_1'$, $x_1 + x_1' > 0$, $p_1 \geq 0$ — so equalities hold in case $x_1 + x_1' = 0$. Then u is symmetric in x_1.

Proof. Fixing $x_1 < 0$ and letting $x_1' \to -x_1$ in (2.20) we find

$$(2.22) \qquad u(x_1, y) \leq u(-x_1, y) \quad \text{for } -a < x_1 < 0 \,, \, y \in \omega \,.$$

On the other hand conditions of the corollary enable us to apply the theorem to the function $v(x_1, y) = u(-x_1, y)$. But then (2.22) holds for v, which implies that equality holds everywhere in (2.22). $\square$

Remarks. In some cases, it is possible to reduce a problem where condition (2.17) is not met to one where it is. Typically, this can be achieved through the introduction of a function $\chi = \chi(y) > 0$ of y alone and by working with the function $v = u/\chi$ instead of u.

General boundary conditions like (2.14) come up naturally in some applications. This is the case for the problem of solitary water waves. Symmetry and monotonicity away from a single crest have been shown for the solitary water waves problem by Craig and Sternberg [CS] using the moving planes method. There, after a change of variable, the problem is reduced to an elliptic equation in an *infinite* strip with nonlinear boundary conditions of the type (2.14). Li [L2] has shown how a further change of unknown of the type $v = u/\chi$ allows one to

suitably modify this boundary condition in order to satisfy a condition of the type (2.14).

Finally, here is the analogue of Theorem 2.3 for the cylinder S_a. It is an improvement of Theorem 3.3 of [BN1].

Theorem 2.6. *Theorem 2.4 holds if condition (1.24) is replaced by (1.25) which is assumed to hold whenever $x_1 < x_1'$ and $p_1 \geq 0$. In addition, $u_1 > 0$ in Ω.*

3. A Result of Varadhan

In this and the next section, we derive some auxiliary results which will be needed in the proofs of some of the results of section 2. We start with a result of Varadhan which shows how to construct a function g in a given set, which is simultaneously a "supersolution" for a whole class of elliptic operators. In the following, we set $B_r = \{|x| < r\}$.

Theorem 3.1. (Varadhan.). *Given $b \geq 1$, $\exists \delta > 0$ such that for every closed set Q in B_1 with measure $\leq \delta$, there exists a C^∞ function g in $\overline{B}_1$, with $1 \leq g \leq 2$ such that for every positive definite matrix $\{a_{ij}\}$ with*

$$(3.1) \qquad \det(a_{ij}) \geq 1 \,,$$

one has

$$(3.2) \qquad a_{ij}g_{ij}(x) + b(|\nabla g| + g) < 0 \,, \quad \forall x \in Q \,.$$

Remark. We emphasize that g depends on the set Q but not on the operator $a_{ij}\partial_{ij}$ as long as (3.1) holds. In particular, it does not involve upper bounds on the a_{ij}.

Proof. Let f be a positive C^∞ function on $\overline{B}_2$ satisfying

$$f \geq 1 \quad \text{on } Q \text{ and } \int_{B_2} f < 2\delta \,.$$

In B_2, solve the Dirichlet problem for the Monge-Ampère equation

$$\det(-u_{ij}) = f \quad \text{in } B_2$$
$$u = 0 \quad \text{on } \partial B_2 \,.$$

There is such a unique, concave positive solution in B_2 which belongs to $C^\infty(B_2)$; see [CNS] and references there, in particular [CY]. We now apply the Alexandroff, Bakelman, Pucci inequality (which, incidentally, is also the main ingredient in the proof of (1.8)): For some $C = C(n)$,

$$\|u\|_{L^\infty(B_2)} \leq C\left[\int_{B_2} \det(-u_{ij})\, dx\right]^{1/n}$$
$$< C(2\delta)^{1/n} \,.$$

Since u is concave and positive it follows that

$$(3.3) \qquad u \,, \ |\nabla u| \leq C(2\delta)^{1/n} \quad \text{in} \quad B_1 \,.$$

Using the arithmetic-geometric mean theorem we find for our $\{a_{ij}\}$,

$$(3.4) \qquad \begin{aligned} -\frac{1}{n}a_{ij}u_{ij} &\geq [\det(-u_{ij})\ \det(a_{ij})]^{1/n} \\ &\geq [\det(-u_{ij})]^{1/n} \qquad \text{by (3.1),} \\ &\geq 1 \quad \text{on } Q \,. \end{aligned}$$

Set $g = 2bu + 1$. Then

$$1 \leq g \leq 1 + 2bC(2\delta)^{1/n} \leq 2 \quad \text{if } 2bC(2\delta)^{1/n} \leq 1 \,.$$

On Q we have by (3.4) and (3.3),

$$\begin{aligned} a_{ij}g_{ij} + b(|\nabla g| + g) &\leq -2bn + 4b^2C(2\delta)^{1/n} + b \\ &\leq -b + 4b^2C(2\delta)^{1/n} \\ &< 0 \quad \text{if } 4bC(2\delta)^{1/n} < 1 \,. \end{aligned}$$

The theorem is proved if, for instance, we choose δ so that

$$(3.5) \qquad 4bC(2\delta)^{1/n} = \frac{1}{2} \,.$$

4. Some auxiliary facts

To demonstrate the general results of section 2 we rely on Theorem 3.1. It is used in proving Proposition 4.1 below. The setup is the following: Let V be a bounded domain in $\mathbb{R}^{n+1}$, with coordinates (x, t), lying in $|x| < r$, $t_0 < t < T$. With

$$V_t := \{x; \ (x, t) \in V\}$$

we assume

$$(4.1) \qquad V_t \subset V_s \text{ for } t < s .$$

Set

$$V_T = \bigcup_{t<T} V_t .$$

The parabolic boundary of V is

$$J = \overline{\partial V \cap \{t < T\}} .$$

Set

$$\widehat{V}_T = \partial V \setminus J = \{(x,T); x \in V_T\} .$$

In V we are given a uniformly elliptic operator in the x variables with L^∞ coefficients depending on t:

$$L = a_{ij}(x,t)\partial_{ij} + b_i(x,t)\partial_i + c(x,t) .$$

Here $a_{ij} \in C(V)$ and the coefficients satisfy

$$(4.2) \qquad c_0|\xi|^2 \le a_{ij}\xi_i\xi_j \le C_0|\xi|^2 , \quad c_0, C_0 > 0 , \quad \forall \xi \in \mathbb{R}^n .$$

$$(4.3) \qquad \sqrt{\Sigma b_i^2} , \quad c \le b .$$

Proposition 4.1. *Let z be a function with $D_x z$, $D_x^2 z$ and z_t continuous in $V \cup \widehat{V}_T$. Assume $z \in C(\overline{V})$ and $z \le 0$ on J. Assume furthermore the following*

$$(4.4) \qquad Lz \ge 0 \quad \text{wherever } z_t \ge 0 \text{ in } V .$$

Then $\exists \delta > 0$ depending only on n, r, c_0 and b such that if the measure $|V_T|$ of V_T is less than δ then $z \le 0$ in V.

Remarks. Condition (4.4) seems somewhat strange. It would be interesting to know if it can be replaced by other conditions. Clearly, the conditions

$$Lz \ge 0 \quad \text{where } z_t + \alpha z \ge 0$$

(for some real α) suffices. This simply follows from the result by considering $e^{-\alpha t}z$. Condition (4.4) clearly holds if z satisfies a degenerate parabolic inequality $\beta z_t - Lz \le 0$ with $\beta \ge 0$. It is easily seen that the result need not hold if the assumption that V_T is small in measure is dropped.

Proof. Without loss of generality we may suppose $r = 1$ and $c_0 = 1$ — the constant b is then replaced by another which we may choose to be not less than 1.

Suppose $N = \max_{\overline{V}} z > 0$. Set

$$W = \{(x,t) \in \overline{V} ; \ z(x,t) \ge \frac{N}{3}\} \text{ and}$$

$$Q = \{x ; \{x,t\} \in W \text{ for some } t\} .$$

Clearly $|Q| < \delta$.

With this Q, choose δ as in Theorem 3.1 and let $g = g(x)$ be the function in the conclusion of the theorem. It satisfies

$$1 \le g \le 2$$

and the important condition (3.2). By that condition it follows that

$$(4.5) \qquad Lg < 0 \text{ at every point in } W .$$

Here we are using condition (4.1), and Theorem 3.1 for $a_{ij} = a_{ij}(x,t)$ for each fixed t.

We use g in a standard way. Maximize $\varsigma = z/g$ in $\overline{V}$. At a point $(\bar{x}, \bar{t})$ where the maximum is assumed,

$$\max \varsigma = \max \frac{z}{g} \ge \frac{N}{2} , \quad \text{hence} \quad z(\bar{x}, \bar{t}) \ge N/2 .$$

Thus $(\bar{x}, \bar{t})$ is an interior point of W, or else $\bar{t} = T$. In either case, $\varsigma_t \ge 0$ whence $z_t \ge 0$, there. Consequently by (4.4), $Lz \ge 0$ at $(\bar{x}, \bar{t})$. This means

$$M'\varsigma + \frac{Lg}{g}\varsigma \ge 0 \quad \text{at } (\bar{x}, \bar{t})$$

where M' is an elliptic operator with no zero order term. But at this maximum point $(\bar{x}, \bar{t})$, $M'\varsigma \le 0$. Since $g, \varsigma > 0$ there, and (4.5) holds, we obtain a contradiction. $\square$

In the proof of Theorem 2.4 we will make use of an auxiliary function which plays the same role as the function g in the preceding proof. With ω as in Theorem 2.4, (ω smooth) set $\omega_\tau = \{y \in \omega ; d(y, \partial\omega) < \tau\}$ for τ small. In the cylinder $S = (-a, \rho) \times \omega$, let $\widetilde{S}$ be the region near the boundary defined by:

$$(4.6) \quad \widetilde{S} = \{(-a, -a+\tau) \times \omega\} \cup \{(\rho-\tau, \rho) \times \omega\} \cup \{[a+\tau, \rho-\tau] \times \omega_\tau\} .$$

Lemma 4.1. *Given $c_0, b > 0$, $\exists \tau_0 > 0$, with $2\tau_0 < \rho + a$ such that for any τ in $0 < \tau < \tau_0$ there exists a $C^{1,1}$ function $g \geq 1$ in $\widetilde{S}$ satisfying*

$$(4.7) \qquad \nabla_y g(x,y) = \nu(y) \quad for \quad y \in \partial\omega , \quad |g_{x_1}| \leq \tau^{1/2} ,$$

and such that for any positive definite symmetric matrix $\{a_{jk}\}$ satisfying (1.3),

$$(4.8) \qquad J[g] := a_{jk}\partial_{jk}g + b(|\nabla g| + g) < 0 \quad in \ \widetilde{S} .$$

Proof: For $y \in \bar{\omega}$, $d(y) = d(y, \partial\omega)$ is smooth if $d(y) < \tau_0$ for τ_0 small. Let h be a C^2 function ≥ 1 in $\bar{\omega}$ with

$$\nabla h = 2\nu \quad \text{on} \quad \partial\omega .$$

(Near the boundary, $h(x)$ is just $-2d+$ constant.) We take

$$g(x_1, y) = h(y) + \ell(y) + m(x_1)$$

where

$$\ell(y) = \begin{cases} \tau - \dfrac{\tau}{2}\left(1 - \dfrac{d(y)}{\tau}\right)^2 & \text{if} \quad d(y) \leq \tau \\ \tau & \text{if} \quad d(y) > \tau \end{cases}$$

$$m(x_1) = \tau^{3/2}\begin{cases} 1 - \dfrac{1}{2}\left(1 - \dfrac{x_1 + a}{\tau}\right)^2 & \text{if} \quad -a \leq x_1 \leq -a + \tau \\ 1 & \text{if} \quad -a + \tau \leq x_1 \leq \rho - \tau \\ 1 - \dfrac{1}{2}\left(1 - \dfrac{\rho - x}{\tau}\right)^2 & \text{if} \quad \rho - \tau \leq x_1 \leq \rho . \end{cases}$$

For $1 < j, k \leq n$ and $d(y) < \tau$ we have

$$\ell_j = \left(1 - \dfrac{d}{\tau}\right)d_j , \qquad \ell_{jk} = -\dfrac{1}{\tau}d_j d_k + \left(1 - \dfrac{d}{\tau}\right)d_{jk} .$$

Hence for $y \in \partial\omega$,

$$\nabla_y \ell = -\nu(y) , \quad \text{i.e. } \nabla_y g = \nu(y) .$$

Furthermore for $d(y) < \tau$, one readily verifies that for τ small,

$$a_{jk}\partial_{jk}g \leq \sum_{j,k=2}^n a_{jk}\partial_{jk}h - \dfrac{c_0}{\tau} + C\,C_0$$

where C depends on bounds on $\partial_{jk}d$ in $d \leq \tau$. Similarly, if $x_1 < -a + \tau$ or $x_1 > \rho - \tau$, we have

$$|\partial_1 m| \leq \tau^{1/2} , \qquad \partial_{11}m = -\tau^{-1/2} .$$

Note indeed that $\partial_{11}m = 0$ where $-a + \tau \leq x_1 \leq \rho - \tau$. These inequalities yield (4.7) and (4.8) for τ small. $\square$

As in [BN1], the proof of the theorems in Section 2 will rely on some results for parabolic inequalities which we restate here. Note that we have changed the sign of w from [BN1].

In $\mathbb{R}^{n+1}$, with coordinates (x,t), $x \in \mathbb{R}^n$, let V be a bounded domain lying in $\{T_1 < t < T\}$. In V consider a function $w \in C(\bar{V})$ having continuous derivatives up to second order in x and first order in t, satisfying a (degenerate) parabolic inequality:

$$(4.9) \ \ (L - \beta\partial_t)w = a_{ij}(x,t)\,w_{x_i x_j} + b_i(x,t)w_{x_i} + c(x,t)w - \beta(x,t)w_t \leq 0 .$$

Here the a_{ij} are continuous, bounded and satisfy the uniform ellipticity condition

$$(4.10) \qquad c_0|\xi|^2 \leq a_{ij}\xi_i\xi_j , \quad c_0 > 0 , \quad \forall\,\xi \in \mathbb{R}^n ,$$

and the coefficients b_i, c are in L^∞, and satisfy (1.4). In addition, we assume $\beta \geq 0$, $\beta \leq 4b$.

We first state Lemma 4.1 of [BN1], a parabolic form of the Hopf lemma. With

$$\widehat{V}_T = \partial V \setminus J \quad \text{where} \quad J = \overline{\partial V \cap \{t < T\}}$$

as before, set

$$V \cup \widehat{V}_T = \widetilde{V} .$$

We will denote by P a paraboloid

$$P = \{(x,t) \,;\, t - T + \delta > |x - x^0|^2\} , \quad \delta > 0$$

for which the parabolic cap

$$Q = P \cap \{t \leq T\} \text{ lies in } \widetilde{V} .$$

We will also consider parabolic caps with T replaced by some other value.

Lemma 4.2. *Let V, w and Q be as above. Suppose $w > 0$ in Q and equals zero at a point $(\bar{x}, T) \in \partial Q \cap \partial P$. Then the exterior normal derivative there, in the space variables, is negative in the sense that*

$$(4.11) \qquad \overline{\lim}_{s \nearrow 1} \dfrac{w(x_0 + s(\bar{x} - x_0), T)}{1 - s} < 0 .$$

A simple corollary of this is (essentially Lemma 4.2 in [BN1]):

Lemma 4.3. *Let V and w be as above. Assume $w \geq 0$ in V and $w > 0$ at some point $(x^0, t^0) \in \tilde{V}$. Then $w > 0$ on the component of V_{t^0} containing (x^0, t^0).*

For proofs, see [BN1].

5. Sliding Method.

In this section we prove Theorems 2.1 and 2.4, proceeding as in the proofs of Theorems 3.5 and 4.1 of [BN1] but using the "maximum principle in narrow domains", Proposition 1.1. We may assume, in Theorem 2.1, that the longest open interval in Ω parallel to $e_1 = (1, 0, \dots, 0)$ has length $2a$ and that $x_1 = \pm a$ at its end points. For $-a < \lambda < a$ let

$$\Sigma(\lambda) = \{x \in \Omega \; ; \; x + (a - \lambda)e_1 \in \Omega\} \, .$$

This is the intersection of Ω with the set obtained by sliding Ω in the direction $-e_1$ by the distance $a - \lambda$. Clearly $\Sigma(\lambda) \subset \Sigma(\mu)$ for $\lambda < \mu$. In case Ω is the cylinder S_a, (for Theorem 2.4)

$$\Sigma(\lambda) = \{x \in S_a \; ; \; x_1 < \lambda\} \, .$$

The proofs of uniqueness and monotonicity in [BN1] make use of a (degenerate) parabolic inequality and we will now derive a more general version of it. Here, we take the time variable t to be λ. In Theorems 2.1 or 2.4, suppose $\underline{u}$ is another solution satisfying the same conditions as u. For $x \in \Sigma(\lambda)$ we define

$$(5.1) \qquad v(x) = u(x_1 + a - \lambda, y) \, , \quad w(x, \lambda) = v(x) - \underline{u}(x) \, .$$

Our aim will be to prove that for $-a < \lambda < a$,

$$(5.2) \qquad w(x, \lambda) > 0 \quad \text{for} \quad x \in \Sigma(\lambda) \, .$$

To do this, we derive parabolic inequalities of the form:

(5.3)

$$Lw = a_{ij}(x, \lambda)w_{x_i x_j} + b_i(x, \lambda)w_{x_i} + c(x, \lambda)w \leq 0 \quad \text{wherever } w_\lambda \leq 0 \, ,$$

and, in fact, more precisely, for some $\beta \geq 0$:

$$(5.3)' \quad (L - \beta \partial_\lambda)w := a_{ij}(x, \lambda)w_{x_i x_j} + b_i(x, \lambda)w_{x_i} + c(x, \lambda)w - \beta w_\lambda \leq 0$$

These inequalities will be shown to hold in the region in $\mathbb{R}^{n+1}$ defined by

$$(5.4) \qquad \mathcal{V} = \{(x, \lambda) \; ; \; x \in \Sigma(\lambda) \, , \, -a < \lambda < a\} \, .$$

Furthermore, with c_0, C_0, b as in (1.22), (1.23) we will have

$$c_0|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq C_0|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \, , \, \forall (x, \lambda) \in \mathcal{V}$$

$$(5.5) \qquad \sqrt{\Sigma b_j^2} \, , \, |c| \leq b \, , \qquad\qquad 0 \leq \beta \leq 2b \, .$$

In $\Sigma(\lambda)$ the function v satisfies

$$F(x_1 + a - \lambda, y, v, Dv, D^2v) = 0 \, .$$

By (1.24), it follows that

$$(5.6) \qquad I := F(x_1, y, v, v_i, v_{jk}) \leq 0 \quad \text{if } v_1(x) \geq 0 \, ,$$

while if $v_1(x) < 0$ we have

$$\begin{aligned} I = &F(x_1, y, v, v_i, v_{jk}) - F(x_1, y, v, 0, v_2, \dots, v_n, v_{jk}) \\ &+ F(x_1, y, v, 0, v_2, \dots, v_n, v_{jk}) - F(x_1 + a - \lambda, y, v, 0, v_2, \dots, v_{jk}) \\ &+ F(x_1 + a - \lambda, y, v, 0, v_2, \dots, v_{jk}) - F(x_1 + a - \lambda, y, v, v_i, v_{jk}) \end{aligned}$$

i.e.,

$$(5.6)' \qquad\qquad I \leq -2bv_1 \, ,$$

again by (1.24) (applied to the middle two terms above) and Lipschitz continuity in p_1. Thus, in any case we have

$$F(x_1, y, v, v_i, v_{jk}) \leq -\beta v_1$$

where β is a nonnegative L^∞ function bounded by $2b$ (in view of (1.23)). Using the integral form of the theorem of the mean, the inequalities (5.6), (5.6)' and the identity $v_1 = -\partial_\lambda w$, we obtain (5.3), (5.3)'. These hold in $\mathcal{V}$ and the coefficients satisfy (5.5).

We set

$$V^T = \{(x, \lambda) \in \mathcal{V} \; ; \; \lambda < T\} \, .$$

Note that $V = V^T$ satisfies the condition (4.1), for $(V^T)_\lambda = \Sigma(\lambda)$.

Now we turn to the

Proof of Theorem 2.1. First we prove uniqueness; suppose u and $\underline{u}$ are solutions satisfying all the conditions of the theorem. We wish to prove (5.2):

$$w > 0 \quad \text{in} \quad \mathcal{V} \, .$$

Indeed, from this, letting $\lambda \to a$, we infer that $u \geq \underline{u}$ in Ω. But u and $\underline{u}$ may be interchanged, hence $u = \underline{u}$. In addition if we set $\underline{u} = u$ in the argument we see that (5.2) implies that u is strictly increasing in x_1. Finally, if F is smooth, so is u, by regularity theory; we may differentiate (2.1) with respect to x_1 and conclude from the maximum principle that $u_1 > 0$ in Ω. Thus to prove the theorem we have only to verify (5.2).

The condition (2.3) implies

$$(5.7) \qquad \begin{array}{l} \text{for } \lambda \text{ fixed, } w(\cdot, \lambda) \geq_{\neq} 0 \text{ on the boundary of each} \\ \text{connected component of } \Sigma(\lambda). \end{array}$$

(i) First we verify (5.2) for $(x, \lambda) \in V = V^T$ with $0 < T + a < \epsilon$ small. Clearly $V_T = (V^T)_T = \Sigma(T)$ has small measure. Using (5.7) we may apply Proposition 4.1 to $z = -w$ and infer that $w \geq 0$ in V. Strict inequality (5.2) in V follows from (5.7) and Lemma 4.3. (It was in this step that our proof of Theorem 4.3 in [BN1] was wrong; see the Remarks after Theorem 2.1 in this paper.)

(ii) Now we wish to increase λ. Suppose $w > 0$ in V^T for $-a < T < \mu < a$. By continuity, $w \geq 0$ in V^μ, and with the aid of Lemma 4.3 we see that $w > 0$ in V^μ. We will show that for every positive ϵ_0 sufficiently small, $w > 0$ in V^T for $T = \mu + \epsilon_0$. This then yields (5.2).

Ω lies in some ball $\{|x| < R\}$. Let $\delta = \delta(n, R, c_0, b)$ be as in Proposition 4.1. Here however, δ is associated with the ellipticity constant c_0 rather than 1 as in that proposition. Let K be a closed set in $\Sigma(\mu)$ such that $|\Sigma(\mu) \setminus K| \leq \delta/2$. Clearly $w(x, \mu) > 0$ for $x \in K$. By continuity, for $0 < \epsilon \leq \epsilon_0$ small, we have

$$(5.8) \qquad |\Sigma(\mu + \epsilon_0) \setminus K| \leq \delta \quad \text{and}$$

$$(5.9) \qquad w(x, \mu + \epsilon) > 0 \quad \text{for } x \in K .$$

Let

$$Z = \{(x, \lambda) \in \mathcal{V}; \ x \notin K , \ \mu < \lambda < \mu + \epsilon_0\} .$$

It is easily verified that Z satisfies all the conditions of Proposition 4.1: (4.1) holds; from (5.7), (5.9) and the fact that $w > 0$ in V^μ we see that $w \geq 0$ on the parabolic boundary J of Z. Finally, set $Z_{\mu+\epsilon} = \Sigma(\mu + \epsilon) \setminus K$, hence by (5.8),

$$|Z_{\mu+\epsilon_0}| \leq \delta .$$

Applying Proposition 4.1 to $z = -w(x, \lambda)$ we conclude that $w \geq 0$ in Z and hence in $V^{\mu+\epsilon_0}$. By (5.7) and Lemma 4.3 we conclude once more that $w > 0$ in $V^{\mu+\epsilon_0}$. The proof of Theorem 2.1 is complete. $\square$

Proof of Theorem 2.4. The proof is similar to that of Theorem 2.1. But here the geometry is simpler since $\Sigma(\lambda) = \{x \in \Omega; \ x_1 < \lambda\}$. On the other hand, we have the nonlinear boundary condition (2.14) on the lateral boundary which makes things more complicated. As in the preceding proof, for $x \in \Sigma(\lambda)$, we consider $w(x, \lambda)$ given by (5.1); once more it suffices to establish (5.2) for $-a < \lambda < a$.

In $\mathcal{V}$, given by (5.4), w satisfies (5.3), (5.3)$'$ but it satisfies different conditions on various parts of the parabolic boundary of $\mathcal{V}$, i.e. on $\partial \mathcal{V} \cap \{\lambda < a\}$. This "parabolic boundary" of $\mathcal{V}$ is the union of two disjoint sets:

$$J_1 = \{(x_1, y, \lambda); \ y \in \bar{\omega} , \ \lambda < a , \ x_1 = -a \text{ or } x_1 = \lambda\} ,$$
$$J_2 = \{(x, \lambda) \in \partial \mathcal{V} ; \ -a < x_1 < \lambda < a , \ y \in \partial \omega\} .$$

Observe that J_2 is a smooth hypersurface on which the exterior normal to $\mathcal{V}$ is $\{\nu, 0\}$ with $\nu_1 = 0$. In view of (2.15) and (2.16) we see that

$$w \geq 0 , \quad w \not\equiv 0 , \quad \text{on } J_1 .$$

On J_2 we have

$$\sigma(x_1 + a - \lambda, y, v, \nabla v) = \sigma(x_1, y, \underline{u}, \nabla \underline{u}) = 0 .$$

Using (2.18), and arguing as above in our derivation of (5.3)$'$, we find

$$\sigma(x_1, y, v, \nabla v) \geq \tilde{\beta} v_1$$

with $0 \leq \tilde{\beta} \in L^\infty, \ \tilde{\beta} \leq C$. Consequently

$$\sigma(x, v, \nabla v) - \sigma(x, \underline{u}, \nabla \underline{u}) - \tilde{\beta} v_1 \geq 0 .$$

Hence, by the integral form of the theorem of the mean, and the identity $v_1 = -\partial_\lambda w$,

$$(5.10) \qquad \sum_1^n a_j w_{x_j} + \gamma w + \tilde{\beta} w_\lambda \geq 0 \quad \text{on } J_2 .$$

From (2.17) we see that

$$(5.11) \qquad \gamma \geq 0 , \quad \sum_2^n \nu_j a_j \geq 0 \quad \text{and} \quad \gamma + \sum \nu_j a_j > 0 .$$

We proceed as before, but in place of Proposition 4.1 we use the function of Lemma 4.1. The proof is again divided into two steps.

Step 1. Start the method i.e., prove (5.2) in $V = V^T$ with $0 < T + a$ small. We use the function g of Lemma 4.1 (with $\rho = T$). For $T + a$ small, the region $\Sigma(T)$ lies in $\tilde{S}$ of the lemma (here τ is small), and hence $Lg < 0$ in $\Sigma(T)$. It follows that in V the function $z = w/g$ satisfies

$$(5.12) \qquad (M' + \frac{Lg}{g} - \beta\partial_\lambda)z \leq 0 \quad \text{in} \quad V ,$$

where M' is an elliptic operator in the x variables with no zero order terms. For (x, λ) on the parabolic boundary of V we have

$$(5.13) \qquad w \overset{\geq}{\not\equiv} 0 \quad \text{if} \quad x_1 = -a , \quad \text{and} \quad w \geq 0 \text{ if } x_1 = \lambda ,$$

while if $-a < x_1 < \lambda$, $y \in \partial\omega$ we find from (5.10):

$$(5.14) \qquad \sum_1^n a_j z_{x_j} + (\gamma + g^{-1}\sum_1^n a_j g_j)z + \tilde{\beta}z_\lambda \geq 0 .$$

If $w < 0$ somewhere in V then z achieves a negative minimum at some point (x, λ) in $\overline{V}$. At that point $z_\lambda \leq 0$, and if $x \in \Sigma(\lambda)$ then we would have $M'z \geq 0$ there, contradicting (5.12) — for $Lg/g < 0$. Thus (x, λ) must lie on the parabolic boundary. By (5.13) we must have $-a < x_1 < \lambda$, $y \in \partial\omega$. But then at (x, λ), $z_{x_1} = 0$ and it follows from (5.14) that

$$0 \geq g\gamma + \Sigma a_j g_j .$$

By (4.7) and (5.11), however,

$$g\gamma + \Sigma a_j g_j \geq \gamma + \sum_2^n a_j \nu_j - b\tau^{1/2}$$
$$> 0 \quad \text{for } \tau \text{ small}.$$

Impossible. Thus $w \geq 0$ in V.

With the aid of Lemma 4.3 we find as before that $w > 0$ in V^T for $T + a$ small.

Step 2. Suppose now $w > 0$ in V^T for $-a < T < \mu < a$. As before we see that $w > 0$ in V^μ and we wish to show that for every positive small ϵ, $w > 0$ in V^T for $T = \mu + \epsilon$. This would imply (5.2) for $-a < \lambda < a$.

We will use the function g of Lemma 4.1 with $\rho = \mu + \epsilon$, $0 < \epsilon$ small and with some $\tau > 2\epsilon$. Then we proceed as in the proof of Theorem 2.1. Let

$$K = \{x \in \Sigma(\mu) \, ; \, d(x, \partial\Sigma(\mu) \geq \tau/2\} .$$

Then $w(x, \mu) > 0$ for $x \in K$, and so for every positive ϵ sufficiently small (5.9) holds in this K. That (5.2) holds in $V^{\mu+\epsilon}$ is then proved by using as in step (i), the function g of Lemma 4.1. On the parabolic boundary region

$$\tilde{\Omega} = \{(x, \lambda) \, , \, x \in \Sigma(\lambda) \setminus K \, , \, \mu < \lambda < \mu + \epsilon\} ,$$

$w \geq 0$. One sees as before that for τ small, the function $z = w/g$ cannot have a negative minimum in $\tilde{\Omega}$. Thus $w \geq 0$ in $V^{\mu+\epsilon}$ and as before, $w > 0$ there. $\square$

6. The Method of Moving Planes

Sketch of proofs of Theorems 2.2 and 2.5. We work as in [GNN1] and [BN1] with a domain $\Sigma(\lambda)$ which is different from the one used in Section 5. It is the same as that used in the proof of Theorem 1.3. For $-a < \lambda \leq 0$ we take

$$\Sigma(\lambda) = \{x \in \Omega \, ; \, x_1 < \lambda\}$$

and let T_λ be the plane $x_1 = \lambda$. In the proofs of both theorems, we consider in $\Sigma(\lambda)$ the functions

$$v(x) := u(x^\lambda) = u(2\lambda - x_1, y) ,$$
$$(6.1) \qquad w(x, \lambda) := v(x) - u(x) .$$

We have to prove that $\forall \lambda \in (-a, 0)$, $x \in \Omega$,

$$(6.2) \qquad w(x, \lambda) > 0 \quad \text{for} \quad x \in \Sigma(\lambda) ,$$

$$(6.3) \qquad -2u_1 = \partial_1 w(x, y) < 0 \quad \text{if} \quad x_1 = \lambda .$$

Note that $w(\lambda, y, \lambda) = 0$.

As in the preceding section we make use of parabolic inequalities (5.3), (5.3)' for w which we now derive as in [BN1]. In $\Sigma(\lambda)$ the function v satisfies

$$(6.4) \quad F(2\lambda - x_1, y, v, -v_1, v_2, \ldots, v_n, v_{11} - v_{12}, \ldots, -v_{1n}, v_{22}, \ldots) = 0 .$$

It follows from (1.25) that

$$I := F(x_1, y, v, v_i, v_{jk}) \leq 0 \quad \text{if} \quad v_1(x) \geq 0 ,$$

while if $v_1(x) < 0$ we have using (6.4), Lipschitz continuity in p_1 and (1.25) again,

$$
\begin{aligned}
I = {} & F(x_1, y, v, v_i, v_{jk}) - F(x_1, y, v, 0, v_2, \ldots, v_n, v_{jk}) \\
& + F(x_1, y, v, 0, v_2, \ldots, v_n, v_{jk}) \\
& - F(2\lambda - x_1, y, v, 0, v_2, \ldots, v_n, v_{11}, -v_{12}, \ldots, -v_{1n}, v_{22}, \ldots) \\
& + F(2\lambda - x_1, y, v, 0, v_2, \ldots, v_n, v_{11}, -v_{12}, \ldots, -v_{1n}, v_{22}, \ldots) \\
& - F(2\lambda - x_1, y, v, -v_1, v_2, \ldots, v_n, v_{11}, -v_{12}, \ldots, -v_{1n}, v_{22}, \ldots) \\
& \leq -2bv_{x_1} \; .
\end{aligned}
$$

(Lipschitz continuity is used on the first pair and last pair of terms of the right hand side above. The difference between the two middle terms is ≤ 0 by (1.25).) Thus in any case

$$
F(x_1, y, v, v_i, v_{jk}) \leq -2\beta \, v_{x_1}
$$

where β is a nonnegative L^∞ function bounded by $4b$. Using the integral theorem of the mean and the identity $\partial_\lambda w = -2v_{x_1}$ we obtain (5.3)':

$$
(6.5) \quad (L - \beta\partial_\lambda)w = a_{ij}(x,\lambda)w_{x_i x_j} + b_i(x,\lambda)w_{x_i} + c(x,\lambda)w - \beta w_\lambda \leq 0
$$

in the region

$$
\mathcal{V} = \{(x,\lambda) \; ; \; x \in \Sigma(\lambda) \, , \, -a < \lambda < 0\} \, ,
$$

with coefficients satisfying (5.5), with the exception that now $0 \leq \beta \leq b$. Hence, similarly (5.3) holds.

To prove Theorems 2.2 and 2.5 we have to establish that for $-a < T < 0$,

$$
(6.6) \qquad w > 0 \text{ in } V^T = \{(x,\lambda) \; ; \; x \in \Sigma(\lambda) \, , \, -a < \lambda < T\} \; .
$$

We remark that if $w > 0$ in V^T for $-a < T \leq \mu$ then it follows from Lemma 4.2 that

$$
(6.7) \qquad 0 > w_1(\lambda, y, \lambda) = -2u_1(\lambda, y) \; .
$$

In both theorems, (6.6) is proved following the arguments used in the above proofs of Theorems 2.1 and 2.4 respectively. The last statement of Theorem 2.5 follows from (6.7).

We omit the proofs of Theorems 2.3 and 2.6 which are improvements of Theorems 3.2 and 3.3 of [BN1] — see the proofs there. One argues as above

with $\Sigma(\lambda)$ replaced by the set

$$
\{(x_1, y) \in \Omega \text{ with } x_1 < \lambda \; ; \; (2\lambda - x_1, y) \in \Omega\} \; .
$$

7. An existence and uniqueness theorem in a cylinder.

We conclude with an application, Theorem 7.2 below, of Theorem 2.4. We state it in a more general form than needed since it may prove useful on other occasions. See also Remark 7.1 below.

In the finite cylinder $\Omega = S_a$ we wish to solve the boundary value problem

$$
(7.1) \qquad Mu + f(x,u) = a_{ij}(x)u_{ij} + b_i(x)u_i + f(x,u) = 0 \quad \text{in } S_a
$$

$$
(7.2) \qquad u_\nu(x_1, y) = 0 \quad \text{for} \quad -a < x_1 < a \, , \, y \in \partial\omega
$$

$$
(7.3) \qquad u(-a, y) = \psi_1(y) \, , \qquad u(a, y) = \psi_2(y) \; .
$$

The coefficients a_{ij} are continuous in $\overline{S}_a$ and satisfy the usual condition (1.3) of uniform ellipticity:

$$
(7.4) \qquad c_0|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq C_0|\xi|^2 \, , \quad c_0, C_0 > 0 \, , \, \forall \, \xi \in \mathbb{R}^n \; .
$$

The functions ψ_1, ψ_2 are assumed to belong to $W^{2,\infty}(\omega)$ and to satisfy conditions compatible with (7.2):

$$
(7.5) \qquad \partial_\nu\psi_j = 0 \quad \text{on} \quad \partial\omega \, , \quad j = 1, 2 \; .
$$

We assume that $f(x,u)$ is continuous in all arguments and is Lipschitz continuous in $u \; \forall \, x \in \overline{S}_a$, with Lipschitz constant k. Set $\Sigma = \{(x_1, y); x_1 = \pm a \, , \, y \in \partial\omega\}$.

Theorem 7.1. *Let $\underline{u} \leq \overline{u}$ be sub and super solutions of the problem (7.1)–(7.3) belonging to $C^1(\overline{S}_a) \cap C^2(S_a)$. More precisely, assume they satisfy*

$$
(7.6) \qquad M\underline{u} + f(x,\underline{u}) \geq 0 \geq M\overline{u} + f(x,\overline{u}) \quad \text{in } S_a
$$

$$
(7.7) \qquad \underline{u}_\nu(x_1, y) \leq 0 \leq \overline{u}_\nu(x_1, y) \quad \text{for} \quad -a < x_1 < a \, , \quad \forall \, y \in \partial\omega \, ,
$$

$$
(7.8) \qquad \left\{ \begin{array}{l} \underline{u}(-a, y) \leq \psi_1(y) \, , \quad \underline{u}(a, y) \leq \psi_2(y) \\ \overline{u}(-a, y) \geq \psi_1(y) \, , \quad \overline{u}(a, y) \geq \psi_2(y) \end{array} \right\} \quad \text{for } y \in \omega \; .
$$

Then there is a solution u in $C(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \setminus \Sigma)$ of (7.1)–(7.3) with

$$
(7.9) \qquad \underline{u} \leq u \leq \overline{u} \; .
$$

This is proved in a standard way using monotone iteration. Since the domain has corners, and different boundary conditions are imposed on parts of the boundary which touch each other — a case not usually treated — we present a complete proof. But first, an interesting consequence, which relies on Theorem 2.4.

Theorem 7.2. *In Theorem 7.1 assume in addition that $\psi_1 \leq \psi_2$ and that*

$$\psi_1(y) = \underline{u}(-a, y) \leq \underline{u}(x_1, y) \leq \psi_2(y)$$

$$\psi_1(y) \leq \overline{u}(x_1, y) \leq \overline{u}(a, y) = \psi_2(y)$$

(7.10)

$$\text{for} - a < x_1 < a, \forall y \in \omega,$$

$\underline{u}$ *and* $\overline{u}$ *are not solutions of* (7.1)–(7.3).

Assume also that the coefficients a_{ij} are independent of x_1, and that they and the b_i are continuous in S_a. Assume furthermore that f is Lipschitz continuous in all arguments and that

(7.11) b_1 *and* f *are nondecreasing in* x_1 .

Then there is only one solution $u \in W^{2,p}_{loc}(\overline{S}_a \setminus \Sigma) \cap C(\overline{S}_a)$ of (7.1)–(7.3) *satisfying* (7.9). *Furthermore, if the coefficients b_i are Lipschitz continuous in x_1, then*

(7.12) $u_{x_1}(x, y) > 0$ for $-a < x_1 < a$, $y \in \overline{\omega}$.

Proof of Theorem 7.2. Consider any solution $u \in C(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \setminus \Sigma)$ of (7.1)–(7.3) satisfying (7.9). Using the maximum principle and the Hopf lemma we see easily that

$$\underline{u} < u < \overline{u} \quad \text{in } S_a ;$$

note that we are using here the last assertion in (7.10). It follows from (7.10) that u satisfies the condition (2.16) in Theorem 2.4. In view of (7.11), and the Remark after Theorem 2.4, the conclusions of Theorem 2.4 hold, and they imply Theorem 7.2.

Remark 7.1. The proof we present of Theorem 7.1 is a fairly standard argument. Using a slight modification of an argument of Amann and Crandall [AC] one may prove a different form of Theorem 7.1. in which (7.1) takes the form

$$\sum_1^n a_{ii}(y) u_{x_i x_i} + f(x, u, \nabla u) .$$

The result of the theorem holds if $f(x, u, p)$ is defined and continuous for $x \in \overline{S}_a$, $\underline{u}(x) \leq u \leq \overline{u}(x)$, $p \in \mathbb{R}^n$, f is locally Lipschitz in (u, p), and for some constant K,

$$|f(x, u, p)| \leq K(1 + |p|^2) .$$

Proof of Theorem 7.1.

We begin first with a lemma for a linear problem; it will be used several times.

Consider the linear problem

(7.13) $Lu := Mu + cu = a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)u = g(x)$

(7.14) $u_\nu(x_1, y) = 0$ for $-a < x_1 < a$, $y \in \partial\omega$.

(7.15) $u(-a, y) = \psi_1(y)$, $u(a, y) = \psi_2(y)$.

Here the a_{ij}, ψ_1, ψ_2 are as in Theorem 7.1, $g \in L^\infty(S_a)$, and the coefficients b_i, c satisfy

(7.16) $|b_i|$, $|c| \leq C$, $c \leq 0$.

Lemma 7.1. *Under the conditions above, the problem* (7.13)–(7.15) *has a unique solution $u \in W^{2,p}_{loc}(\overline{S}_a \setminus \Sigma) \cap C(\overline{\Omega})$, and*

(7.17) $\max |u| \leq C_1(\|g\|_{L^\infty(S_a)} + \sum_j \|\psi_j\|_{W^{2,\infty}(\omega)})$

with C_1 depending only on ω, a, c_0, C and p.

Proof. Uniqueness follows from the maximum principle (see p. 241 in [BN1]). To prove existence we write

(7.18) $u = v + \dfrac{a - x_1}{2a}\psi_1(y) + \dfrac{a + x_1}{2a}\psi_2(y)$;

then for v we obtain the problem

(7.19) $Lv = \tilde{g} \in L^\infty(S_a)$
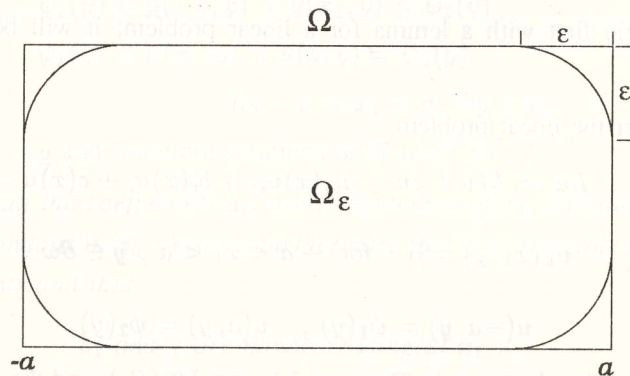
(7.20) $v_\nu = 0$ for $-a < x_1 < a$, $y \in \partial\omega$,

(7.21) $v = 0$ for $x_1 = \pm a$,

with $\tilde{g}$ obtained from g, ψ_1 and ψ_2.

Because of the difficulties at the corners we consider an approximate problem in a subdomain in which the corners have been rounded off. For $0 < \epsilon$ small consider a domain $\Omega_\epsilon \subset \Omega$ with smooth boundary as in the diagram (in case $n = 2$) such that $\Omega \setminus \Omega_\epsilon$ lies within a distance ϵ of the corner set:

$$\Sigma = \{(\pm a, y) , \ y \in \partial\omega\} .$$

In Ω_ϵ we solve the boundary value problem for $v = v_\epsilon$ of

$$(7.22) \qquad \begin{cases} Lv = \tilde{g} \\ (\epsilon + \sigma)v_\mu + (1 - \sigma)v = 0 \qquad \text{on } \partial\Omega_\epsilon , \end{cases}$$

where μ is the outward unit normal to Ω_ϵ on $\partial\Omega_\epsilon$ and where $\sigma(x_1)$ is a smooth function of x_1, $0 \le \sigma \le 1$, which vanishes on $[-a, -a + \epsilon]$ and on $[a - \epsilon, a]$ and satisfies

$$(7.23) \qquad \sigma(x_1) \equiv 1 \quad \text{on } [-a + 2\epsilon , a - 2\epsilon] .$$

This problem has a solution in $W^{2,p}(\Omega_\epsilon)$. We wish now to let $\epsilon \to 0$ and obtain a limit function which is continuous in $\overline{S}_a$ and is the desired solution.

For this purpose we make use of a concave symmetric positive barrier function $h(x_1)$ on $(-a, a)$ vanishing at the end points and satisfying $Lh \le -1$. For example we may take

$$h = \frac{c_0}{b^2}e^{ba/c_0}[1 - e^{-(b/c_0)(x_1+a)}] - \frac{1}{b}(x_1 + a) , \quad \text{on } [-a, 0]$$

and extend it to be symmetric. Then on $(-a, 0)$,

$$\begin{aligned} Lh &= a_{11}\ddot{h} + b_1\dot{h} + ch \\ &\le c_0\ddot{h} + b\dot{h} = -1 . \end{aligned}$$

(Without loss of generality we may always assume $b > 0$.)

We claim that the solution v of (7.22) satisfies

$$(7.24) \qquad |v| \le \|\tilde{g}\|_{L^\infty}(h + \epsilon k) \quad \text{in } S_a ,$$

where $k = \max|\dot{h}|$. In verifying this we may suppose $|\tilde{g}|_{L^\infty} = 1$. Then $L(v - h - \epsilon k) \ge 0$ in Ω_ϵ and hence if $v - h - \epsilon k$ is positive somewhere it achieves its maximum at the boundary. At that point, $(v - h)_\mu > 0$ by the Hopf lemma, and hence, there,

$$(\epsilon + \sigma)(v - h)_\mu + (1 - \sigma)(v - h - \epsilon k) > 0 .$$

Because of the boundary condition in (7.22), this means

$$(\epsilon + \sigma)h_\mu + (1 - \sigma)(h + \epsilon k) < 0$$

there. At this point σ cannot be positive, for if it were then h_μ would be zero there; impossible. Thus $\sigma = 0$ there; that again is impossible since $k \ge h_\mu$.

Having established (7.24) we now let $\epsilon \to 0$. Using standard local $W^{2,p}$ estimates (up to smooth parts of the boundary) we find that for a sequence $\epsilon_j \to 0$, $v_j \to v$ uniformly in compact subsets of $(-a, a) \times \bar{\omega}$, $v \in W_{loc}^{2,p}((-a, a) \times \bar{\omega})$ and satisfies (7.19)–(7.21). Furthermore v satisfies

$$|v| \le \|\tilde{g}\|_{L^\infty} h$$

and is therefore continuous in $\overline{S}_a$. Using local $W^{2,p}$ theory again one sees that $v \in W_{loc}^{2,p}(\overline{S}_a \setminus \Sigma)$. The lemma is proved.

With its aid, Theorem 7.1 is proved by a well known argument. Rewrite the equation (7.1) as

$$Lu := Mu - ku = -f(x, u) - ku$$

where k is the Lipschitz constant of f in u. Starting with $u_0 = \underline{u}$ this is solved by monotone iteration: Let u_1 be the solution given by Lemma 7.1 of (7.13–7.15) with

$$g(x) = -f(x, u_0) - ku_0 .$$

By the lemma it is continuous. Now

$$L(u_1 - u_0) \le 0 ,$$

and it follows from the maximum principle and the Hopf lemma (it is important that $u_1 \in C(\overline{S}_a)$) that

$$u_1 \ge u_0 .$$

Using the fact that

$$(7.25) \qquad f(x,v) + kv \text{ is nondecreasing in } v \,,$$

we find again by the maximum principle that $u_1 \leq \overline{u}$.

Now continue iteratively: u_{j+1} is the solution (via Lemma 7.1) of

$$Lu_{j+1} = -f(x, u_j) - ku_j \quad \text{in } S_a$$

satisfying (7.2) and (7.3). Using the maximum principle and (7.25) recursively we find

$$\underline{u} = u_0 \leq u_1 \leq u_2 \leq \cdots \leq \overline{u} \,.$$

Furthermore, by Lemma 7.1, the functions u_j all satisfy

$$(7.26) \qquad \left| u_j - \frac{a - x_1}{2a}\psi_1 - \frac{a + x_1}{2a}\psi_2 \right| \leq Ch$$

for a fixed constant C. As usual one verifies that the u_j converge uniformly on compact subsets of $\overline{S}_a \setminus \Sigma$ to a solution $u \in W^{2,p}_{loc}(\overline{S}_a \setminus \Sigma)$ of (7.1–7.3). From (7.26) it follows that u is continuous in $\overline{S}_a$, thus u is a solution with all the desired properties. $\square$

References

[AC]   H Amann, M. G. Crandall, On some existence theorems for semilinear elliptic equations, Indiana Univ. Math. J. 27 (1978) 779-790.

[B]   A. Bakelman, Convex functions and elliptic equations, Book, to appear.

[BN1]   H. Berestycki, L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, J. Geometry and Physics 5 (1988) 237-275.

[BN2]   ———, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, Analysis et cetera, ed. P. Rabinowitz et al., Academic Pr. (1990) 115-164.

[BN3]   ———, Travelling front solutions of semilinear equations in n dimensions, to appear in volume dedicated to J. L. Lions.

[BN4]   ———, KPP in higher dimensions, to appear.

[BNV]   H. Berestycki, L. Nirenberg, S. R. S. Varadhan, On principal eigenvalues of second order elliptic operators in general domains, to appear.

[Bo]   J. M. Bony, Principe du maximum dans les espaces de Sobolev, C. R. Acad. Sci. Paris, Ser. A, 265 (1967) 333-336.

[CNS]   L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations I. Monge-Ampère equation, Comm. Pure Appl. Math. 38 (1984) 369-402.

[CS]   W. Craig, P. Sternberg, Symmetry of solitary waves, Comm. P.D.E. 13 (1988) 603–633.

[CY]   S. Y. Cheng, S. T. Yau, On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$, Comm. Pure Appl. Math. 30 (1977) 41-68.

[GNN1]   B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209-243.

[GNN2]   ———, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Math. Anal. and Applic., Part A, Advances in Math. Suppl. Studies 7A, ed. L. Nachbin, Academic Press (1981) 369-402.

[GT]   D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Ed'n. Springer Verlag, 1983.

[L1]   Li, Cong Ming, Monotonicity and symmetry of solutions of fully nonlinear elliptic equation I, Bounded domains, Comm. Partial Diff. Eq'ns., to appear.

[L2]   ———, Ibid., II, Unbounded domains, Comm. Part. Diff. Eq'ns., to appear.

[Lio]   P. L. Lions, A remark on Bony's maximum principle, Proc. Amer. Math. Soc. 88 (1983) 503-508.

[S]   J. Serrin, A symmetry problem in potential theory. Arch. Rat. Mech. 43 (1971) 304–318.

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