

On a Construction Related to the Non-abelian Tensor Square of a Group

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Abstract. Let G and G^p be isomorphic groups. We introduce and study a quotient $\mathcal{V}(G)$ of the free product $G * G^p$ which is a group extension of the non-abelian tensor square $G \otimes G$. This seems to bring computational advantages to calculate this last group. Looking over $\mathcal{V}$ as an operator in the class of groups we prove that it preserves properties of the argument G such as finiteness, set of prime divisors, nilpotency and solvability. For a finite p -group G we find a good polynomial bound for the order of $\mathcal{V}(G)$.

1. Introduction

The non-abelian tensor product $G \otimes H$ of the groups G and H , as introduced by R. Brown and J.-L. Loday [2], generalises the usual tensor product $\frac{G}{G'} \otimes_{\mathbb{Z}} \frac{H}{H'}$ of the abelianized groups, on the assumption that each of G and H acts on the other.

Specifically, given groups G, H each of which acts on the other (on the right)

$$G \times H \rightarrow G, (g, h) \mapsto g^h; H \times G \rightarrow H, (h, g) \mapsto h^g$$

in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$(1) \quad g^{hg_1} = g^{g_1^{-1}hg_1} \quad \text{and} \quad h^{gh_1} = h^{h_1^{-1}gh_1}$$

where G and H acts on itself by conjugation, then the *non-abelian tensor product* $G \otimes H$ is defined to be the group generated by all symbols $g \otimes h, g \in G, h \in H$, subject to the relations

$$(2) \quad gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h)$$

$$(3) \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

for all $g, g_1 \in G, h, h_1 \in H$, where the action of G on itself is the conjugation $g^{g_1} = g_1^{-1} g g_1$, and similarly for H .

In particular, as the conjugation action of a group G on itself satisfies (1), the *tensor square* $G \otimes G$ of a group G may always be defined. This tensor square is the focus of attention of [1] and [3], and constructions related to the general non-abelian tensor product are focused in [4].

The purpose of this article is to study a group which is also related to the above construction, defined as follows:

Let G and G^φ be isomorphic groups through $\varphi, g \mapsto g^\varphi, \forall g \in G$. We define the group

$$\mathcal{V}(G) := \langle G, G^\varphi \mid [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi}, \quad \forall g_1, g_2, g_3 \in G \rangle$$

(here we keep in mind that for elements h, k of any group, $h^k = k^{-1} h k$ and $[h, k] = h^{-1} h^k$).

Our motivation to introduce $\mathcal{V}(G)$ is that its *subgroup* $[G, G^\varphi]$ is *actually isomorphic to the non-abelian tensor square* $G \otimes G$ (Proposition 2.6).

Another construction related to $\mathcal{V}(G)$ is the one introduced by S. Sidki [10],

$$\chi(G) = \langle G, G^\varphi \mid [g, g^\varphi] = 1, \quad \text{for all } g \in G \rangle,$$

which has, among other attributes, the property of being a finite group when G is finite. Considering the subgroup $\Delta(G)$ of $\mathcal{V}(G)$, generated by all $[g, g^\varphi], g \in G$, we obtain $\Delta(G) \leq \mathcal{V}(G)' \cap \mathcal{Z}(\mathcal{V}(G))$. The finiteness of $\mathcal{V}(G)$ then follows from the fact that $\frac{\mathcal{V}(G)}{\Delta(G)}$ is isomorphic to a certain natural factor of $\chi(G)$ (Proposition 2.4).

By using techniques similar to those used in [5] and [9] we describe the lower central series and the derived series of $\mathcal{V}(G)$ in terms of the corresponding series of G . Our main results are the following:

Theorem A. *Let G be a nilpotent group of class c (resp. a solvable group of derived length ℓ). Then $\mathcal{V}(G)$ is a nilpotent group of class at most $c + 1$ (resp. a solvable group of derived length at most $\ell + 1$).*

Theorem B. *Let G be a finite p -group of order p^n with G' of order p^m . Then $\mathcal{V}(G)$ is a p -group of order dividing $p^{n^2+2n-mn}$.*

In particular we obtain bounds for $G \otimes G$ similar to those of Jones [6] for the Schur Multiplier:

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-m)}$$

where $d = d(G)$ denotes the minimal number of generators of G .

2. Basic Results

In this section we derive some properties of the group $\mathcal{V}(G)$ and identify $G \otimes G$ as a subgroup of it. We use some standard commutator identities without reference (see, for instance, D. Robinson [8]):

For elements x, y, z in a group G , the conjugate of x by y is $x^y = y^{-1} x y$; the commutator of x and y is $[x, y] = x^{-1} x^y$ and our commutators are left normed, $[x, y, z] = [[x, y], z]$. The following identities hold:

$$\begin{aligned} [x, y] &= [x, y^{-1}]^{-y} = [x^{-1}, y]^{-x}; \\ [xy, z] &= [x, z]^y [y, z] = [x, z][x, z, y][y, z]; \\ [x, yz] &= [x, z][x, y]^z = [x, z][x, y][x, y, z]; \\ [x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x &= 1. \end{aligned}$$

We simplify the definition of $\mathcal{V}(G)$ as

$$\mathcal{V}(G) = \langle G, G^\varphi \mid [g, h^\varphi]^{k^\epsilon} = [g^k, (h^k)^\varphi], \quad \text{for all } g, h, k \in G, \epsilon \in \{1, \varphi\} \rangle,$$

where $\varphi: G \rightarrow G^\varphi, g \mapsto g^\varphi$ is a group isomorphism.

2.1 Lemma. *The following relations hold in $\mathcal{V}(G)$:*

- (i) $[g_1, g_2^\varphi]^{[g_3, g_4^\varphi]} = [g_1, g_2^\varphi]^{[g_3, g_4]}, \quad \forall g_1, g_2, g_3, g_4 \in G;$
- (ii) $[g_1, g_2^\varphi, g_3] = [g_1, g_2, g_3^\varphi] = [g_1, g_2^\varphi, g_3^\varphi]$ and $[g_1^\varphi, g_2, g_3] = [g_1^\varphi, g_2, g_3^\varphi] = [g_1^\varphi, g_2^\varphi, g_3], \quad \forall g_1, g_2, g_3 \in G;$
- (iii) $[g, g^\varphi]$ is central in $\mathcal{V}(G), \quad \forall g \in G;$
- (iv) $[g_1, g_2^\varphi][g_2, g_1^\varphi]$ is central in $\mathcal{V}(G), \quad \forall g_1, g_2 \in G;$
- (v) $[g, g^\varphi] = 1, \quad \forall g \in G'.$

Proof. (i) The defining relations of $\mathcal{V}(G)$ yield:

$$\begin{aligned} [g_1, g_2^\varphi]^{[g_3, g_4^\varphi]} &= [g_1, g_2^\varphi]^{g_3^{-1} g_4^{-\varphi} g_3 g_4^\varphi} \\ &= [g_1^{g_3^{-1}}, (g_2^{g_3^{-1}})^\varphi]^{g_4^{-\varphi} g_3 g_4^\varphi} \\ &= \dots \\ &= [g_1^{g_3^{-1} g_4^{-1} g_3 g_4}, (g_2^{g_3^{-1} g_4^{-1} g_3 g_4})^\varphi] \\ &= [g, g_2^\varphi]^{[g_3, g_4]}; \end{aligned}$$

(ii) From $[x, y] = x^{-1}x^y$ and commutator calculus we get

$$\begin{aligned} [g_1, g_2, g_3^\varphi] &= [g_1^{-1} g_1^{g_2}, g_3^\varphi] \\ &= [g_1^{-1}, g_3^\varphi]^{g_1^{g_2}} \cdot [g_1^{g_2}, g_3^\varphi] \\ &= [g_1^{-1}, g_3^\varphi]^{g_2^{-1} g_1 g_2} [g_1, (g_1^{g_2^{-1}})^\varphi]^{g_2} \\ &\quad (\text{by defining relations of } \mathcal{V}(G)) \\ &= [g_1, g_3^\varphi]^{-g_1^{-1} g_2^{-1} g_1 g_2} [g_1, (g_2 g_3 g_2^{-1})^\varphi]^{g_2} \\ &= [g_1, g_3^\varphi]^{-[g_1, g_2]} \cdot [g_1, (g_2^{-1})^\varphi]^{g_2} [g_1, (g_2 g_3)^\varphi] \\ &= [g_1, g_3^\varphi]^{-[g_1, g_2]} [g_1, g_2^\varphi]^{-1} [g_1, g_3^\varphi] [g_1, g_2^\varphi]^{g_3} \\ &= [g_1, g_3^\varphi]^{-[g_1, g_2^\varphi]} [g_1, g_2^\varphi]^{-1} [g_1, g_3^\varphi] [g_1, g_2^\varphi]^{g_3} \quad (\text{by (i)}) \\ &= [g_1, g_2^\varphi]^{-1} [g_1, g_3^\varphi]^{-1} [g_1, g_3^\varphi] [g_1, g_2^\varphi]^{g_3} \\ &= [g_1, g_2^\varphi]^{-1} [g_1, g_2^\varphi]^{g_3} \\ &= [g_1, g_2^\varphi, g_3]; \end{aligned}$$

Now we observe that

$$\begin{aligned} [g_1, g_2^\varphi, g_3^\varphi] &= [g_1 g_2^\varphi]^{-1} [g_1, g_2^\varphi]^{g_3^\varphi} \\ &= [g_1, g_2^\varphi]^{-1} [g_1, g_2^\varphi]^{g_3} \quad (\text{by defining relations}) \\ &= [g_1, g_2^\varphi, g_3] \end{aligned}$$

The last two relations in (ii) follow by a symmetric argument.

(iii) It follows from (ii) that for all $g, h \in G$,

$$[g, g^\varphi, h] = [g, g, h^\varphi] = 1;$$

But

$$\begin{aligned} [g, g^\varphi, h^\varphi] &= [g, g^\varphi]^{-1} \cdot [g, g^\varphi]^{h^\varphi} \\ &= [g, g^\varphi]^{-1} [g, g^\varphi]^h \\ &= [g, g^\varphi, h], \end{aligned}$$

so that (iii) is proved:

(iv) For $g_1, g_2 \in G$ we get

$$\begin{aligned} [g_1 g_2, (g_1 g_2)^\varphi] &= [g_1, (g_1 g_2)^\varphi]^{g_2} [g_2, (g_1 g_2)^\varphi] \\ &= [g_1, g_2^\varphi]^{g_2} [g_1, g_1^\varphi]^{g_2^\varphi g_2} [g_2, g_2^\varphi] [g_2, g_1^\varphi]^{g_2^\varphi} \\ &= [g_1, g_2^\varphi]^{g_2} [g_1, g_1^\varphi] [g_2, g_2^\varphi] [g_2, g_1^\varphi]^{g_2^\varphi} \quad (\text{by (iii)}) \end{aligned}$$

Therefore, again by (iii), we can write

$$[g_1 g_2, (g_1 g_2)^\varphi] [g_1, g_1^\varphi]^{-1} [g_2, g_2^\varphi]^{-1} = [g_1, g_2^\varphi]^{g_2} [g_2, g_1^\varphi]^{g_2^\varphi}$$

As the first member is central in $\mathcal{V}(G)$, on conjugating by $g_2^{-\varphi}$ and using the definition of $\mathcal{V}(G)$ we obtain

$$[g_1, g_2^\varphi] [g_2, g_1^\varphi] = [g_1 g_2, (g_1 g_2)^\varphi] [g_1, g_1^\varphi]^{-1} [g_2, g_2^\varphi]^{-1},$$

which belongs to the center of $\mathcal{V}(G)$;

As for (v), we first observe that when $g \in G'$ is a simple commutator, say $g = [x, y]$, then by (i) and (ii),

$$\begin{aligned} [[x, y], [x, y]^\varphi] &= [x, y, (x^{-1}x^y)^\varphi] \\ &= [x, y^\varphi, x^{-1}x^y] \\ &= [x, y^\varphi]^{-1} [x, y^\varphi]^{[x, y^\varphi]} \\ &= [x, y^\varphi]^{-1} [x, y^\varphi] = 1. \end{aligned}$$

Now for a general element $g \in G'$, say $g = [x_1, y_1] \dots [x_r, y_r]$, we use (i), (ii) and make induction on $r \geq 1$ to get

$$\begin{aligned} [g, g^\varphi] &= [[x_1, y_1] \dots [x_r, y_r], [x_1, y_1]^\varphi \dots [x_r, y_r]^\varphi] \\ &= \dots \\ &= [[x_1, y_1^\varphi] \dots [x_r, y_r^\varphi], [x_1, y_1^\varphi] \dots [x_r, y_r^\varphi]] = 1, \end{aligned}$$

proving (v). $\square$

2.2 Lemma. Let a, b, x be elements in G such that $[x, a] = 1 = [x, b]$. Then

$$[a, b, x^\varphi] = 1 = [[a, b]^\varphi, x].$$

Proof. By Lemma 2.1 (ii) we obtain

$$\begin{aligned} [a, b, x^\varphi] &= [a, b^\varphi, x] \\ &= [a, b^\varphi]^{-1} \cdot [a, b^\varphi]^x \\ &= [a, b^\varphi]^{-1} [a^x, (b^x)^\varphi] \\ &= [a, b^\varphi]^{-1} [a, b^\varphi] = 1. \end{aligned}$$

The other identity follows by the symmetry in part (ii) of Lemma 2.1. $\square$

2.3 Lemma. Let x, y be elements of G such that $[x, y] = 1$. Then

- (i) $[x^n, y^\varphi] = [x, y^\varphi]^n = [x, (y^\varphi)^n]$, for all $n \in \mathbb{Z}$;
- (ii) If x and y are torsion elements of orders $o(x)$ and $o(y)$, then $o([x, y^\varphi])$ divides the $g.c.d.(o(x), o(y))$.

Proof. (i) is proved by induction for $n \geq 0$, while

$$[x, y^\varphi]^{-1} = [x^{-1}, y^\varphi]^x = [x^{-1}, (y^x)^\varphi] = [x^{-1}, y^\varphi];$$

(ii) is a consequence of (i). $\square$

Remark 1. By the symmetry between the defining relations of $\mathcal{V}(G)$ we note that the isomorphism φ extends uniquely to an automorphism Ψ of $\mathcal{V}(G)$ sending $g \mapsto g^\varphi, g^\varphi \mapsto g$ and $[g_1, g_2^\varphi] \mapsto [g_2, g_1^\varphi]^{-1}$, for all $g, g_1, g_2 \in G$.

Remark 2. For a finite group G , we can get the finiteness of $\mathcal{V}(G)$ making use of the finiteness of the following group $\chi(G)$ (cf. S. Sidki [10]):

For the given isomorphic pair G, G^φ , consider the group

$$\chi(G) := \langle G, G^\varphi \mid [g, g^\varphi] = 1, \quad \forall g \in G \rangle.$$

Then we quote the following results [10] on $\chi(G)$ (see also [5,9]): "Let G be a finite π -group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\chi(G)$ is also a finite π -group, finite nilpotent or solvable of finite degree". It should be noted that $\chi(G)$ has a subgroup $R(G)$ such that the relations $[g_1, g_2^\varphi]^{g_3^\varphi} = [g_1^{g_3}, (g_2^{g_3})^\varphi]$ hold in $\frac{\chi(G)}{R(G)}$ for all $g_1, g_2, g_3 \in G$ ([10], Lemma 4.11 (iii)). Here $R(G) = [G, L(G), G^\varphi]$, where $L(G)$ is given by $L(G) = [G, \varphi] := \langle g^{-1}g^\varphi, \forall g \in G \rangle$.

Returning to our group $\mathcal{V}(G)$ we note that on introducing the relations $[g, g^\varphi] = 1$ for all $g \in G$ it renders an epimorphism $\rho: \mathcal{V}(G) \rightarrow \frac{\chi(G)}{R(G)}$ defined by $g \mapsto gR(G), g^\varphi \mapsto g^\varphi R(G), \forall g \in G, \forall g^\varphi \in G^\varphi$, whose Kernel $\Delta(G)$ is contained in $Z(\mathcal{V}(G)) \cap \mathcal{V}(G)'$, by Lemma 2.1 (iii). This implies that $\Delta(G)$ is a homomorphic image of the Schur Multiplier of $\frac{\chi(G)}{R(G)}$ which, together with the above quoted results, gives

2.4 Proposition. Let G be a finite π -group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\mathcal{V}(G)$ is also a finite π -group, finite nilpotent or solvable of finite degree.

Let N be a normal subgroup of G . We set $\bar{G}$ for the quotient group $\frac{G}{N}$ and note that the canonical epimorphism $\pi = G \rightarrow \bar{G}$ gives rise to an epimorphism $\tilde{\pi}: \mathcal{V}(G) \rightarrow \mathcal{V}(\bar{G})$ such that $g \mapsto \bar{g}, g^\varphi \mapsto \bar{g}^\varphi$, where $\bar{G}^\varphi = \frac{G^\varphi}{N^\varphi}$ is identified with $\bar{G}^\varphi$.

2.5 Proposition. With the above notation we have

- (i) $[N, G^\varphi] \trianglelefteq \mathcal{V}(G), [G, N^\varphi] \trianglelefteq \mathcal{V}(G)$;
- (ii) $\text{Ker } \tilde{\pi} = \langle N, N^\varphi \rangle \cdot [N, G^\varphi] \cdot [G, N^\varphi]$.

Proof. (i) For elements $x \in N$ and $g, h \in G$, it follows that

$$\begin{aligned} [x, g^\varphi]^h &= [x, g^\varphi][x, g^\varphi, h] \\ &= [x, g^\varphi][x, g, h^\varphi] \quad (\text{by Lemma 2.1}). \end{aligned}$$

This implies that G normalizes $[N, G^\varphi]$, and similarly G^φ normalizes $[N, G^\varphi]$, from what we get $[N, G^\varphi] \trianglelefteq \mathcal{V}(G)$. An analogous argument shows that $[G, N^\varphi] \trianglelefteq \mathcal{V}(G)$.

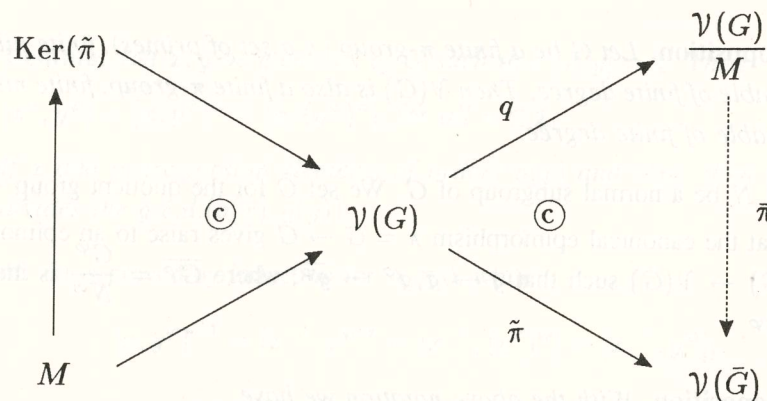
To prove (ii) we set $M = \langle N, N^\varphi \rangle \cdot [N, G^\varphi] \cdot [G, N^\varphi]$, so that $M \leq \text{Ker } \tilde{\pi}$. Furthermore M is a normal subgroup of $\mathcal{V}(G)$; thus we can define a function $\theta: \bar{G} \cup \bar{G}^\varphi \rightarrow \frac{\mathcal{V}(G)}{M}$ by setting $(\bar{g})\theta = Mg$ and $(\bar{g}^\varphi)\theta = Mg^\varphi$, which is well defined since $N, N^\varphi \subseteq M$. The restrictions of θ to $\bar{G}$ and $\bar{G}^\varphi$ are both homomorphisms, so that there is a unique homomorphism θ^* which extends θ to the free product $\bar{G} * \bar{G}^\varphi$. We see that the relations

$$[\bar{g}_1 \bar{g}_2, \bar{g}_3^\varphi] = [(\bar{g}_1^{\bar{g}_2}), (\bar{g}_3^{\bar{g}_2})^\varphi][\bar{g}_2, \bar{g}_3^\varphi]$$

and

$$[\bar{g}_1, (\bar{g}_2 \bar{g}_3)^\varphi] = [\bar{g}_1, \bar{g}_3^\varphi][\overline{(g_1^{g_3})}, \overline{(g_2^{g_3})}^\varphi]$$

are preserved by θ^* . Consequently, θ induces a homomorphism $\tilde{\theta}: \mathcal{V}(\bar{G}) \rightarrow \frac{\mathcal{V}(G)}{M}$. Since $M \leq \text{Ker}(\tilde{\pi})$ this yields an epimorphism $\bar{\pi}: \frac{\mathcal{V}(G)}{M} \rightarrow \mathcal{V}(\bar{G})$



such that $(Mg)\bar{\pi} = \bar{g}$ and $(Mg^\varphi)\bar{\pi} = \bar{g}^\varphi$. By composition of $\tilde{\theta}$ and $\bar{\pi}$ we get $(\bar{g})\tilde{\theta}\bar{\pi} = \bar{g}$ and $(\bar{g}^\varphi)\tilde{\theta}\bar{\pi} = \bar{g}^\varphi$, $\forall g \in G$. Thus $\tilde{\theta}\bar{\pi} = 1_{\mathcal{V}(\bar{G})}$, and this in turn shows that $\tilde{\theta}$ is an isomorphism. $\square$

Now we want to consider the subgroup

$$\Upsilon(G) = [G, G^\varphi]$$

which is normal in $\mathcal{V}(G)$.

By the early definition of the non-abelian tensor square $G \otimes G$ we see that the map $\tau: G \otimes G \rightarrow \Upsilon(G)$ defined on the generators by $(g_1 \otimes g_2)^\tau = [g_1, g_2^\varphi]$ extends to an epimorphism from $G \otimes G$ to $\Upsilon(G)$. In fact we have

2.6 Proposition. τ is an isomorphism.

Proof. Firstly we look at the free product $G * G^\varphi$. Its subgroup $[G, G^\varphi]$ is free, freely generated by the commutators $[g_1, g_2^\varphi]$ where $1 \neq g_1 \in G, 1 \neq g_2^\varphi \in G^\varphi$. (See for instance [7], chap. 4). As a normal subgroup of $G * G^\varphi$, $[G, G^\varphi]$ admits

the actions of G and G^φ by conjugation and the following identities hold

$$(I) \begin{cases} [g_1, g_2^\varphi]^g = [g_1 g, g_2^\varphi][g, g_2^\varphi]^{-1} \\ [g_1, g_2^\varphi]^{g^\varphi} = [g_1, g^\varphi]^{-1} \cdot [g_1, (g_2 g)^\varphi], \end{cases}$$

for all $g, g_1, g_2 \in G$.

Now the map $\mu: [G, G^\varphi] \rightarrow G \otimes G$ defined on the free generator $[g_1, g_2^\varphi]$ by $[g_1, g_2^\varphi]^\mu = g_1 \otimes g_2$ extends to an epimorphism from the (free) group $[G, G^\varphi]$ ($\leq G * G^\varphi$) onto $G \otimes G$. Consequently, the introduction in $G * G^\varphi$ of the defining relations of $\mathcal{V}(G)$ takes us to describe $\Upsilon(G)$ as the quotient of $[G, G^\varphi]$ (still a subgroup of $G * G^\varphi$) by the relations

$$(II) \begin{cases} [g_1 g_2, g_3^\varphi] = [g_1^{g_2}, (g_3^{g_2})^\varphi][g_2, g_3^\varphi] \\ [g_1, (g_2 g_3)^\varphi] = [g_1, g_3^\varphi] \cdot [g_1^{g_3}, (g_2^{g_3})^\varphi], \end{cases}$$

for all $g_1, g_2, g_3 \in G$. But relations (II) are mapped by μ in the defining relations of $G \otimes G$, from what we get that μ induces an epimorphism from $\Upsilon(G)$ onto $G \otimes G$. We have $\mu\tau = 1_{\Upsilon(G)}$ and $\tau\mu = 1_{G \otimes G}$, thus proving our assertion. $\square$

Remark 3. An argument similar to that used in Proposition 2.5 (ii) may be used to show if N is a normal subgroup of G and $\tilde{\pi}: \mathcal{V}(G) \rightarrow \mathcal{V}\left(\frac{G}{N}\right)$ is the epimorphism induced by the projection $\pi: G \rightarrow \frac{G}{N}$, then $\text{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [N, G^\varphi] \cdot [G, N^\varphi]$.

We close this section by proving

2.7 Proposition. Let

$$G = G_1 \triangleright G_2 (= G') \triangleright \dots \triangleright G_j \triangleright \dots,$$

$$1 = \xi_0(G) \trianglelefteq \xi_1(G) (= Z(G)) \trianglelefteq \dots \trianglelefteq \xi_j(G) \trianglelefteq \dots,$$

and

$$G = \gamma_1(G) \triangleright \gamma_2(G) \triangleright \dots \triangleright \gamma_j(G) \triangleright \dots$$

be respectively the derived series, the upper central series and the lower central series of G . Then

- (i) $[\xi_j(G), G_{j+1}^\varphi] = 1$, for all $j \geq 0$;
- (ii) $[\xi_{j+1}(G), \gamma_j(G^\varphi)] \cdot [\gamma_j(G), \xi_{j+1}(G^\varphi)]$ is central in $\Upsilon(G)$ for all $j \geq 1$;
- (iii) $[\xi_j(G), \gamma_j(G^\varphi)]$ is central in $\mathcal{V}(G)$, for all $j \geq 1$.

Proof. (i) is trivial for $j = 0$ while the general case follows directly from Lemma 2.2, since $G_j \leq \gamma_j(G)$ and $[\xi_j(G), \gamma_j(G)] = 1$ for all $j \geq 1$.

(ii) for $j \geq 1$, $z \in \xi_{j+1}(G)$, $g \in \gamma_j(G)$ and $g_1, g_2 \in G$ we have

$$\begin{aligned} [[z, g^\varphi], [g_1, g_2^\varphi]] &= [z, g^\varphi]^{-1} [z, g^\varphi]^{[g_1, g_2^\varphi]} \\ &= [z, g^\varphi]^{-1} [z, g^\varphi]^{[g_1, g_2]} \quad (\text{Lemma 2.1 (i)}) \\ &= [z, g^\varphi, [g_1, g_2]] \\ &= [z, g, [g_1, g_2]^\varphi] \quad (\text{Lemma 2.1 (ii)}) \\ &= 1 \quad (\text{by Lemma 2.2, since } [\xi_{j+1}(G), \gamma_j(G)] \leq \xi_1(G)). \end{aligned}$$

This implies that $\Upsilon(G)$ centralizes $[\xi_{j+1}(G), \gamma_j(G^\varphi)]$ and by symmetry $\Upsilon(G)$ also centralizes $[\gamma_j(G), \xi_{j+1}(G^\varphi)]$.

(iii) This part follows directly from Lemma 2.1 (ii) since $[\xi_j(G), \gamma_j(G)] = 1$, for all $j \geq 1$. $\square$

3. The Main Results

The description of $\mathcal{V}(G)$ as the product $\mathcal{V}(G) = \Upsilon(G) \cdot G \cdot G^\varphi$, which comes from the fact that $\Upsilon(G) \trianglelefteq \mathcal{V}(G)$, gives an elegant description for the lower central series and the derived series of $\mathcal{V}(G)$.

3.1 Theorem. For $i \geq 2$ the i -th term of the lower central series of $\mathcal{V}(G)$ is given by

$$\gamma_i(\mathcal{V}(G)) = \gamma_i(G) \gamma_i(G^\varphi) [\gamma_{i-1}(G), G^\varphi] [G, \gamma_{i-1}(G^\varphi)]$$

Proof. For $i = 2$, $\gamma_2(\mathcal{V}(G)) = [\mathcal{V}(G), \mathcal{V}(G)] = [\Upsilon(G) \cdot G \cdot G^\varphi, \Upsilon(G) \cdot G \cdot G^\varphi]$. By using the defining relations of $\mathcal{V}(G)$ together with Lemma 2.1 and Proposition 2.5 (i) we get

$$[\Upsilon(G) \cdot G \cdot G^\varphi, \Upsilon(G) \cdot G \cdot G^\varphi] \leq \Upsilon(G) \cdot \gamma_2(G) \cdot \gamma_2(G^\varphi).$$

This shows that $\gamma_2(\mathcal{V}(G)) = \gamma_2(G) \gamma_2(G^\varphi) \cdot \Upsilon(G)$. Suppose, by induction on $i \geq 2$, that

$$\gamma_i(\mathcal{V}(G)) \leq \gamma_i(G) \gamma_i(G^\varphi) [\gamma_{i-1}(G), G^\varphi] \cdot [G, \gamma_{i-1}(G^\varphi)].$$

Then by Proposition 2.5 (i),

$$[\gamma_i(\mathcal{V}(G)), G] \leq \gamma_{i+1}(G) \cdot [\gamma_i(G^\varphi), G] \cdot [\gamma_{i-1}(G), G^\varphi, G] \cdot [G, \gamma_{i-1}(G^\varphi), G],$$

and once more invoking Lemma 2.1 (i) we obtain

$$[\gamma_{i-1}(G), G^\varphi, G] = [\gamma_i(G), G^\varphi] = [G, \gamma_{i-1}(G^\varphi), G].$$

Therefore $[\gamma_i(\mathcal{V}(G)), G] \leq \gamma_{i+1}(G) \cdot [\gamma_i(G), G^\varphi] \cdot [G, \gamma_i(G^\varphi)]$. By symmetry it follows that

$$[\gamma_i(\mathcal{V}(G)), G^\varphi] \leq \gamma_{i+1}(G^\varphi) [\gamma_i(G), G^\varphi] [G, \gamma_i(G^\varphi)],$$

and these last two inclusions show that

$$\gamma_{i+1}(\mathcal{V}(G)) \leq \gamma_{i+1}(G) \cdot \gamma_{i+1}(G^\varphi) [\gamma_i(G), G^\varphi] [G, \gamma_i(G^\varphi)],$$

so that our theorem is proved by induction. $\square$

3.2 Corollary. Let G be a nilpotent group of class c . Then $\mathcal{V}(G)$ is a nilpotent group of class at most $c + 1$.

The next theorem is proved using, step by step, similar arguments as in the proof of Theorem 3.2. We will omit its proof.

3.3 Theorem. For $i \geq 2$ the i -th term of the derived series of $\mathcal{V}(G)$ is given by

$$\mathcal{V}(G)_i = G_i G_i^\varphi [G_{i-1}, G_{i-1}^\varphi],$$

where G_i , denotes the i -th term of the derived series of G .

3.4 Corollary. Let G be a solvable group of derived length ℓ . Then $\mathcal{V}(G)$ is solvable of derived length at most $\ell + 1$.

3.5 Proposition. Let $G = N \cdot H$ be a semidirect product of its subgroups $N \trianglelefteq G$ and $H \leq G$. Then

$$(i) \quad \mathcal{V}(G) = \langle N, N^\varphi \rangle [N, H^\varphi] [H, N^\varphi] \cdot \langle H, H^\varphi \rangle;$$

$$(ii) \quad \langle H, H^\varphi \rangle \cong \mathcal{V}(H).$$

Proof. (i), (ii). It follows easily from Proposition 2.5 that $[N, H^\varphi]$ and $[H, N^\varphi]$ are both normal subgroups of $\mathcal{V}(G)$; also, $\langle N, N^\varphi \rangle [N, H^\varphi] [H, N^\varphi]$ is actually the Kernel of $\tilde{\pi}: \mathcal{V}(G) \rightarrow \mathcal{V}\left(\frac{G}{N}\right) (\cong \mathcal{V}(H))$. On writing $\mathcal{V}(G) = \mathcal{V}(NH) = [NH, N^\varphi H^\varphi] \cdot NH \cdot N^\varphi H^\varphi$ we see that

$$[NH, N^\varphi H^\varphi] \leq [N, N^\varphi] [N, H^\varphi] [H, N^\varphi]$$

and thus $\mathcal{V}(G)$ has the desired expression. As for (ii), $\langle H, H^\varphi \rangle^{\tilde{\pi}} = \mathcal{V}\left(\frac{G}{N}\right) (\cong \mathcal{V}(H))$, while on the other hand $\mathcal{V}(H)$ is mapped onto $\langle H, H^\varphi \rangle$. Therefore $\text{Ker}(\tilde{\pi}) \cap \langle H, H^\varphi \rangle = \{1\}$ and $\langle H, H^\varphi \rangle \cong \mathcal{V}(H)$. $\square$

3.6 Proposition. Let $G = N \times H$ be the direct product of its normal subgroups N and H . Then

- (i) $\mathcal{V}(G) = \langle N, N^\varphi \rangle \cdot [N, H^\varphi] \cdot [H, N^\varphi] \cdot \langle H, H^\varphi \rangle$
- (ii) $\langle N, N^\varphi \rangle \cong \mathcal{V}(N)$; $\langle H, H^\varphi \rangle \cong \mathcal{V}(H)$
- (iii) $\Upsilon(G) = \Upsilon(N) \times \Upsilon(H)$.

Proof. Parts (i) and (ii) follows from double application of Proposition 3.5. As for (iii), we get from Proposition 2.7 (i) that the four subgroups $[N, H^\varphi]$, $[N, N^\varphi]$, $[H, N^\varphi]$ and $[H, H^\varphi]$ are mutually centralized in $\Upsilon(G)$, since $[N, H] = 1$. Also, normality of $[N, H^\varphi]$ and $[H, N^\varphi]$ in $\mathcal{V}(G)$ give

$$\Upsilon(G) = [N, N^\varphi] \cdot [N, H^\varphi][H, N^\varphi][H, H^\varphi].$$

Lastly we observe that part (ii) implies $[N, N^\varphi] \cong \Upsilon(N)$ and $[H, H^\varphi] \cong \Upsilon(H)$. $\square$

Remark 4. The result in Part (iii) is Proposition 11 of [1].

In fact, by arguments similar to those used in Proposition 2.6 we can prove that when H and K are groups which act trivially on each other (but by conjugation on themselves) then the subgroup $[H, K^\varphi]$ of $\mathcal{V}(H \times K)$ is isomorphic to $H \otimes K$ which in turn is the usual tensor product $H \otimes_{\mathbb{Z}} K$ (this follows from Lemma 2.1; see also Remark 2 of [1]).

Remark 5. In case of abelian groups A and B we have therefore the known decomposition of the ordinary tensor product: $(A \times B) \otimes_{\mathbb{Z}} (A \times B) \cong \Upsilon(A \times B) \cong (A \otimes_{\mathbb{Z}} A) \times (A \otimes_{\mathbb{Z}} B) \times (B \otimes_{\mathbb{Z}} A) \times (B \otimes_{\mathbb{Z}} B)$.

3.7 Corollary. Let $G = P_1 \times \cdots \times P_n$ be a finite nilpotent group where $\{P_1, \dots, P_n\}$ is the set of distinct Sylow p -subgroups of G . Then,

- (i) $\mathcal{V}(G) \cong \mathcal{V}(P_1) \times \cdots \times \mathcal{V}(P_n)$
- (ii) $\Upsilon(G) \cong \Upsilon(P_1) \times \cdots \times \Upsilon(P_n)$

Proof. For any prime p dividing $|G|$, let P be a Sylow p -subgroup of G and N be a normal p -complement in G . We have by Lemma 2.3 (ii) that $[N, P^\varphi] = [P, N^\varphi] = 1$.

The previous proposition then yields $\mathcal{V}(G) \cong \mathcal{V}(N) \times \mathcal{V}(P)$ and $\Upsilon(G) \cong \Upsilon(N) \times \Upsilon(P)$. Parts (i), (ii) now follow by induction on $n \geq 2$. $\square$

From now on we restrict ourselves to the case of a finite p -group G .

3.8 Lemma. Let G be a finite p -group and $c \in Z(G) \cap G'$ be an element of order p . If $\phi(G)$ denotes the Frattini subgroup of G , then

$$|\mathcal{V}(G)| \text{ divides } p^2 \left| \frac{G}{\phi(G)} \right| \left| \mathcal{V}\left(\frac{G}{\langle c \rangle}\right) \right|$$

Proof. By Proposition 2.7 (i) we get $[c, g^\varphi] = 1$ for all $g \in G'$. On the other hand, if $x \in G$ then, by Lemma 2.3 (i), $[c, (x^p)^\varphi] = [c, x^\varphi]^p = [c^p, x^\varphi] = 1$, so that $[c, g^\varphi] = 1$ for all $g \in G^p = \langle x^p | x \in G \rangle$. It follows that $[c, \phi(G)^\varphi] = 1$ since $\phi(G) = G'G^p$. If we set $\lambda: G \rightarrow [c, G^\varphi]$, $g \mapsto [c, g^\varphi]$ then λ is an epimorphism, as $[c, G^\varphi]$ is central in $\mathcal{V}(G)$. Also, $\phi(G) \leq \text{Ker}(\lambda)$. Let $\pi: G \rightarrow \frac{G}{\langle c \rangle}$ be the canonical projection and $\tilde{\pi}$ its induced in $\mathcal{V}(G)$, whose kernel is $\text{Ker}(\tilde{\pi}) = \langle c \rangle \langle c^\varphi \rangle [c, G^\varphi][G, c^\varphi]$. Let $\bar{a}$ be a generator of $\frac{G}{\phi(G)}$. If c is a simple commutator, say $c = [x, y]$, then we get

$$\begin{aligned} [a, c^\varphi] &= [a, [x, y]^\varphi] \\ &= [[x, y]^\varphi, a]^{-1} \\ &= [x, y^\varphi, a]^{-1} \quad (\text{by Lemma 2.1 (ii)}) \\ &= [a, [x, y^\varphi]]. \end{aligned}$$

In general, if c is a product of commutators, say $c = [x_1, y_1][x_2, y_2] \cdots [x_r, y_r]$, then by induction we get $[a, c^\varphi] = [a, [x_1, y_1]^\varphi] \cdots [a, [x_r, y_r]^\varphi]$.

Analogously, $[c, a^\varphi] = [[x_1, y_1]^\varphi \cdots [x_r, y_r]^\varphi, a]$. Since $c \in Z(G) \cap G'$, it follows from the above identities that, in $[c, G^\varphi][G, c^\varphi]$, the elements of the form $[c, a^\varphi][a, c^\varphi]$ are all trivial.

On the other hand if $\frac{G}{\phi(G)}$ is generated by $\{\bar{a}_1, \dots, \bar{a}_d\}$, we have

$$[c, (a_1^{i_1} \cdots a_d^{i_d})^\varphi] = [c, a_1^\varphi]^{i_1} \cdots [c, a_d^\varphi]^{i_d}$$

and

$$[(a_1^{j_1} \dots a_d^{j_d}), c^\varphi] = [a_1, c^\varphi]^{j_1} \dots [a_d, c^\varphi]^{j_d},$$

by Lemma 2.3 (i). This means that $[c, G^\varphi][G, c^\varphi]$ is generated by the $2d$ elements

$$[a_1, c^\varphi], \dots, [a_d, c^\varphi], [c, a_1^\varphi], \dots, [c, a_d^\varphi].$$

But since $[a_i, c^\varphi][c, a_i^\varphi] = 1$ for $i = 1, \dots, d$, it results that $[c, G^\varphi][G, c^\varphi] = [c, G^\varphi]$, which is generated by $[c, a_1^\varphi], \dots, [c, a_d^\varphi]$. This together with the fact that λ is an epimorphism gives

$$|\text{Ker}(\tilde{\pi})| \leq p^2 \cdot |[c, G^\varphi]| \leq p^2 \cdot \left| \frac{G}{\phi(G)} \right|.$$

Therefore $|\mathcal{V}(G)|$ divides $p^2 \left| \frac{G}{\phi(G)} \right| \cdot \left| \mathcal{V} \left(\frac{G}{\langle c \rangle} \right) \right|$. $\square$

3.9 Proposition. *Let G be a finite p -group of class 2. Then $|\Upsilon(G)|$ divides*

$$\left| G' \otimes_{\mathbb{Z}} \frac{G}{G'} \right| \cdot \left| \Upsilon \left(\frac{G}{G'} \right) \right|.$$

Proof. Let $\tilde{\pi}: \mathcal{V}(G) \rightarrow \mathcal{V} \left(\frac{G}{G'} \right)$ be the epimorphism induced by the canonical map $\pi: G \rightarrow \frac{G}{G'}$. By Remark 3 we have $\text{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [G', G^\varphi][G, (G')^\varphi]$, while $\Upsilon(G)^{\tilde{\pi}} = \Upsilon \left(\frac{G}{G'} \right)$. Thus it remains to evaluate $|\text{Ker}(\tilde{\pi}) \cap \Upsilon(G)|$. Since $G' \leq Z(G)$, Proposition 2.7 (i) gives $[G', G'^\varphi] = 1$. Hence, for $c \in G'$ and $g = dh \in G$, where $h \in G'$, $[c, (dh)^\varphi] = [c, h^\varphi][c, d^\varphi]^{h^\varphi} = [c, d^\varphi]$.

As $[G', G^\varphi]$ is central in $\mathcal{V}(G)$ (Proposition 2.7 (iii)), this implies that $[G', G^\varphi]$ is a homomorphic image of $G' \otimes_{\mathbb{Z}} \frac{G}{G'}$ through the map $c \otimes \bar{d} \mapsto [c, d^\varphi]$, where $c \in G'$ and $\bar{d} = d^\pi$.

Therefore $|[G', G^\varphi]|$ divides $\left| G' \otimes_{\mathbb{Z}} \frac{G}{G'} \right|$. Suppose $G' = \langle c_1, \dots, c_m \rangle$ and $\frac{G}{G'} = \langle \bar{d}_1, \dots, \bar{d}_n \rangle$. Then $[G', G^\varphi]$ is generated by the set $\{[c_i, d_j^\varphi], 1 \leq i \leq m, 1 \leq j \leq n\}$ and similarly $[G, (G')^\varphi]$ is generated by $\{[d_j, c_i^\varphi], 1 \leq j \leq n, 1 \leq i \leq m\}$. But each c_i is a product of commutators so that we get, as in the proof of Lemma 3.8, $[c_i, d_j^\varphi][d_j, c_i^\varphi] = 1$, for all pairs (i, j) . This in turn gives $[G', G^\varphi][G, (G')^\varphi] = [G', G^\varphi]$, and consequently $|\text{Ker}(\tilde{\pi}) \cap \Upsilon(G)|$ divides $\left| G' \otimes \frac{G}{G'} \right|$. $\square$

3.10 Corollary. *Let G be a p -group of class ≤ 2 with $|G| = p^n$ and $|G'| = p^m$. Then $|\Upsilon(G)|$ divides $p^{n(n-m)}$.*

Proof. We observe that $\left| G' \otimes_{\mathbb{Z}} \frac{G}{G'} \right|$ divides $p^{m(n-m)}$ and

$$\left| \Upsilon \left(\frac{G}{G'} \right) \right| = \left| \frac{G}{G'} \otimes_{\mathbb{Z}} \frac{G}{G'} \right|$$

divides $p^{(n-m)^2}$. $\square$

3.11 Theorem. *Let G be a finite p -group with $|G| = p^n$ and $|G'| = p^m$. Then $|\mathcal{V}(G)|$ divides $p^{n^2+2n-mn}$.*

Proof. Since $|\mathcal{V}(G)| = |\Upsilon(G)| \cdot |G|^2$, all we need is to evaluate $|\Upsilon(G)|$. If G has nilpotence class ≤ 2 then we are done with Corollary 3.8.

Suppose G has class at least 3 and let $c \in \gamma_3(G) \cap Z(G)$ be an element of order p . We argue by induction on $|G|$. Since

$$\left| \frac{G}{\langle c \rangle} \right| = p^{n-1} \quad \text{and} \quad \left| \left(\frac{G}{\langle c \rangle} \right)' \right| = p^{m-1},$$

our hypothesis give that $\left| \Upsilon \left(\frac{G}{\langle c \rangle} \right) \right|$ divides $p^{(n-1)(n-m)}$.

On the other hand $\left| \frac{G}{\phi(G)} \right|$ divides $\left| \frac{G}{G'} \right| = p^{n-m}$, so that by Lemma 3.8 we finally obtain $|\Upsilon(G)|$ divides $p^{n(n-m)}$. $\square$

3.12 Corollary. *Let $|G| = p^n$, $|G'| = p^m$ and $d = d(G)$ be the minimal number of generators of G . Then*

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-m)}$$

Proof. We observe that on making $N = \phi(G)$ and

$$\tilde{\pi}: \mathcal{V}(G) \rightarrow \mathcal{V} \left(\frac{G}{\phi(G)} \right)$$

in Proposition 2.5, then by Remark 3 it results that

$$\text{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [\phi(G), G^\varphi][G, \phi(G)^\varphi],$$

so that the restriction of $\tilde{\pi}$ to $\Upsilon(G)$ renders

$$|\Upsilon(G)| \geq \left| \Upsilon \left(\frac{G}{\phi(G)} \right) \right| = \left| \left[\frac{G}{\phi(G)}, \left(\frac{G}{\phi(G)} \right)^\varphi \right] \right|.$$

But $\frac{G}{\phi(G)}$ is elementary abelian of order p^d and (as observed in Remarks 4 and 5.)

$$\left[\frac{G}{\phi(G)}, \left(\frac{G}{\phi(G)} \right)^\varphi \right]$$

is precisely the usual tensor product

$$\frac{G}{\phi(G)} \otimes_{\mathbb{Z}} \frac{G}{\phi(G)},$$

of order p^{d^2} .

On the other hand the last theorem gives the upper bound

$$|\Upsilon(G)| = \frac{|\mathcal{V}(G)|}{|G|^2} \leq p^{n^2 - mn}.$$

Our proof is now finished by the isomorphism of Proposition 2.6. $\square$

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