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On the existence of periodic solutions for the equation $\ddot{x} + f(x)\,\dot{x} + g(x) = 0$

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Abstract. We establish in this work sufficient conditions for the existence of periodic solutions for the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$.

1. The Definite Positive Function V_α. Auxiliary Lemmas.

Throughout this work we assume $f,\ g : \mathbb{R} \to \mathbb{R}$ are functions satisfying the following conditions:

a) f is continuous and g is of class C^1;

b) $xg(x) > 0$ for $x \neq 0$;

c) $\int_0^{+\infty} g(x)dx = +\infty = \int_0^{-\infty} g(x)dx$.

Let α be a given real. We indicate by Ω_α the following open set:

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 \mid y > -\frac{1}{\alpha}\} \qquad \text{for } \alpha > 0;$$

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 \mid y < -\frac{1}{\alpha}\} \qquad \text{for } \alpha < 0;$$

$$\Omega_\alpha = \mathbb{R}^2 \qquad \text{for } \alpha = 0.$$

We indicate by V_α the definite positive function given by

$$V_\alpha(x,y) = \int_0^x g(u)du + \int_0^y \frac{s}{\alpha s + 1}ds, \quad (x,y) \in \Omega_\alpha.$$

It can be immediately verified that, for $\alpha \neq 0$,

$$\int_0^{+\infty} \frac{s}{\alpha s + 1}ds = +\infty = \int_0^{-\frac{1}{\alpha}} \frac{s}{\alpha s + 1}ds.$$

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It can also be immediately verified that the level curves of V_α are all closed curves and that $V_\alpha(x,0)$ is strictly increasing in $[0,+\infty[$. Such curves show the aspect of Figure 1: (see [1])

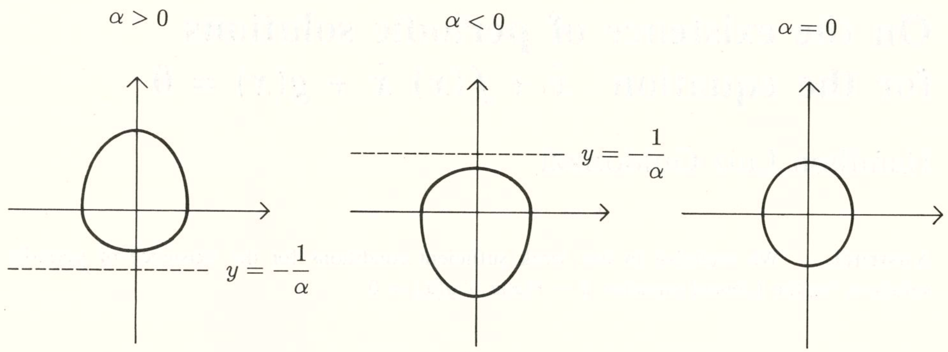

Figure 1

The equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

is equivalent to the system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - g(x) \end{aligned} \tag{2}$$

The condition a) ensures existence and uniqueness of solution of (2). The condition b) ensures that $(0,0)$ is the only point of equilibrium for system (2). It can be immediately verified that the derivative of V_α relative to system (2) is:

$$\dot{V}_\alpha(x,y) = -\frac{[f(x) - \alpha g(x)]}{\alpha y + 1} y^2, \quad (x,y) \in \Omega_\alpha. \tag{3}$$

Because $\alpha y + 1 > 0$ holds for all $(x,y) \in \Omega_\alpha$, it follows that the sign of $\dot{V}_\alpha$ depends only of $f(x) - \alpha g(x)$.

Lemma 1. *Assume there are* $\alpha > 0$ *and* $b > 0$ *such that for all* $x \geq b$, $f(x) \geq \alpha g(x)$.

Let $y_0 > 0$, $L = V_\alpha(b, y_0)$ *and*

$$K = \{(x,y) \in \Omega_\alpha \mid x \geq b \ \text{ and } \ V_\alpha(x,y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ *be the solution of (2) so that* $\gamma(t_0) = (b, y_1)$, *with* $0 < y_1 < y_0$. *Then, there is* $t_1 > t_0$ *such that*

$$\gamma(t) \in K, \ t_0 \leq t \leq t_1$$

and

$$\gamma(t_1) = (b, y_2),$$

with $-\frac{1}{\alpha} < y_2 < 0$.

Proof. From $\dot{x}(t_0) = y_1 > 0$, it follows there is $t_2 > t_0$ so that

$$\gamma(t) \in K, \ t_0 \leq t \leq t_2.$$

On the other hand, being $\dot{x}(t) > 0$ on the half plane $y > 0$, $\dot{x}(t) < 0$ on the half plane $y < 0$, $\dot{y}(t) < 0$ on the positive half-axis $0x$ and $(0,0)$ the only point of equilibrium, there must exist $t_3 > t_2$ such that $\gamma(t_3) \notin K$.

Let

$$t_1 = \max\{u > t_0 \mid \gamma(t) \in K, \ t_0 \leq t \leq u\}.$$

From the hypothesis

$$f(x) \geq \alpha g(x), \quad x \geq b,$$

and from (3) it follows that

$$\dot{V}_\alpha(\gamma(t)) \leq 0, \quad t_0 \leq t \leq t_1.$$

Since

$$V_\alpha(\gamma(t_0)) = V_\alpha(b, y_1) < L,$$

it follows that $V(\gamma(t_1)) < L$. So, $\gamma(t_1)$ does not belong to the arc given by

$$x \geq b \ \text{ and } \ V_\alpha(x,y) = L.$$

Because $\dot{x}(t) > 0$ on the $y > 0$ half-plane, it follows that

$$\gamma(t_1) = (b, y_2), \quad \text{with} \quad -\frac{1}{\alpha} < y_2 < 0.$$

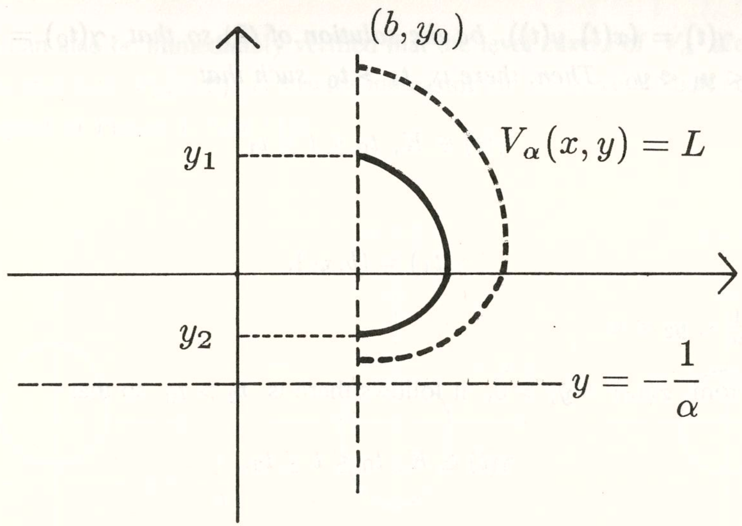

Figure 2

In a similar way, we can demonstrate the following lemmas:

Lemma 2. *Assume there exist* $\alpha < 0$ *and* $a < 0$ *such that, for all* $x \leq a$, $f(x) \geq \alpha g(x)$.

Let $y_0 < 0$, $L = V_\alpha(a, y_0)$ *and*

$$K = \{(x, y) \in \Omega_\alpha \mid x \leq a \quad \text{and} \quad V_\alpha(x, y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ *the solution of (2) such that* $\gamma(t_0) = (a, y_1)$, *with* $y_0 < y_1 < 0$. *Then there is* $t_1 > t_0$ *so that*

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1$$

and

$$\gamma(t_1) = (a, y_2)$$

with $0 < y_2 < -\frac{1}{\alpha}$.

Lemma 3. *Assume there exists* $a < 0$ *such that for all* $x \leq a$, $f(x) \geq 0$.

Let $y_0 < 0$, $L = V_0(a, y_0)$ *and*

$$K = \{(x, y) \in \mathbb{R}^2 \mid x \leq a \quad \text{and} \quad V_0(x, y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ *the solution of (2) such that* $\gamma(t_0) = (a, y_1)$, *with* $y_0 < y_1 < 0$. *Then, there is* $t_1 > t_0$ *such that* $\gamma(t) \in K$, $t_0 \leq t \leq t_1$

and

$$\gamma(t_1) = (a, y_2)$$

with

$$0 < y_2 < |y_0|.$$

Lemma 4. *Assume there is* $b > 0$ *such that*

$$f(x) \geq 0, \quad x \geq b.$$

Let $y_0 > 0$, $L = V_0(b, y_0)$ *and*

$$K = \{(x, y) \in \mathbb{R}^2 \mid x \geq b \quad \text{and} \quad V_0(x, y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ *be the solution of (2) such that* $\gamma(t_0) = (b, y_1)$, *with* $0 < y_1 < y_0$. *Then there is* $t_1 > t_0$ *such that*

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1$$

and

$$\gamma(t_1) = (b, y_2)$$

with $-y_0 < y_2 < 0$.

To close this section, we prove that the solutions of (2) do not admit vertical asymptotes. It is enough, to this end, to show that all solutions of the equation

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y}, \quad y \neq 0 \tag{4}$$

do not admit vertical asymptotes.

Let us assume that (4) has a solution

$$y = y(x), \quad a \leq x < b$$

such that

$$\lim_{x \to b^-} y(x) = +\infty. \tag{5}$$

We can assume with no loss of generality, that $0 < y(a) \leq y(x)$ for $a \leq x < b$. Let

$$A \geq \max_{a \leq x \leq b} |f(x)| \quad \text{and} \quad B \geq \max_{a \leq x \leq b} |g(x)|.$$

It follows from the mean value theorem that, for $a < x < b$,

$$y(x) - y(a) \leq \left[A + \frac{B}{y(a)}\right](b-a)$$

which is in clear contradiction with (5). The other situations can be analyzed in a similar way.

2. Sufficient conditions for the existence of periodic solutions

Theorem 1. *Consider the equation*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

where f, g *satisfy the conditions a), b) and c) of the previous section. Assume also, that the following hypotheses are satisfied:*

1) *There are* $\alpha > 0$ *and* $b > 0$ *such that for all* $x \geq b$, $f(x) \geq \alpha g(x)$;
2) *The origin is repulsive;*
3) *There is* $a < 0$ *such that for all* $x \in [c, a]$, $f(x) \geq 0$ *where* $V_0(c, 0) = V_0(a, r)$, $r = \frac{1}{\alpha} + (A + \alpha B)(b - a)$,

$$A \geq \max_{a \leq x \leq b} |f(x)| \quad \text{and} \quad B \geq \max_{a \leq x \leq b} |g(x)|.$$

Under these conditions, the equation (1) will admit at least one non trivial periodic solution.

Proof. The equation (1) is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{2}$$

Let $\gamma(t) = (x(t), y(t))$ the solution of (2) that at time $t = 0$ is at the position $\gamma(0) = (b, -\frac{1}{\alpha})$. Because γ does not admit vertical asymptotes and the origin is repulsive, there is a smallest time $t_1 > 0$ such that

$$\gamma(t_1) = (a, y_1), \quad y_1 < 0,$$

or

$$\gamma(t_1) = (x_1, 0), \quad a < x_1 < 0.$$

It can be immediately shown that

$$-\frac{1}{\alpha} - [A + \alpha B](b - a) < y_1 < 0.$$

(Indeed: Assuming $y_1 < -\frac{1}{\alpha}$, let $y = y(x)$ the solution of

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y}$$

such that $y(a) = y_1$ and $y(b) = -\frac{1}{\alpha}$. There is $x_0 \in]a, b]$ such that $y(x_0) = -\frac{1}{\alpha}$ and $y(x) < -\frac{1}{\alpha}$, $a \leq x \leq x_0$; by the mean value theorem, $y(x_0) - y(a) < [A + \alpha b](x_0 - a)$.)

Let $t_2 > 0$ the smallest value of t when γ crosses the y negative half-axis: $\gamma(t_2) = (0, y_2)$, $y_2 < 0$. The hypotheses 1), 2) and 3) together with lemmas 1 and 3 ensure that $\gamma(t)$ will again cross the y negative half-axis at a point $(0, y_3)$ with $y_2 < y_3 < 0$.

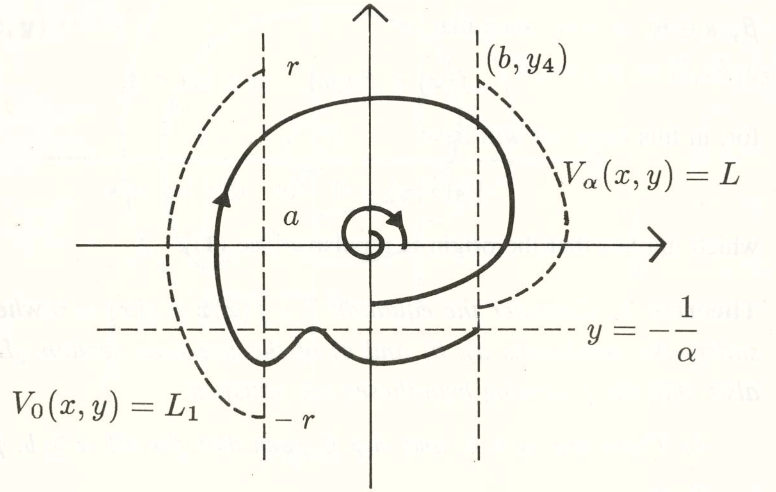

Figure 3

$(L_1 = V_0(c, 0) = V_0(a, r)$ and $L = V_\alpha(b, y_4))$. From Theorem of Poincaré-Bendixson, the equation will admit at least one periodic solution.

Remark 1. One possible value for y_4 is

$$y_4 = r + [A + \frac{B}{r}](b - a).$$

Let $m > b$ such that $V_\alpha(m, 0) = V_\alpha(b, y_4)$. The hypothesis 1) can be weakened: it is enough to assume

$$f(x) \geq \alpha g(x), \quad b \leq x \leq m.$$

Remark 2. The hypotheses 1) and 3) can be replaced by:

1') There are $\alpha < 0$ and $a < 0$ such that for all $x \leq a$,

$$f(x) \geq \alpha g(x);$$

3') There are $b > 0$ such that, for all $x \in [b, c]$,

$$f(x) \geq 0$$

where $V_0(c, 0) = V_0(b, r)$, $r = -\frac{1}{\alpha} + [A - \alpha B](b - a)$,

$A \geq \max_{a\leq x\leq b} |f(x)|$ and $B \geq \max_{\leq x\leq b} |g(x)|$.

Remark 3. A sufficient condition for the origin to be repulsive is that there exist β, $s \in \mathbb{R}$, $s > 0$, such that

$$f(x) < \beta g(x), \quad 0 < |x| < s,$$

for, in this case, we will have

$$\dot{V}_\beta(x, y) < 0 \quad \text{for} \quad 0 < |x| < s$$

which implies that the origin is repulsive [see (1)].

Theorem 2. *Consider the equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ where f and g satisfy the conditions a), b) and c) of the previous section. Let us assume, also, that the following hypotheses are satisfied:*

1) There are $\alpha > 0$ and $b > 0$ such that, for all $x \geq b$, $f(x) \geq \alpha g(x)$;

2) The origin is repulsive;

3) There is $a < 0$ such that, for all $x \leq a$, $f(x) \geq \beta g(x)$ where $\frac{1}{\beta} \geq \frac{1}{\alpha} + (A + \alpha B)(b - a)$, $A \geq \max_{a\leq x\leq b} |f(x)|$ and $B \geq \max_{a\leq x\leq b} |g(x)|$.

Under these conditions, the equation will admit at least one non trivial periodic solution.

Proof. Let $\gamma(t) = (x(t), y(t))$ be the solution of (2) that at time $t = 0$ is at the position $\gamma(0) = (b, -\frac{1}{\alpha})$.

By the same reasoning as in Theorem 1, there will be a smallest value $t_1 > 0$ such that

$$\gamma(t_1) = (x_1, 0), \quad a < x_1 < 0$$

or

$$\gamma(t_1) = (a, y_1)$$

where $-\frac{1}{\beta} < y_1 < 0$. Let $-\frac{1}{\beta} < y_4 < y_1$. Suppose $\gamma(t_1) = (a, y_1)$. The hypothesis 3) ensures that $\gamma(t)$ cannot leave the compact set

$$K = \{(x, y) \in \Omega_\beta \mid x \leq a, \ V_\beta(x, y) \leq V_\beta(a, y_4)\}$$

by crossing the arc

$$x \leq a \quad \text{and} \quad V_\beta(x, y) = V_\beta(a, y_4) = L_2$$

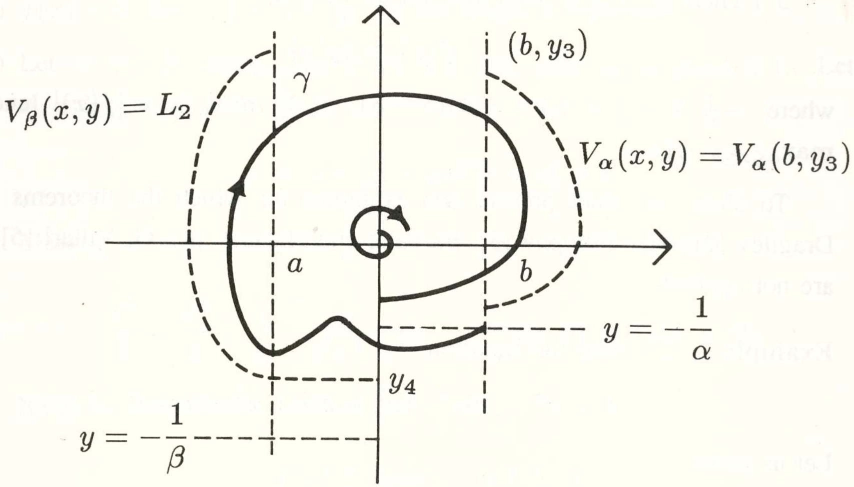

Figure 4

The proof is completed following the same reasoning as in Theorem 1.□

Remark 4. The hypothesis 3) of Theorem 2 can be replaced by:

3') There is $a < 0$ such that, for all $x \in [c, a]$,

$$f(x) \geq \beta g(x)$$

where $\frac{1}{\beta} > r \geq \frac{1}{\alpha} + (A + \alpha\beta)(b - a)$, $c < a$ is such that $V_\beta(c, 0) = V_\beta(a, -r)$, $A \geq \max_{a\leq x<b} |f(x)|$ and $B \geq \max_{a\leq x\leq b} |g(x)|$.

When the hypothesis 3') is satisfied, we can make y_3 equal to

$$y_3 = y_5 + (A + \frac{B}{y_5})(b - a)$$

where $y_5 > 0$ is such that $V_\beta(a, -r) = V_\beta(a, y_5)$.

In this case, it is enough to assume in hypothesis 1) that

$$f(x) \geq \alpha g(x), \quad b \leq x \leq m$$

where $m > b$ is such that $V_\alpha(b, y_3) = V_\alpha(m, 0)$.

Remark 5. The hypotheses 1) and 3) of Theorem 2 can be replaced by:

1") There are $\alpha < 0$ and $a < 0$ such that, for all $x \leq a$,

$$f(x) \geq \alpha g(x);$$

3") There is $b > 0$ such that, for all $x \geq b$,

$$f(x) \geq \beta g(x)$$

where $-\frac{1}{\beta} \geq -\frac{1}{\alpha} + (A - \alpha B)(b - a)$, $A \geq \max_{a \leq x \leq b} |f(x)|$ and $B \geq \max_{a \leq x \leq b} |g(x)|$.

To close, we shall present two examples for which the theorems of A.V. Dragilëv [2], A.F. Filippov [3], Barbalat and Halanay [4], G. Villari [5] and [6] are not applied.

Example 1. Consider the equation

$$\ddot{x} + [x^5 + 16x^4 - x^2 + x]\dot{x} + x^5 + x = 0.$$

Let us make

$$f(x) = x^5 + 16x^4 - x^2 + x \quad \text{and} \quad g(x) = x^5 + x.$$

We have

1) $f(x) \geq \alpha g(x)$ for $x \geq b$, where $\alpha = 1$ and $b = \frac{1}{4}$;

2) $f(x) < g(x)$ for $0 < |x| < \frac{1}{4}$; from remark 3 the origin is repulsive;

3) Let $a = -\frac{1}{2}$; $\max_{a \leq x \leq b} |f(x)| \leq 1$ and $\max_{a \leq x \leq b} |g(x)| \leq 1$. Let $A = 1$ and $B = 1$. So,

$$r = \alpha + (A + \alpha B)(b - a) = \frac{5}{2}.$$

From

$$V_0(x, y) = \frac{x^6}{6} + \frac{x^2}{2} + \frac{y^2}{2},$$

it follows that

$$\frac{c^6}{6} + \frac{c^2}{2} = \frac{a^6}{6} + \frac{a^2}{2} + \frac{r^2}{2} \qquad (V_0(c, 0) = V_0(a, r),\ c < 0)\ .$$

It can be easily verified that $f(x) > 0$ for all $x \in [c, -\frac{1}{2}]$. From theorem 1, the equation admits at least one non trivial periodic solution.

Example 2. Consider the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

where $f(x) = (2x - 1)e^{x^2 + 99x - 100}$ and $g(x) = x$. We have

1) $f(x) \geq \alpha g(x)$ for $x \geq b$, where $\alpha = 1$ and $b = 1$;

2) $f(x) < 0$ for $-\frac{1}{2} < x < \frac{1}{2}$; so, the origin is repulsive;

3) Let $a = -\frac{1}{2}$; $\max_{a \leq x \leq b} |f(x)| \leq 1$ and $\max_{a \leq x \leq b} |g(x) \leq 1$. Let $A = 1$ and $B = 1$. So,

$$r = \alpha + [A + \alpha B](b - a) = 4.$$

Making $\beta = \frac{1}{5}$, we have $\frac{1}{\beta} > r$. Let $c < 0$ be such that

$$\frac{c^2}{2} = \frac{a^2}{2} + \int_0^{-r} \frac{s}{\beta s + 1} ds \qquad (V_\beta(c, 0) = V_\beta(a, -r)).$$

It can be immediately verified that

$$f(x) \geq \beta g(x), \quad x \in [c, a].$$

From theorem 2 and remark 4, the equation admits at least one non trivial periodic solution.

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