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# A method of desingularization for analytic two-dimensional vector field families

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**Abstract.** It is well known that isolated singularities of two dimensional analytic vector fields can be desingularized: after a finite number of blowing up operations we obtain a vector field that exhibits only elementary singularities. In the present paper we introduce a similar method to simplify the periodic limit sets of analytic families of vector fields. Although the method is applied here only to reduce to families in which the zero set has codimension at least two, we conjecture that it can be used in general. This is related to the famouss Hilbert's problem about planar vector fields.

## Introduction

Let $X = \{X_\lambda\}$ be a real analytic family of vector fields on a two-dimensional surface $S^2$, with parameter $\lambda \in \Lambda$, where $\Lambda$ is some compact real-analytic manifold. To avoid complicated recurrence phenomena we suppose that $S^2$ is either a two-dimensional sphere or any compact two-dimensional analytic submanifold (with boundary) of the sphere.

The set of singular points of $X$, denoted by $Z(X)$, is an analytic subset of $S^2 \times \Lambda$. The aim of this paper is to describe a method of desingularization of $X$. This desingularization is done to simplify the family and obtain in this way a better knowledge of it, for instance concerning its limit cycles, i.e. isolated periodic orbits.

In order to motivate the interest of such desingularization we recall that Hilbert's $16^{th}$ problem can be formulated as follows: given any $n \geq 2$, prove the existence of an upper bound $H(n)$ on the number of limit cycles of any polynomial vector field on $\mathbb{R}^2$ of degree less than $n$.

The family of all polynomial vector fields of degree smaller than $n$ may be

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replaced by an analytic family $X$ as above after extending the given vector field to the 2-sphere $S^2$ and restricting the parameters to the unit sphere in the coefficient space.

It was suggested in [**R1**] to replace Hilbert's $16^{th}$ problem by a more general conjecture about the family $X$, called

**Finiteness Conjecture.** *For any analytic family $X$ as above, there exists a bound $C$ on the number of limit cycles of each vector field $X_\lambda$. We say that the family $X$ has the finiteness property.*

It was explained in [**R1**] how this conjecture is implied by a local one, concerning the existence of such a bound for analytic unfoldings. To state the local conjecture, we need the following definition of a *limit periodic set* ([**FP**]):

**Definition.** Let $X$ be an analytic family of vector fields on $S^2$, with $\lambda \in \Lambda, \Lambda$ not necessarily compact. A *limit periodic set* of $X$ is a compact set $\Gamma \subset S^2$, invariant by $X_{\lambda_o}$ for some $\lambda_o \in \Lambda$, and such that there exist a sequence $\{\lambda_n\}_n$ converging to $\lambda_o$ and for each $\lambda_n$ a limit cycle $\gamma_n$ of $X_{\lambda_n}$, with the property that $\{\gamma_n\}$ converges to $\Gamma$ in the Hausdorff metric on $S^2$.

Then, the existence of a local bound is made precise, using the notion of *cyclicity*, introduced in [**R1**]:

**Definition.** We will say that a limit periodic set $\Gamma$ of the family $X$ has finite cyclicity, if there is a neighbourhood $V$ of $\lambda_o$, and there exists $\varepsilon > 0, N \in \mathbb{N}$ such that for each $\lambda \in V$ the number of limit cycles of the field $X_\lambda$, whose Hausdorff distance to $\Gamma$ is less than $\varepsilon$, is not bigger than $N$. The *cyclicity* of $\Gamma$ in $X$ is the infimum of such $N$ as the diameter of $V$ and $\varepsilon$ tend to 0.

It is shown in [**R1**] that the Finiteness Conjecture is implied by the following:

**Finite Cyclicity Conjecture.** ([**R1**].) *Every limit periodic set of any analytic family $X$ has finite cyclicity.*

So finally Hilbert's $16^{th}$ problem will be a consequence of the finite cyclicity of any analytic unfolding of any limit periodic set.

Our desingularization method is aimed to further the simplification of limit periodic sets of analytic families. In particular, at each step of this desingularization we will keep track of all the limit cycles. This seems to distinguish the ideas



presented here from a similar work recently published by Trifonov ([**Tr**]).

A step of desingularization will consist in making one of the following three *operations*, locally performed around limit periodic sets:

*Induction* of a family by another one (for instance by versal unfolding when it exists),

*Local division* by factor functions. This operation needs the generalization of vector fields into what is called *local vector fields* (see Chapter I),

*Generalized blowing up*. This operation consists in making a *weighted* blowing up along analytic submanifolds. It is the most important of the three operations.

Now, the source of difficulties is the following: when applied to a family $X$, the generalized blowing up operation, as conceived here, destroys the family structure, because it is done globally in $S^2 \times \Lambda$. For instance, see an example in [**R3**], where this operation was first used. The blown up family is no longer a family but a new object, consisting of a vector field tangent to a two-dimensional singular foliation and called a *foliated local vector field.*

This object is studied in detail in Chapter I.

It is clear that, because we want to use it in a recurrent way, we have to define the operation of generalized blowing up directly for foliated local vector fields. This is done in Chapter II.

In the present paper we limit ourselves to the presentation of the method and we do not give a proof of any desingularization theorem. In Chapter III we only indicate what such a theorem could be. By analogy to the desingularization of a single analytic vector field (see [**D**], [**S**], [**V**]) we introduce *elementary limit periodic sets* (all their points are *elementary* singular points, i.e. at least one of their eigenvalues is non-zero) in the context of foliated local vector fields. We conjecture that for any analytic family $X$ there is a finite number of steps, using one of our three operations, which produce foliated local vector fields with all their limit periodic sets elementary.

In Chapter IV we make the first step toward the proof of such a result, showing that any analytic family can be desingularized to obtain a family whose singular set $Z(X)$ is of codimension at least two.

This paper deals with the method but we do not provide any examples of successful desingularization here. The rescaling formulae (**[Ta]**, **[B]**, **[DRS1]**) provide the first example of this type. The global blowing up described in this paper has already been applied in **[R3]**, where it is proved that analytic unfoldings of cusp loops can be desingularized in the sense of Chapter III. Some new applications of such desingularization to particular cases are forthcoming (**[DR1]**, **[DR2]**, **[DRS]**), along with the desingularization of families with nilpotent points (**[R4]**).

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## 1. Foliated Local Vector Fields

In the next chapter we will describe an operation of generalized blowing up, which transforms a family of vector fields into a new object, called foliated local vector field and defined in the present chapter. Because of the fact that we want to use this desingularization in a recurrent way, we have to define the generalized blowing up for this new object i.e. for a foliated local vector field. In order to enlighten a little the subject, we begin with a very simple example of a blowing up that produces a foliated local vector field:

**(I.1) Example.** Let us take a family of analytic vector fields, parameterized analytically by

$$X_\lambda = F(x,y,\lambda)\frac{\partial}{\partial x} + G(x,y,\lambda)\frac{\partial}{\partial y}$$

with $(x,y) \in \mathbb{R}^2$ and a two-dimensional parameter $\lambda$. It can also be regarded as an analytic vector field $X$ in $\mathbb{R}^4$. Suppose $X(0) = 0$ and take the simplest blowing up, i.e. the blowing up of $\mathbb{R}^4$ with centre 0:

$$\phi: S^3 \times \mathbb{R}^+ \ni (\bar{x}, \bar{y}, \bar{\lambda}_1, \bar{\lambda}_2, r) \to (r\bar{x}, r\bar{y}, r\bar{\lambda}_1, r\bar{\lambda}_2) \in \mathbb{R}^4.$$

Throughout the paper we denote by $\mathbb{R}^+$ the set $[0,\infty)$ of nonnegative real numbers.

For such a blowing up there exists (cf. **[D2]**) an analytic vector field $\hat{X}$ on $S^3 \times \mathbb{R}^+_\circ$ such that $\phi_*(\hat{X}) = X$. We have the surjection $p = \pi \circ \phi$ which is no longer regular. The blown up field $\hat{X}$ is tangent to regular fibers of $p$, which fill an open dense set $U_\circ$ in $S^3 \times \mathbb{R}^+_\circ$. The complement of this set consists of the singular fiber $p^{-1}(0,0) = S^3 \times \{0\} \cup S^1 \times \mathbb{R}^+$. The regular fibers form a two-dimensional foliation of $U_\circ$. This foliation extends in a unique way to a maximal foliation $\mathcal{F}$ with the singular set $\sum$ which is the circle $S^1$ embedded in the exceptional divisor $S^3 \times \{0\}$ of $\phi$. The leaves of $\mathcal{F}$ in $S^3 \setminus \sum$ are all equal to the interior of an open disc with boundary $\sum$. For more precise similar description see **[R3]**.



As we will use iterations of blowing ups, we need to generalize the example above, which amounts to generalizing the field and the foliation in the blown up space.

**(I.2) Definition.** A local vector field $X$ is defined on a compact (maybe with boundary) analytic manifold $E$ by an open finite covering $\{U_i\}$ of $E$ and a collection $\{X_i\}$ of analytic vector fields on $U_i$, verifying the following compatibility condition: for each pair of indices $i, j$ such that $U_i \cap U_j \neq \phi$, there is an analytic function $g_{ij}$ defined and strictly positive in $U_i \cap U_j$ such that:

$$X_i = g_{ij} X_j \quad \text{in} \quad U_i \cap U_j.$$

Two collections $\{U_i, X_i\}$ and $\{V_j, Y_j\}$ as above are said to be equivalent if there exists a collection of strictly positive analytic functions $f_{ij}$, defined on $U_i \cap V_j$, such that $X_i = f_{ij} Y_j$ in $U_i \cap V_j$.

*A local vector field on* $E$ is an equivalence class of this relation.

We will denote by $Z(X)$ the set $\cup Z_i(X)$ of singular points of all $X_i$. It does not depend on the choice of the defining collection.

**Remark.** The notion of a local vector field is not new. It is what has been sometimes called an oriented singular foliation of dimension one (cf. for instance **[R1]**). The notion of oriented orbits and limit cycles is thus well known for this object. If we use a different name here, it is because we reserve the name of singular foliation for another object, defined later in this chapter.

**(I.3) Definition.** We will call *a singular fibration* a triplet $(E, \pi, \Lambda)$ consisting of:

- a compact real analytic manifold $E$ of dimension $k+2$,
- a compact real analytic manifold $\Lambda$ of dimension $k$,
- an analytic surjective mapping $\pi: E \to \Lambda$ such that for each $x \in E$ there are local coordinates $x_1, \dots, x_{k+2}$ in a neighbourhood of $x$, sending $x$ to 0, and

there are local coordinates $\lambda_1, \dots, \lambda_k$ in a neighbourhood of $\pi(x)$, sending $\pi(x)$ to 0, such that in these coordinates $\pi$ takes the form:

$$\lambda_1 = \prod_1^{k+2} x_i^{p_1^i}, \dots, \lambda_k = \prod_1^{k+2} x_i^{p_k^i}, \quad \text{with} \quad p_j^i \in \mathbb{N}. \tag{1.1}$$

We suppose, moreover, that $\pi$ is regular $(\mathrm{rank}(\pi) = \mathrm{rk}\ \pi = k)$ on an open dense set $U_\circ$ in $E$, which is equivalent to saying that the matrix $P = (p_j^i)$ of the definition above has the rank $k$. We suppose also that each regular fiber of $\pi$ in $U_\circ$ is diffeomorphic to a two-dimensional compact submanifold of the 2-sphere.

Each manifold in this definition can be a manifold with or without boundary. In the case where there are boundaries, we suppose that $\pi^{-1}(\partial\Lambda) \subset \partial E$.

Observe that for a foliation defined on an open dense set in $E$ any two extensions coincide on the intersection of their domains, thus there exists the unique maximal foliation extending the given one.

We will denote by $\mathcal{F}$ the *maximal foliation* extending the foliation $\mathcal{F}_\circ$ defined by the connected components of the regular fibers of $\pi$ in $U_\circ$.

The domain of $\mathcal{F}$ will be denoted by $U$ and the singular set of $\mathcal{F}$, equal to $E/U$, will be denoted by $\sum$.

**(I.4) Proposition.** *The maximal foliation $\mathcal{F}$ associated to a singular fibration $(E, \pi, \Lambda)$ verifies the following properties:*

1. *The set $\sum$ is an analytic subset of E and it is decomposed into two analytic manifolds, $\sum_1$ and $\sum_\circ$, with codim $\sum_1 \geq 1$ and codim $\sum_\circ \geq 2$, such that $\partial \sum_1 = \sum_\circ$.*
2. *Each leaf of $\mathcal{F}$ is homeomorphic to a closed submanifold of $S^2$, with boundary and corners. For a given leaf L let $\partial_\circ L$ stands for the set of corners and $\partial_1 L = \partial L \backslash \partial_\circ L$.*
3. *We have $\bigcup_L \partial_1 L = \sum_1$ and $\{\partial_1 L, L \in \mathcal{F}\}$ define an analytic one-dimensional foliation of $\sum_1$. Also, $\cup \partial_\circ L = \sum_\circ$, which can be regarded as a 0-dimensional foliation of $\sum_\circ$.*
4. *Let $x \in \sum$ and $\mathcal{L}_x$ denote the collection of all the leaves L of $\mathcal{F}$ such that $\bar{L}$ is an analytic manifold with boundary in some neighbourhood of $x$. Suppose that $x \in \sum_1$ and let $l_x$ denote the leaf of $\sum_1$ through $x$. We have:*

$$\cap\{T_x L; L \in \mathcal{L}_x \text{ and } x \in \bar{L}\} = T_x l_x.$$

*Similarly, for $x \in \sum_\circ$ we have: $\cap\{T_x L; L \in \mathcal{L}_x \text{ and } x \in \bar{L}\} = \{0\}$.*



**Proof.** First observe that the interior of each leaf $L$ of $\mathcal{F}$ is homeomorphic to an open subset of $S^2$. Indeed, if this was not true, we would have a compact handle in $L$. Since in an open dense set $U_\circ$ the foliation is without holonomy, some leaves in $U_\circ$ would also include a handle, which contradicts the condition put on regular fibers in definition (I.3).

All the other properties are local, so let us now take a point $x \in E$ and local coordinates $x_1, \dots, x_n, n = k+2$, in a neighbourhood of $x$ such that in these coordinates $\pi$ is given by formula (1.1). We then have:

$$d\lambda_j = \lambda_j \sum p_j^i / x_i dx_i, i = 1, \dots, n \text{ and } j = 1, \dots, k. \tag{1.2}$$

Let $Q$ be the set $\{x; x_1 \dots x_n \neq 0\}$ and $P$ the matrix $(p_j^i)$. By hypothesis, the set $U_\circ$ of regular points of $\pi$ is open and dense. This implies that rk $P = k$ and that $Q \subseteq U_\circ$.

Let us take the Abelian Lie algebra $A$ of diagonal vector fields $\sum_1^n \gamma_i x_i \partial / \partial x_i$. This algebra is isomorphic to $\mathbb{R}^n$, with $X$ corresponding to $\gamma = (\gamma_1, \dots, \gamma_n)$ by this isomorphism.

For a function $\mu$ of the form $\mu = \prod_1^n x_i^{\alpha_i}$ and field $X \in A$ we have:

$$x \cdot \mu = (\gamma \cdot \alpha)\mu,$$

where $\gamma \cdot \alpha$ stands for the Euclidean scalar product of $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$ in $\mathbb{R}^n$.

Now we define

$$A_\pi = \{X \in A; X \cdot \lambda \equiv 0\},$$

where $\lambda$ is the function of definition (I.3).

Obviously, $A_\pi$ is a two-dimensional Lie subalgebra of $A$, corresponding to vectors $\gamma \in \mathrm{Ker}\, P$.

For $x \in Q$ the orbits of $A_\pi$ are two-dimensional and coincide with the leaves of $\mathcal{F}_\circ$. Thus extending $\mathcal{F}_\circ$ amounts to extending the foliation by orbits defined by $A_\pi$ in $Q$.

In the sequel we will use the symbol $W(i_1, \ldots, i_r)$ for the subspace $\{x; x_i = 0 \text{ for } i \notin \{i_1, \ldots, i_r\}\}$ and we will choose for our reasoning appropriate basis of $A_\pi$, denoted

$$X_1 = \sum \gamma_i x_i \frac{\partial}{\partial x_i}, \quad X_2 = \sum \delta_i x_i \frac{\partial}{\partial x_i}.$$

We will have to consider two cases:

**Case (a).** Suppose there exist a basis $X_1, X_2$ with $X_2 = \delta_n \partial/\partial x_n, \delta_n \neq 0$. Then we can choose $X_1$ with $\gamma_n = 0$.

As $[X_1, \partial/\partial x_n] = 0$, the fields $X_1, \partial/\partial x_n$ define also $\mathcal{F}_0$ in $Q$, therefore we can replace in our reasoning $X_1, X_2$ by $X_1, \partial/\partial x_n$. After a suitable permutation of coordinate variables we can suppose that

$$X_1 = \gamma_1 x_1 \frac{\partial}{\partial x_1} + \ldots + \gamma_m x_m \frac{\partial}{\partial x_m},$$

with $m < n$ and all $\gamma_1 \neq 0$. Now, if $m = 1$, we can replace again $X_1, X_2$ by $\partial/\partial x_1, \partial/\partial x_n$ and in this case the foliation $\mathcal{F}$ is the everywhere regular foliation defined by the last two fields.

If, however, $m > 1$, we only know that the restriction of $X_1$ to the space $W(1, \ldots, m)$ is a hyperbolic field. Zero is the only singular point of this restriction.

Since $\dim W(m, \ldots, n) \geq 2$, the foliation by orbits cannot be extended to the origin, as in the case where $m = 1$ above.

The field $X_1$ restricted to the hyperplane $W(1, \ldots, n-1)$ admits, as the singular set, the space $W(m+1, \ldots, n-1)$ and is normally hyperbolic to it.

The foliation defined by $X_1, \partial/\partial x_n$ is obtained as the Cartesian product of the foliation given by $X_1$ on $W(1, \ldots, n-1)$ by the axis $W(n)$.

This is a two-dimensional analytic foliation outside the space $W(m+1, \ldots, n)$ and cannot be extended to it. Thus this is the maximal foliation $\mathcal{F}$. Its singular set $\Sigma$ is reduced to $\Sigma_1 = W(m+1, \ldots, n)$, foliated in dimension 1 by the orbits of $\partial/\partial x_n$.

The boundaries of the leaves L of $\mathcal{F}$ are leaves of this foliation. In particular, the products of $W(n)$ by $\{x_i \geq 0\}, i = 1, \ldots, m$, are leaves of $\mathcal{L}$ (i.e. analytic manifolds with boundary), which proves property 4.



**Case (b).** Suppose now that we are not in the case (a). Denote by $D_\circ$ the set $D_\circ = \{x: \dim\{X_1(x), X_2(x)\} = 0\}$. We have two subcases:

**(b1).** Suppose we can choose $X_1$ hyperbolic i.e. $X_1 = \sum_1^n \gamma_i x_i \partial/\partial x_i$ with all $\gamma_i \neq 0$. This is equivalent to saying that $\dim D_\circ = 0$.

We will denote by $X_1(t, x)$ the flow of $X_1$. Let $P_i^+, P_i^-$ stand for the sets $\{x_i = +1\}, \{x_i = -1\}$, respectively.

Let $V_i^+ = \bigcup_t X_1(t, P_i^+)$ and $V_i^- = \bigcup_t X_1(t, P_i^-)$.

Observe that $X_1$ is transversal to $P_i^+, P_i^-$, the sets $V_i^+, V_i^-$ are open in $\mathbb{R}^n$, invariant by $A_\pi$, and $(\bigcup V_i^+) \cup (\bigcup V_i^-) = \mathbb{R}^n \setminus \{0\}$.

We will establish the properties of $\mathcal{F}$ by looking at it in restriction to the sets $V_i^\pm$. As the proof is the same for each $V_i^\pm$, suppose we are in $V_1^+$. After a suitable permutation of coordinates, we can write $X_2 = \sum_m^n \delta_i x_i \partial/\partial x_i$, with all $\delta_i \neq 0, i = m, \ldots, n$ and $1 < m < n$.

The field $X_2$ is then obviously tangent to $P_1^+$ and hyperbolic in restriction to $W(m, \ldots, n)$.

As $\dim W(m, \ldots, n) \geq 2$, in this space we can reason as in case (a) for $m > 1$. Take

$$\sum_1 \cap V_1^+ = \bigcup_t X_1(t, W(1, \ldots, m-1)).$$

This set is foliated in dimension 1 by the orbits of $X_1$. All properties of $\sum_1$ are verified like in case (a). This proves property 4 for $\sum_1$. We still have to prove it for $\sum_\circ$, which in this case is equal to $\{0\}$ and was empty in case (a). Property 4 for $\sum_\circ$ follows directly from the properties of the restrictions of $A_\pi$ to the planes $W(i, j)$.

**(b2).** In this subcase we have $\dim D_\circ > 0$. After a suitable permutation of coordinates we can write

$$X_1 = \sum_1^m \gamma_i x_i \frac{\partial}{\partial x_i} \quad \text{and} \quad X_2 = \sum_1^m \delta_i x_i \frac{\partial}{\partial x_i},$$

with $1 < m < n$ and all $\gamma_i \neq 0$, for $i = 1, \ldots, m$.

The foliation $\mathcal{F}$ is the same, up to a translation, in every space parallel to $W(1, \ldots, m)$, where we have the situation of $(\text{b}_1)$.

The set $\sum_\circ$ is thus equal to the space $W(m+1,\dots,n)$ and all the required properties follow from the properties verified in the spaces parallel to $W(1,\dots,m)$. □

**(I.5) Definition.** Given a singular fibration $(E,\pi,\Lambda)$ and a point $x \in E$ *a leaf through* $x$ is:

the 2-dimensional leaf of $\mathcal{F}$ containing $x$, if $x \in U$,

the 1-dimensional leaf of $\sum_1$ containing $x$, if $x \in \sum_1$,

the point $x$, if $x \in \sum_\circ$.

We will introduce now the main object of our study, generalizing the notation of a family of vector fields:

**(I.6) Definition.** A *foliated local vector field* $\mathcal{E} = (E,\pi,\Lambda,X)$ is an object consisting of a singular fibration $(E,\pi,\Lambda)$ and of a local vector field $X$ defined on $E$ such that $X$ is tangent to the fibers, i.e. $d\pi(x)[X(x)] = 0$ for all $x \in E$.

If $F$ is a compact analytic submanifold of $E$ such that $\pi(F)$ is a submanifold of $\Lambda$, we will denote by $\mathcal{E}_F$ the restriction $(F,\pi_F,\pi(F),X_F)$, if it is again a foliated local vector field.

If $\Delta$ is a compact submanifold of $\Lambda$, we will denote by $\mathcal{E}_\Delta$ the restriction $\mathcal{E}_{\pi^{-1}(\Delta)}$, whenever the associated restricted object for $F = \pi^{-1}(\Lambda)$ is a foliated local vector field.

**Remark.** A family $\{X_\lambda\}, \lambda \in \Lambda$, of analytic vector fields defined on $S^2$ is a foliated vector field with $E = S^2 \times \Lambda$ and $\pi$ the natural projection of $E$ on $\Lambda$.

**(I.7) Definition.** *A limit cycle* of a foliated local vector field $\mathcal{F}$ is a limit cycle of a restriction $X_L$ of $X$ to one of the leaves of $\mathcal{F}$.

**(I.8) Proposition.** *A foliated local vector field $X$ is tangent at each point $x \in E$ to the leaf through $x$.*

**Proof.** The field $X$ is tangent to the regular fibers by definition, so, by continuity and by the density of $Q$, it is tangent to all the leaves of the maximal foliation $\mathcal{F}$. Now take a point $x$ in $\sum_1$. By continuity, $X(x) \in T_x l_x$, where $l_x$ is the leaf



through $x$. The same reasoning holds for the points of $\sum_\circ$. □

## 2. Operations of Desingularization

In this chapter we will define three operations of desingularization in the class of foliated local vector fields, associating to a certain foliated local vector field $\mathcal{E} = (E,\pi,L,X)$ a new one $\tilde{\mathcal{F}} = (\tilde{E},\tilde{\pi},\tilde{\Lambda},\tilde{X})$.

First we will generalize the notion of a uniform bound on the number of limit cycles of $X_\lambda$ in a family $\{X_\lambda\}$ to the notion of a uniform bound by leaves for a foliated local vector field.

**(II.1) Definition.** We say that $K$ is a uniform bound on the number of limit cycles by leaves for a foliated local vector field $\mathcal{E}$ if, for each $L$ belonging to the maximal foliation $\mathcal{F}$, the number of limit cycles of the restricted field $X_L$ is not greater than $K$.

If such a bound exists for a given foliated local vector field $\mathcal{E}$, we say that $\mathcal{E}$ *has the finiteness property.*

The most important property of the three operations defined below is that they preserve the existence of a uniform bound, i.e. if a uniform bound on the number of limit cycles by leaves exists for $\tilde{\mathcal{E}}$ and is equal to $K$, then $N.K$, for some $N \in \mathbb{N}$, is a uniform bound on the number of limit cycles by leaves for $\mathcal{E}$.

### Induction

**(II.2) Definition.** Let $\{X_\lambda\}, \{\tilde{X}_{\tilde{\lambda}}\}$ be two analytic families of vector fields on the same analytic surface $S$, with parameters $\lambda \in \Lambda$ and $\tilde{\lambda} \in \tilde{\Lambda}$, respectively, where $\Lambda, \tilde{\Lambda}$ are two analytic manifolds.

We say that the family $\{\tilde{X}_{\tilde{\lambda}}\}$ is *induced* from $\{X_\lambda\}$ by an analytic map $h\colon \Lambda \to \tilde{\Lambda}$, if for each $\lambda \in \Lambda$, $X_\lambda$ is topologically equivalent to $\tilde{X}_{h(\lambda)}$.

If we regard $\{X_\lambda\}$ and $\{X_{\tilde{\lambda}}\}$ as foliated local vector fields $\mathcal{E} = (S \times \Lambda, \pi, \Lambda, X)$ and $\tilde{\mathcal{E}} = (S \times \tilde{\Lambda}, \tilde{\pi}, \tilde{\Lambda}, \tilde{X})$, where $\pi$ and $\tilde{\pi}$ are the natural projections, it is easy to see that if there is a uniform bound by leaves for $\tilde{\mathcal{E}}$, then the same number is a uniform bound by leaves for $\mathcal{E}$.

## Local Division

**(II.3) Definition.** Given two local vector fields $X, Y$ on $(E, \pi, \Lambda)$, we say that $Y$ is *the result of local division of* $X$ if there is a finite open covering $\{V_i\}$ of $E$, defining both $X$ and $Y$, and analytic functions $f_i: V_i \to \mathbb{R}$ such that

$$X_i = f_i Y_i \quad \text{in} \quad V_i. \tag{2.1}$$

A uniform bound on the number of limit cycles is preserved by this operation too, as the leaves of $(E, \pi, \Lambda, X)$ and $(E, \pi, \Lambda, Y)$ are the same. It is sufficient to remark that no limit cycle for $X$ intersects the set $Z(X)$ and then $Z(Y) \subseteq Z(X)$. Thus, if there is a uniform bound on the number of limit cycles in leaves for $Y$, the same bound is good for $X$.

## Generalized Blowing Up

This operation will be the most important one and, as we have already mentioned, it is the reason for introducing the notion of foliated local vector fields.

In order to explain why we have to use a generalized blowing up, we begin with recalling an example, which is dealt with in [**R3**]:

Let us take the two-parameter family (known as the Bogdanov-Takens family [Ta], [B]) near zero in $\mathbb{R}^2$:

$$X_\lambda = y\frac{\partial}{\partial x} + (x^2 + \mu + y(\nu + x))\frac{\partial}{\partial y}, \tag{2.2}$$

where $\lambda = (\mu, \nu)$ is the parameter.

Take the following rescaling formulae:

$$x = u^2\bar{x}, \quad y = u^3\bar{y}, \quad \mu = u^4\bar{\mu}, \quad \nu = u\bar{\nu}. \tag{2.3}$$

The family (2.2) can be desingularized with the use of these formulae as a singular change of variables and parameters $\phi$, transforming points $((\bar{x}, \bar{y}), u, (\bar{\mu}, \bar{\nu})) \in \bar{D} \times \mathbb{R}^+ \times S^1$ into $(x, y, \mu, \nu) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $\bar{D}$ is a fixed compact domain in the $(\bar{x}, \bar{y})$ space.

The family $X_\lambda$ can then be replaced by the new family $X_{(u,\bar{\lambda})} = \frac{1}{u}\phi_*^{-1}(X_\lambda)$, depending on the parameter $(u, \bar{\lambda}) \in \mathbb{R}^+ \times S^1$, where $\bar{\lambda} = (\bar{\mu}, \bar{\nu})$, and has to be studied on $\bar{D}$.



It can be shown that the new family is desingularized in the sense that each of its singular points has at least one non-zero eigenvalue.

Such a desingularization by rescaling the family allowed a complete study of the unfoldings defined by the family (2.2) near $0 \in \mathbb{R}^4$ (see [B], [Ta] for more details).

Suppose now that the singular point $0 \in \mathbb{R}^2$ of $X_\circ$ belongs to some polycycle $\Gamma$. To study the unfolding of $\Gamma$ we have to glue the preceding local study to the study of the deformations of the remaining part of $\Gamma$. Here a major problem arises: the domain in the $(x, y)$-plane which is covered by the rescaling formulae is equal to $D_u = \{u^2\bar{x}, u^3\bar{y}) : (\bar{x}, \bar{y}) \in \bar{D}\}$ and its diameter tends to zero as $u$ tends to zero.

To bypass this difficulty, the following idea was used in [**R3**]: to regard the rescaling (2.3) as formulae for global rescaling with weights. This means that now we are going to take $(\bar{x}, \bar{y}, \bar{\mu}, \bar{\nu}) \in S^3$ and $u \in \mathbb{R}^+$ and we will apply them to $X_\lambda$ regarded as a global vector field in $\mathbb{R}^4$. The result is no longer a family, but precisely a foliated local vector field (and here we have a global vector field). It is exactly with this global blowing up that we will work.

Since we want to use this blowing up in a recurrent way, we will have to define it in the context of foliated local vector fields. To make the presentation clear, we begin explaining this operation in detail for a one-point centre $a \in E$. This is a mere extension of the particular case presented in [**R3**]. Next, we will describe our blowing up operation in all generality. This is what is called in the sequel *a generalized blowing up of $E$ along an analytic submanifold $C \subset E$* of codimension $n$. This operation is associated to weights $\alpha = (\alpha_1, \ldots, \alpha_n)$ and we will also need some restriction on the choice of trivializations of tubular neighbourhoods of $C$ in $E$. The general idea is to have locally good trivializing charts transversally to $C$, so as to be brought back to a situation similar to the one with one-point centre.

## Generalized Blowing Up With One-point Centre

**(II.4) Definition.** Given a point $a \in E$, a coordinate map $T: \mathbb{R}^n \to W$ on a neighbourhood $W$ of $a$ (coordinates: $x_1, \ldots, x_n$) and weights $\alpha = (\alpha_1, \ldots, \alpha_n)$

$\in \mathbb{N}^n$, take

$$\phi : S^{n-1} \times \mathbb{R}^+ \ni (\bar{x}_1, \dots, \bar{x}_n, r) \to r^\alpha \bar{x} = (r^{\alpha_1}\bar{x}_1, \dots, r^{\alpha_n}\bar{x}_n) \in \mathbb{R}^n$$

and put $\tilde{E} = E \backslash \{a\} \cup S^{n-1} \times \mathbb{R}^+$, glued together by $T \circ \phi$, with the natural structure of a compact analytic manifold (cf. **[BJ]** for instance). Let $\Phi$ be the analytic mapping defined by the commuting diagram:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\ \Phi\ } & E \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle T} \\ S^{n-1} \times \mathbb{R}^+ & \xrightarrow{\ \phi\ } & \mathbb{R}^n \end{array}$$

where $i$ stands for the natural inclusion.

We say that $(\tilde{E}, \Phi, E)$ constructed in this way is a *blowing up of $E$ in $a$, associated to the map $T$, with weights $\alpha$.* We will denote by $\mathcal{D}$ the set $i(S^{n-1} \times \{0\})$, which is the exceptional divisor of $\Phi$.

**Remark.** In the case where $a \in \partial E$, we suppose that, in the coordinate system, $\partial E$ is given by $\{x_i = 0\}$, for some $i$.

To be able to work in coordinate systems, we prove the following lemma:

**(II.5) Lemma.** *Let $C \subset \mathbb{R}^n \backslash \{0\}$ be a hypersurface such that the mapping*

$$\phi_C : C \times \mathbb{R}^+ \ni (c_1, \dots, c_n, \tau) \to \tau^\alpha c = (\tau^{\alpha_1} c_1, \dots, \tau^{\alpha_n} c_n) \in \mathbb{R}^n$$

*is an analytic embedding.*

*Then there exist an analytic embedding $\bar{x} : C \to S^{n-1}$ and an analytic function $\mathcal{T} : C \to \mathbb{R}^+$ such that $\psi$ defined as*

$$\psi : C \times \mathbb{R}^+ \ni (c, \tau) \to (\bar{x}(c), \mathcal{T}(c).\tau) \in S^{n-1} \times \mathbb{R}^+$$

*is an analytic embedding and $\phi_S \circ \psi = \phi_C$, where $\phi_S$ stands for the map $\psi$ of Definition* (II.4).

**Proof.** The map $\phi_S$ is an analytic diffeomorphism between $S^{n-1} \times \mathbb{R}^+$ and $\mathbb{R}^n \backslash \{0\}$. Take $\phi_S^{-1} : \mathbb{R}^n \backslash \{0\} \to S^{n-1} \times \mathbb{R}^+$ and denote it by $(\bar{x}, \mathcal{T})$, where



$\bar{x} : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ and $\mathcal{T} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^+$. Observe that the hypothesis about $\phi_C$ is equivalent to $\bar{x}$ being an embedding. We have $\phi_S(\bar{x}(c), \mathcal{T}(c)) = c$, which means $c = \mathcal{T}^\alpha \bar{x}, i = 1, \dots, n$. Now take $\phi_C(c, \tau)$. We have

$$\phi_C(c, \tau) = \tau^\alpha c = \tau^\alpha \mathcal{T}^\alpha \bar{x} = (\tau \mathcal{T})^\alpha \bar{x}.$$

This implies that $\phi_C = \phi_S \circ \psi$. □

**Remark (II.5.1).** When $C$ is the hyperplane $\{x_i = +1\}$ or $\{x_i = -1\}$, the mapping $\phi_C$ is called directional blowing up in the direction $x_i$, associated to $\phi_S$.

Now we look at the effect such a blowing up has on a vector field defined near $a$:

**(II.6) Lemma.** *Suppose we have a blowing up of $E$ in $a$, with weights $\alpha$, associated to $T$ (coordinates $x_1, \dots, x_n$) and $\Phi : \tilde{E} \to E$ (cf. Def. (II.4)). Let $X$ be an analytic vector field defined in the chart domain $W$ and such that $X(a) = 0$. Then there exists $s \in \mathbb{N}_\circ$ such that the field $t^s \Phi_\star^{-1}(X)$, defined on $\Phi^{-1}(W) \backslash \mathcal{D}$, can be extended to the whole set $\Phi^{-1}(W)$ analytically.*

**Proof.** We use the coordinate system $(x_1, \dots, x_n)$. Put $\Omega = dx_1 \wedge \dots \wedge dx_n$ and let $\omega = X \rfloor \Omega$, where $\rfloor$ stands for the interior product.

Denote $\tilde{\omega} = \phi_\star \omega$ and $\tilde{\Omega} = \phi_\star \Omega$. Observe that

$$\tilde{\Omega} = t^{\Sigma \alpha_i - 1}.\bar{\Omega}, \tag{2.4}$$

where $\bar{\Omega}$ is a volume form on $S^{n-1} \times \mathbb{R}^+$.

As $\phi$ is a diffeomorphism on $S^{n-1} \times (\mathbb{R}^+ - \{0\})$, the field $\tilde{X} = \phi_\star^{-1}(X)$ is defined in this set.

We have $\tilde{X} \rfloor \tilde{\Omega} = \tilde{\omega}$ in $S^{n-1} \times (\mathbb{R}^+ - \{0\})$. Substituting $\tilde{\Omega}$ by the formula (2.4) we get $\tilde{X} \rfloor t^s \bar{\Omega} = \tilde{\omega}$, where $s = \Sigma \alpha_i - 1$.

Therefore, since $\tilde{X} \rfloor t^s \bar{\Omega} = t^s \tilde{X} \rfloor \bar{\Omega}$ on $S^{n-1} \times (\mathbb{R}^+ - \{0\})$, the field $t^s \tilde{X}$ is defined and analytic in the whole set $\Phi^{-1}(W)$, also for $t = 0$. □

**Remark (II.6.1).** In the case where $\alpha_i = 1, i = 1, \dots, n$, we can take $s = 0$ and obtain the classical result (cf. for instance **[D2]**).

**Remark (II.6.2).** In the coordinate system $(x_1, \dots, x_n)$ the map $\Phi$ is expressed as $\phi_S$ by definition and the field $\bar{X} = t^s \Phi_*^{-1}(\bar{x})$ is equal to $t^s \phi_*^{-1}(X)$.

Let a collection of manifolds $C_i \subseteq \mathbb{R}^n \backslash \{0\}$ (space of the variable $\bar{x}$) be as in Lemma II.5 and such that the corresponding maps $\psi_i$ form an atlas of $S^{n-1} \times \mathbb{R}^n$ and let

$$\phi_{C_i}(c_i, r_i) = r_i^{\alpha} c_i, \quad \phi_i : C_i \times \mathbb{R}^+ \to \mathbb{R}^n$$

be as in lemma II.5.

Then, proceeding like in lemma II.6, we can define a collection of vector fields $\bar{X}_i = r_i^s \phi_{C_i^*}^{-1}(X)$ in each chart domain $C_i \times \mathbb{R}^+$.

It follows from lemma II.5 that each $\bar{X}_i$ is equivalent to the global $\bar{X} = t^s \phi_{S^*}^{-1}(X)$ in its chart domain and thus the collection $\{\bar{X}_i, C_i \times \mathbb{R}^+\}_i$ is a *local vector field* in $S^{n-1} \times \mathbb{R}^+$.

## Generalized Blowing Up Along a Submanifold

In the sequel $C$ will denote a connected compact analytic submanifold of $E$, with or without boundary. If $C$ is with boundary, it will always be supposed that $\partial C \subset \partial E$. The proofs we give below concern submanifolds without boundary, but each of them extends in an obvious way to $C$ with boundary $\partial C \subset \partial E$.

$n$ will now stand for the codimension of $C$ in $E$ and $k$ will denote the dimension of $C$. Therefore $\dim E = n + k$.

Let $\alpha$ be a system of weights as above. We suppose that it is ordered in the following way: $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. For what follows, we need to introduce the gap values $1 \leq l_1 < \ldots < l_m \leq n$ defined by $\alpha_1 = \ldots = \alpha_{l_1} > \alpha_{l_1+1} = \ldots = \alpha_{l_m} > \alpha_{l_m+1} = \ldots = \alpha_n$. The gap values correspond to the "jumps" in the sequence $\alpha$.

Let $l = \{l_1, \ldots, l_m\}$ with $l = \varnothing$ when weights are equal. We consider the following subspaces of the space $\mathbb{R}^n$

$$F_{l_s} = \{(x_1, \ldots, x_n) : x_i = 0 \quad \text{whenever} \quad i \leq l_s \quad \text{or} \quad i > l_{s+1}\}$$

for $s = 1, \ldots, m-1$. This definition is easily extended to $s = 0, m$ (for instance $F_{l_0} = \mathbb{R}^n$. Next we define $E_{l_s}$ for $s = 0, \ldots, m$ as the direct sums: $E_{l_s} = F_{l_s} \oplus \ldots \oplus F_{l_m}$. The collection $E_l = (E_{l_0}, \ldots, E_{l_m})$ is called a *flag*. We have the inclusions: $E_{l_m} = F_{l_m} \subset \ldots \subset E_{l_2} \subset E_{l_1} \subset E_{l_0} = \mathbb{R}^n$.

Let $GL_l \subset GL(n, \mathbb{R}^n)$ be the subgroup of invertible $n \times n$ matrices for which



$E_{l_s}, s = 0, \ldots, m$ are invariant subspaces, i.e.:

$$GL_l = \{M \in GL(n, \mathbb{R}) : M(E_{l_s}) \subset E_{l_s} \text{ for } s = 0, \ldots, m\}.$$

The subgroup $GL_l$ consists of all invertible matrices $M = (M_{ij})$ such that $M_{ij} = 0$ for $l_s < i \leq l_{s+1}$ and $j > l_{s+1}$. For $l = \varnothing, GL_l$ is the whole linear group $GL(n, \mathbb{R})$.

**(II.7) Definition.** Let $x^i = x_1^{i_1} \ldots x_n^{i_n}$ be a monomial in $x = (x_1, \ldots, x_n)$ and let $i = (i_1, \ldots, i_n)$.

The *$\alpha$-degree* of $x^i$, denoted by $\alpha$-deg $(x^i)$, is defined as $\alpha$-deg $(x^i) = \alpha . i = \sum_{j=1}^{n} \alpha_j i_j$.

More generally, if $f(x)$ is an analytic function, with $f(0) = 0$, we define:

$$\alpha\text{-deg}(f) = \text{Inf}\{s : f(u^\alpha \bar{x}) = O(u^s) \text{ for } (u, \bar{x}) \in \mathbb{R}^+ \times S^{n-1}\}.$$

See (II.4) for the notation $u^\alpha \bar{x}$.

It is easy to compute $\alpha$-deg $(f)$ by looking at the Newton diagram of $f$.

**Remark.** Let $M$ be an invertible matrix $M \in GL(n, \mathbb{R}^n)$. Then $M \in GL_l$ if and only if $\alpha$-deg $(\sum_j M_{ij} x_j) = \alpha_i$, for $i = 1, \ldots, n$.

Now we are in position to define a weighted blowing up transversal to the given submanifold $C$ ($C$ as above) and which will have the form $x = u^\alpha \bar{x}$ in local coordinates, analogous to that of Definition II.4. To this end, we introduce the following:

**(II.8) Definition.** For a sequence of weights $\alpha$ and a submanifold $C$ as above, an *$\alpha$-admissible trivialization* of $C$ is given by a collection of charts $(W_i, \psi_i)$ such that:

(1) $\psi_i : U_i \times \mathbb{R}^n \to E$ is an analytic diffeomorphism on its open image $W_i$ and $\psi_i(U_i \times \{0\}) = W_i \cap C$,

(2) $T = \cup W_i$ is an open tubular neighbourhood of $C$.

(3) If $W_i \cap W_j \neq \varnothing$, let the transition map $g_{ji} = \psi_j^{-1} \circ \psi_i$ be written $g_{ji}(c_i, x_i) = (\phi_{ji}(c_i, x_i), C_{ji}(c_i, x_i)) \in U_i \times \mathbb{R}^n$.

For sake of simplicity, we will write $C(c, x)$ for $C_{ji}(c_i, x_i)$. We suppose that for each $(c, 0)$ in the set $\psi_j^{-1}[\psi_i(U_i \times \{0\})]$ the following conditions are satisfied:

the partial Jacobian matrix

$$\left(\frac{\partial C}{\partial x}(c,0)\right) \in GL_l \quad \text{for all} \quad (c,0) \in U, \tag{2.5}$$

$$\alpha\text{-deg}\left(C_i(x) - < \frac{\partial C_i}{\partial x}(c,0), x >\right) > \alpha_i. \tag{2.6}$$

We say that a trivialization chart $(W, \psi)$, with $W \subset T$ (i.e. a chart that verifies (1)) is *compatible* with the collection $\{W_i, \psi_i\}$ above if it verifies (3) for each $(W_i, \psi_i)$ of the collection. The *$\alpha$-admissible* trivialization of $C$ will be precisely the maximal collection of all trivialization charts compatible with the collection $\{W_i, \psi_i\}$.

**Remark.** If $\alpha_1 = \ldots = \alpha_n$, the condition (3) is verified by any transition map $g_{ji}$, therefore in this case any trivialization of $C$ is $\alpha$-admissible. This is the case of the standard blowing up. In the general situation, we have a list of gap values $l = \{l_\circ, \ldots, l_m\}$ and condition (2.5) above implies that the normal bundle $NC$ of $C$ admits a reduction of its structural group to the group $GL_l$, i.e. that $NC$ admits a filtration by an $l$-flag of bundles. For instance, if the sequence $\alpha$ is strictly decreasing: $\alpha_1 > \cdots > \alpha_n$, the group $GL_l$ is the group of inferior triangular matrices and $NC$ must contain a flag of subbundles of any dimension 1 and $n$.

**(II.9) Proposition.** *Let $C \subset E$ be, as above, a compact connected submanifold with an $\alpha$-admissible trivialization and let $T$ denote the associated tubular neighbourhood.*

*Then there exists an analytic manifold $\tilde{T}$ and an analytic surjective mapping $\phi: \tilde{T} \to T$ such that to each compatible trivialization chart $(W, \psi)$ for $C$, $\psi: U \times \mathbb{R}^n \to T$ corresponds to an analytic diffeomorphism $\tilde{\psi}: U \times S^{n-1} \times \mathbb{R}^+ \to \tilde{T}$ on its image $\tilde{W}$ (we will call it a chart of $\tilde{T}$) which verifies:*

$$\psi^{-1} \circ \phi \circ \tilde{\psi}(c, \bar{x}, u) = (c, u^\alpha \bar{x}) \tag{2.7}$$

*and $\tilde{T}$ is the union of the $\tilde{W}$.*

*The inverse image $\phi^{-1}(C) = D$ will be called the exceptional divisor of $\phi$. In each chart $\tilde{W}$ we have $D \cap \tilde{W} = \tilde{\psi}(U \times S^{n-1} \times \{0\})$, so $D$ is a fiber bundle over $C$ with fiber $S^{n-1}$.*



**Proof.** Suppose that such an analytic manifold $\tilde{T}$ exists, together with the map $\phi$ and charts $(\tilde{W}, \tilde{\psi})$ as stated above.

Take two $\alpha$-admissible trivializing charts for $T$, denoted $(W, \psi)(W', \psi')$. Let $g$ be the transition map between their domains $U \times \mathbb{R}^n$ and $U' \times \mathbb{R}^n$. For simplicity, we do not write down the domain of $g$.

Let $(c, x) \in U \times \mathbb{R}^n, (c', x') \in U' \times \mathbb{R}^n$, and $(c, \bar{x}, u) \in U \times S^{n-1} \times \mathbb{R}^+, (c', \bar{x}', u') \in U' \times S^{n-1} \times \mathbb{R}^+$ be the corresponding coordinates.

Put

$$\phi_U(c, \bar{x}, u) = (c, u^\alpha \bar{x}),$$
$$\phi_{U'}(c', \bar{x}', u') = (c', u'^\alpha \bar{x}').$$

Using the two relations

$$\psi_U^{-1} \circ \phi \circ \tilde{\psi}_U = \phi_U \quad \text{and} \quad \psi_{U'}^{-1} \circ \phi \circ \tilde{\psi}_{U'} = \phi_{U'},$$

we see that the transition map $\tilde{g}$ between the two $\tilde{T}$-charts $(\tilde{W}, \tilde{\psi})$ and $(\tilde{W}', \tilde{\psi}')$ must verify

$$g \circ \phi_U = \phi_{U'} \circ \tilde{g} \tag{2.8}$$

We remark that $\phi$ is invertible for $u' \neq 0$. Therefore, for $u \neq 0$ the map $\tilde{g}$ is defined by the formula:

$$\tilde{g} = \phi_{U'}^{-1} \circ g \circ \phi_U \tag{2.9}$$

To construct $\tilde{T}$ we have to prove first that the map $\tilde{g}$ defined by (2.9) for $u \neq 0$ has an analytic extension to $u = 0$. To this end, take $g(c, x) = (c', x')$, where

$$\begin{cases} c' = \Phi(c, x) \\ x' = C(c, x) = C(c)x + R(c, x), \end{cases} \tag{2.10}$$

where $R(c, x) = O(\|x\|^2)$ and $C(c)$ is an $n \times n$ matrix.

We have to compute $g \circ \phi_U$, so in formulas (2.9) above we substitute $x$ by $u^\alpha \bar{x}$.

Consider first the linear part $A = C(c)(u^\alpha \bar{x})$.

Write $C(c) = (C(c)_{ij})_{ij}, \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n), \bar{x}' = (\bar{x}'_1, \ldots, \bar{x}'_n)$.

Develop the $i^{th}$ line in $A$

$$A_i = \sum_{j=1}^{n} C(c)_{ij}(u^{\alpha_j} \bar{x}_j)$$

and take $l_s < i \le l_{s+1}$. By hypothesis, we have $\alpha_j > \alpha_i$ if $j \le l_s, \alpha_j = \alpha_i$ if $l_s < j \le l_{s+1}$ and $C(c)_{ij} = 0$ if $j \ge l_{s+1}$.

Therefore, if $l_s < i < l_{s+1}$, we have:

$$A_i = u^{\alpha_i} \sum C(c)_{ij} \bar{x}_j + o(u^{\alpha_i}), \tag{2.11}$$

where the sum is taken over all $j$ such that $l_s < j \le l_{s+1}$.

Let $Q(c)$ be the reduced matrix whose coefficients are: $Q(c)_{ij} = C(c)_{ij}$ for $l_s < i \le l_{s+1}$ and $l_s < j \le l_{s+1}$, zero otherwise. Geometrically, the matrix $Q$ can be interpreted in the following way: the matrix $C(c)$ leaves the flag $E_{l_m} \subset \ldots \subset E_{l_\circ}$ invariant and thus induces isomorphisms on each quotient space $F_{l_{s+1}} \cong E_{l_{s+1}}/E_{l_s}$, so the matrix $Q(c)$ is the matrix of this representation of $C(c)$ on the direct sum $\oplus_{s=0}^m F_{l_s}$.

So finally we get the following expression for the linear part:

$$A_i = u^{\alpha_i} \sum_j Q(c)_{ij} \bar{x}_j + o(u^{\alpha_i}).$$

Now, consider the remaining term $R(c, u^\alpha \bar{x})$. Write it as $R = (R_1, \ldots, R_n)$. By the condition (2.6) in Definition (II.8) we have:

$$R_i(c, u^{\alpha_i} \bar{x}) = u^{\alpha_i} \bar{R}_i(c, \bar{x}, u),$$

where $\bar{R}_i$ is an analytic function of $(c, \bar{x}, u)$ such that $\bar{R}_i(c, \bar{x}, 0) = 0$. Then:

$$x'_i = C_i(c, x) = u^{\alpha_i} \left[ (Q(c)\bar{x})_i + \bar{R}_i(c, \bar{x}, u) \right].$$

Denote by $M(c, \bar{x}, u)$ the analytic function inside brackets above. We have:

$$M = Q(c)\bar{x} + \bar{R}, \quad \text{where} \quad \bar{R} = (\bar{R}_1, \ldots, \bar{R}_n).$$

For any $c$, the map $S^{n-1} \ni \bar{x} \to \bar{x}' = Q(c)\bar{x}$ is transversal to the trajectories of the linear vector field $\alpha x' \partial / \partial \bar{x}' = \sum_i \alpha_i \bar{x}'_i \partial / \partial \bar{x}'_i$. This follows easily from the fact that $Q(c)$ preserves each space $F_{l_s}$ and on this space $\alpha \bar{x}' \partial / \partial \bar{x}'$ is radial with a single eigenvalue $\alpha_{l_s}$.

This property remains true for the map $\bar{x} \to M(c, \bar{x}, u)$ with $u$ sufficiently small, thus the composition $G(c, x, u)$ of the map $\bar{x} \to M(c, \bar{x}, u)$ with the projection along the trajectories of the field $\alpha \bar{x}' \partial / \partial \bar{x}'$ onto the unitary sphere $S^{n-1}$ in the $\bar{x}'$-space is a covering map and thus it is an analytic diffeomorphism in variable $\bar{x}$, which depends analytically on $c$ and $u$. The time $u$, needed to go from $G(c, \bar{x}, u)$ to $M(c, \bar{x}, u)$, is also an analytic function, denoted $T(c, \bar{x}, u)$.



By definition, we have:

$$M(c, \bar{x}, u) = T(c, \bar{x}, u)^\alpha G(c, \bar{x}, u),$$

so

$$x' = u^\alpha M = u^\alpha T^\alpha G = (uT)^\alpha G, \tag{2.12}$$

with $uT \in \mathbb{R}^+$ and $G \in S^{n-1}$.

Now, if $(c', \bar{x}', u') = \tilde{g}(c, \bar{x}, u)$, by substitution in (2.10) and identification (2.12) above we get for $\tilde{g}$:

$$\tilde{g} \begin{cases} c' = \Phi(c, u^\alpha \bar{x}) \\ \bar{x}' = G(c, \bar{x}, u) \\ u' = uT(c, \bar{x}, u) \end{cases}$$

with all formulas analytic for $u = 0$.

Now we construct our space $\tilde{T}$ by taking the quotient of the disjoint union $\coprod_i U_i \times S^{n-1} \times \mathbb{R}^+$ by the transition map $\tilde{g}$. In order to obtain a structure of an analytic manifold, we have to verify the cocycle condition $\tilde{g}_{jk} \circ \tilde{g}_{ki} = \tilde{g}_{ji}$ for the maps $\tilde{g}$. Remark that outside of the exceptional divisor $D$ this is obvious, as the maps $\tilde{g}$ are defined there by formula (2.9), so the cocycle condition for the maps $\tilde{g}$ outside $D$ is implied by the analogous condition verified by the maps $g$. By continuity, this condition extends to $D$.

In this way we have constructed the analytic manifold $\tilde{T}$. The chart maps $\tilde{\psi}_i$ are induced by the inclusions of $U_i \times S^{n-1} \times \mathbb{R}^+$ in the disjoint union.

The map $\phi$ is defined now by the condition $\psi_{U_i}^{-1} \circ \phi \circ \psi_{U_i} = \tilde{\phi}_{U_i}$ in each chart domain $U_i \times S^{n-1} \times \mathbb{R}^+$. The consistence of this definition comes from the construction of the maps $\tilde{g}$. □

Now we are in a position to define the operation of generalized blowing up along $C$.

**(II.10) Definition.** Given $\alpha$ and $C$ as above, let an $\alpha$-admissible trivialization of $C$ be taken, $T$ denote the associated tubular neighbourhood of $C$ and $\phi: \tilde{T} \to T$ be the analytic submersion constructed in Proposition (II.9) above.

Put $\tilde{E} = (E \backslash C) \cup \tilde{T}$, the disjoint union obtained by the identification of $\tilde{m} \in \tilde{T}$ with $\phi(\tilde{m}) \in T \subset E$ for all $m \in \tilde{T} \backslash D$. This endows $\tilde{E}$ with a natural structure of a compact analytic manifold. Let $\Phi$ be the analytic mapping

$\Phi: \tilde{E} \to E$ defined by the commuting diagram:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\Phi} & E \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ \tilde{T} & \xrightarrow{\phi} & T \end{array}$$

where $i$ stands for natural inclusions.

We say that the map $\Phi: \tilde{E} \to E$ constructed above is *the $\alpha$-weighted blowing up of $E$ along $C$, relative to the given $\alpha$-admissible trivialization of $C$.*

As in definition (II.4), we will denote by $\mathcal{D}$ the image $i(D)$ and call it *exceptional divisor* of this blowing up. Since $\mathcal{D}$ is diffeomorphism to $D$, it is fibered on $C$ with fiber $S^{n-1}$.

Now we have a lemma, analogous to lemma (II.6):

**(II.11) Lemma.** *Suppose we have a blowing up $\Phi$ in the sense of Definition (II.10) and a local vector field $X$ defined on the associated tubular neighbourhood $T$, such that $C \subset Z(X)$, where $Z(X)$ is the set of zeros of $X$.*

*Then there exists an analytic local vector field $\tilde{X}$ on $\tilde{T}$ which is equal to $\Phi_*^{-1}(X)$ on $\tilde{T}/\mathcal{D}$, where $\mathcal{D}$ is the exceptional divisor.*

**Proof.** It suffices to prove this result locally. So let us choose an open covering of $T$ by a collection of $\alpha$-admissible charts $(W_i, \psi_i), \psi_i: U_i \times \mathbb{R}^n \to T$ as in definition (II.8) and such that $X$ is defined by an analytic vector field $X_i$ on $W_i$ for all $i$. The transition maps from $X_i$ to $X_j$, when $W_i \cap W_j \neq \varnothing$, are denoted $f_{ij}$.

Now, in each chart domain $W_i$, up to a trivial factor $u_i$ we have a situation analogous to that of lemma (II.6). A reasoning similar to the proof of this lemma shows that if $s = \sum \alpha_i - 1$, then in each chart $\tilde{W}_i$ of $\tilde{T}$ the vector field $u_i^s \Phi_*^{-1}(X_i)$ defined on $\tilde{W}_i \backslash \mathcal{D}$ is extended to an analytic vector field $\bar{X}_i$ on $\tilde{W}_i$. The collection $\{\tilde{W}_i, \bar{X}_i\}$ defines a local vector field $\bar{X}$ on $\tilde{T}$, with transition maps $\tilde{f}_{ij} = f_{ij} \circ \Phi$. Clearly, $\bar{X}$ is equal to $\Phi_*^{-1}(X)$ on $\tilde{T} \backslash \mathcal{D}$. □



**Remark.** As in Remark (II.6.2), if we have a covering of $S^{n-1}$ by a collection of charts, $\bar{X}_i$ will be replaced by another representant of its equivalence class and so will $\bar{X}$.

**(II.12) Definition.** Given a blowing up $\Phi: \tilde{E} \to E$ as in Definition (II.10), along a submanifold $C$ of $E$, and a local vector field $X$ such that $C \subset Z(X)$, we will call *the blowing up of $X$* the local vector field $\bar{X}$ on $\tilde{W}$ constructed in Lemma (II.11) above.

By the construction, $\bar{X}$ is equal to $X$ on $\tilde{E} \backslash \mathcal{D}$.

To have the uniqueness in this construction (up to the equivalence relation for local vector fields) we take the minimal $s \in \mathbb{N} \cup \{0\}$ such that the field $u_i^s \Phi_*^{-1}(X_i)$ extends analytically and denote the blown up field for such minimal $s$ by $\bar{X}$.

Clearly, this definition does not depend on the choice of the collection of $\alpha$-admissible charts $(W_i, \psi_i)$ made in the proof of lemma (II.11).

**(II.13) Definition.** Let $C \subset E$ be as above and let $\mathcal{E} = (E, \pi, \Lambda, X)$ be a foliated local vector field. Take an $\alpha$-admissible trivialization of $C$ and let $\Phi$ be the generalized blowing up relative to it.

We say that this blowing up is *compatible* with the given singular fibration $(E, \pi, \Lambda)$ if $\pi_C: C \to \Lambda$ is a submersion onto an analytic submanifold $C' \subset \Lambda$.

For each $\alpha$-admissible chart $(W, \psi)$ of $C$, there exists a trivialization chart $(W', \psi')$ of $C', \psi': U' \times \mathbb{R}^l \to \Lambda$ being a local diffeomorphism on its image $W'$, with $\psi'(U' \times \{0\} = W' \cap C'$ such that if $(x_1, \dots, x_n)$ are coordinates in $\mathbb{R}^n$ and $(\lambda_1, \dots, \lambda_l)$ coordinates in $\mathbb{R}^l$, we have:

$$(\psi')^{-1} \circ \pi \circ \psi(c, x_1, \dots, x_n) = (\pi_C(c), \lambda_1, \dots, \lambda_l),$$

with $\lambda_1, \dots, \lambda_l$ monomials of $x_1, \dots, x_n$. In the above, $l$ denotes the codimension of $C'$ in $\Lambda$.

**Remark.** This definition is clearly independent of the choice of $(W, \psi)$ and $(W', \psi')$.

**(II.14) Proposition.** *Suppose that we have a foliated local vector field $\mathcal{E} = (E, \pi, \Lambda, X)$ and a blowing up $\Phi$ compatible with the singular fibration*

$(E, \pi, \Lambda)$ *along a submanifold* $C \subset Z(X)$. *Such a blowing up, applied to* $\mathcal{E}$, *gives as a result the object* $\tilde{\mathcal{E}} = (\tilde{\mathcal{E}}, \tilde{\pi}, \tilde{\Lambda}, \tilde{X})$, *which is again a foliated local vector field. We have* $\tilde{\Lambda} = \Lambda$ *and* $\tilde{\pi} = \pi \circ \Phi$.

**Proof.** All the required properties are easily verified, except the form of $\tilde{\pi}$. We will restrict the proof to the case where $C$ restricts to one point. The general case is completely analogous, using systems of local coordinates of definition (II.13).

Suppose $C = \{a\}$. In the local coordinates $(x_1, \dots, x_n)$ chosen for $\pi$ (cf. Def. (I.3)), it has the form

$$\lambda_1 = \prod x_i^{p_1^i}, \dots, \lambda_{n-2} = \prod x_i^{p_{n-2}^i}.$$

Putting $x_i = r^{\alpha_i} \bar{x}_i, i = 1, \dots, n$, i.e. substituting $\phi$ into these formulas we obtain (take one $\lambda$)

$$\lambda = r^N \bar{x}_1^{p^1} \cdots \bar{x}_n^{p^n}, \quad \text{where} \quad N = \alpha_1 p^1 + \cdots + \alpha_n p^n.$$

Let us look at $\lambda$ in the coordinate system $\{x_i = cst\}$, for instance $\{x_1 = 1\}$ (cf. Lemma (II.6)). Local coordinates are $(t, x_2, \dots, x_n)$, and in this coordinate system $a = (0, a_2, \dots, a_n)$ (we have supposed $x_1(a) > 0$ to take $\{x_1 = 1\}$).

By the change of coordinates

$$\begin{cases} t = X_1 \\ x_i = a_i + X_i \quad \text{for} \quad i = 2, \dots, n \end{cases}$$

we get

$$\lambda = u \cdot r^N \cdot X_1^{\bar{p}^1} \dots X_n^{\bar{p}^n},$$

with $u$ an analytic strictly positive function and some $\bar{p}^i \neq 0$.

As we have done this for each $\lambda_j, j = 1, \dots, n-2$, we obtain a matrix $\bar{p}$ with entries $\bar{p}_j^i$.

By the hypothesis about $\pi$, we have $rg\lambda = n - 2$. This implies that $rg\bar{P} = n - 2$ too and after a suitable permutation of rangs and columns in $\bar{P}$ we may suppose that $p_i^i \neq 0$ for $i = 1, \dots, n-2$. Now we can take again a coordinate change

$$\begin{cases} y_1 = x_1 \\ y_i = u^{1/\bar{p}_i^i} \cdot X_i, \quad i = 2, \dots, n \end{cases}$$

and the wanted form of $\tilde{\pi}$ is obtained. □



**(II.15) Proposition.** *If there is a uniform bound* $K$ *on the number of limit cycles of* $\tilde{\mathcal{E}}$ *by leaves, then there is a uniform bound* $K \cdot N$, *for some* $N \in \mathbb{N}$, *on the number of limit cycles of* $\mathcal{E}$ *by leaves, where* $\mathcal{E}$ *is a foliated local vector field and* $\tilde{\mathcal{E}}$ *its blowing up (cf. (I.11)).*

**Proof.** Because $C \subset Z(X)$, any limit cycle in a leaf $L$ is in $L \backslash C$. Now by the quasitransversality of $C$ there exists an $N \in \mathbb{N}$ such that the number of connected components of $L \backslash C$ is equal to $N$ for each leaf $L$ intersecting $C$. This proves the proposition. □

Each of the three operations defined above induces a new foliated local vector field $\tilde{\mathcal{E}}$ when applied to a foliated local vector field $\mathcal{E}$. We have shown that, for each operation, the finiteness property of $\tilde{\mathcal{E}}$ implies the finiteness property of $\mathcal{E}$.

## 3. Conjectures

In this chapter we will formulate and discuss the conjectures that we would aim to prove in future works on the subject, using the tools presented in the present paper. In Chapters I and II we introduced the tools we will use (such as foliated local vector fields or the three desingularization operations). Let us first introduce some notions.

**(III.1) Definition.** Let $\mathcal{E} = (E, \pi, \Lambda, X)$ be a foliated local vector field (F.L.V.F.) and let $a \in E$ be a singular point of $X$ $(a \in Z(X))$.

We say that $a$ is *an elementary singular point* if for each leaf $L \in \mathcal{L}_a$, the point $a$ is an elementary singular point of $X_{\bar{L}}$ (i.e. the 1-jet $j_a^1 X_{\bar{L}}$ has at least one nonzero eigenvalue).

**Remark.** If $a \in \Sigma$, it may happen that $a \in \bar{L}$ for several leaves of $\mathcal{L}_a$. Also, we do not require the elementary singular point to be algebraically isolated. As they are hyperbolic or semi-hyperbolic points, in the case they are not isolated, they will form a normally hyperbolic line.

**(III.2) Definition.** Let $\mathcal{E} = (E, \pi, \Lambda, X)$ be a foliated local vector field. We say that a compact invariant set $\Gamma \subset E$ is *a limit periodic set of* $\mathcal{E}$ if there exists a sequence of limit cycles of $\mathcal{E}$ that converges to $\Gamma$ in the Hausdorff metric.

**(III.3) Definition.** A limit periodic set of $\mathcal{E}$ is called *an elementary* limit periodic set if each of its points is either regular ($X(a) \neq 0$) or is an elementary singular point of $\mathcal{E}$.

**Remark.** It is possible to show, using the Poincaré-Bendixon theorem, that an elementary limit periodic set is formed of a finite number of arcs, each of them being either a trajectory or a normally hyperbolic line of $X$. Besides, each of these arcs is contained in the closure of one of the leaves of $\mathcal{F}$. The extremal points of the arcs are in $Z(X)$ and may belong to $\Sigma$.

In particular, an elementary polycycle contained in a leaf $L \in \mathcal{F}$ is an example of an elementary limit periodic set, although in general an elementary limit periodic set need not be contained in a leaf of $\mathcal{F}$. See [**R3**] for more examples.

**(III.4) Definition.** *A step of desingularization* is given by a correspondence

$$\{\mathcal{E}_i\}_{i\in I} \to \{\mathcal{E}_{ij}\}_{(i,j)\in I\times J}$$

between two collections of foliated local vector fields, satisfying the following conditions: let $\mathcal{E}_i = (E_i, \pi_i, \Lambda_i, X_i)$, and suppose that there exists a collection $\{\mathcal{E}^\circ_{ij}\}_{(i,j)\in I\times J}, \mathcal{E}^\circ_{ij} = (E_{ij}, \pi_{ij}, \Lambda_{ij}, X_{ij})$ such that:

(1) $E_{ij} \subset E_i$ for all $(i,j) \in I \times J$,

(2) for each $i \in I$, every non-elementary limit periodic set of $\mathcal{E}_i$ is contained in the interior of one of $E_{ij}$,

(3) for each $(i,j) \in I\times J$, the maximal foliation of $\mathcal{E}^\circ_{ij}$ is the trace on $E_{ij}$ of the maximal foliation of $\mathcal{E}_i$. Moreover, there is an analytic map $\phi_{ij}: \Lambda_{ij} \to \Lambda_i$ such that $\pi_i \circ i = \phi_{ij} \circ \pi_{ij}$, where $i$ is the inclusion $i: E_{ij} \to E_i$,

(4) for each $(i,j) \in I \times J, \mathcal{E}_{ij}$ is either equal to $\mathcal{E}^\circ_{ij}$ or is induced from $\mathcal{E}^\circ_{ij}$ by one of the three desingularization operations of Chapter II.

Now we are in a position to formulate the two conjectures:

**(III.5) Desingularization Conjecture.** *For any analytically parameterized family $\{X_\lambda\}, \lambda \in \Lambda$, of analytic vector fields in $S^2$ a finite number of desingularization steps can be chosen, such that in the resulting final collection $\{\mathcal{E}_i\}$ of foliated local vector field each $\mathcal{E}_i$ has only elementary limit periodic sets.*

**(III.6) Reduced Local Finite Cyclicity Conjecture.** *Each elementary limit periodic set $\Gamma$ of a foliated local vector field $\mathcal{E} = (E, \pi, \Lambda, X)$ has the finite*



*cyclicity property, i.e. there exist $\epsilon > 0$ and $K \in \mathbb{N}$ such that for each leaf $L \in \mathcal{F}$ the number of limit cycles of $\mathcal{E}$, at a E-Hausdorff distance to $\Gamma$ less than $\epsilon$, is bounded by $K$.*

**(III.7) Definition.** We say that a foliated local vector field $\mathcal{E}$ has the *finiteness property* if there exists a number $K$ such that each leaf of the maximal foliation of $\mathcal{E}$ contains less than $K$ limit cycles. In the case of families this definition coincides with the one given in the introduction.

**(III.8) Proposition.** *Desingularization Conjecture (III.5) together with Reduced Local Finite Cyclicity Conjecture (III.6) imply that each family on $S^2 \times \Lambda$ as in the introduction has the finiteness property.*

**Proof.** Suppose that after $k$ desingularization steps we have obtained a collection of foliated local vector fields whose limit periodic set is elementary. We want to prove that all foliated local vector field of this collection, preceding the final one, has the finiteness property.

It follows from (III.6) that each foliated local vector field $\mathcal{E}$ in the final collection has the finite cyclicity property. Using the proposition given in the appendix we conclude that each $\mathcal{E}$ has the finiteness property.

Suppose we have proved that in the $s$-th step of desingularization, $s \geq 1$, all foliated local vector fields have the finiteness property. Let $\mathcal{E}$ be one of the foliated local vector fields of the $(s-1)$-th step. Since the finiteness property is preserved by the three desingularization operations, it follows that each non-elementary limit periodic set of $\mathcal{E}$ has the finite cyclicity property. Each elementary limit periodic set has it too, by (III.6). So $\mathcal{E}$ has the finiteness property. Therefore, by decreasing induction, we obtain the required result. □

## 4. Basic Simplifications

In this chapter we want to eliminate families of vector fields $\{X_\lambda\}$, defined on $S^2 \times \Lambda$ as in the introduction, for which some $X_\lambda$ vanish identically, replacing them by a finite collection of families $\{X_{\tilde{\lambda}}\}, \tilde{\lambda} \in \tilde{\Lambda}_i$, not containing vanishing fields. This result is established in Proposition (IV.3).

Next, we want to replace foliated local vector fields $(E, \pi, \Lambda, X)$ such that codim $Z(X) = 1$ by foliated local vector fields $(E, \pi, \Lambda, \tilde{X})$ for which codim

$\mathcal{Z}(X) \geq 2$. This will be done by local division, under the assumption that $H^1(E, \mathbb{Z}_2) = 0$ (see Proposition (IV.5)).

Combining the two preceding results, it follows for instance that in order to establish the finiteness property for any analytic family on the 2-sphere $S^2$ it suffices to establish this property for the families with no exceptional values as parameters and such that codim $\mathcal{Z}(X) \geq 2$.

**(IV.1) Definition.** Given an analytic family of vector fields on $S^2, \{X_\lambda\}, \lambda \in \Lambda$, we say that $\lambda_\circ \in \Lambda$ is *an exceptional value* if $X_{\lambda_\circ} \equiv 0$.

We will show how to eliminate exceptional values. In what follows, families will be identified with local vector fields on the total space $S^2 \times \Lambda$ and denote $X$ instead of $X_\lambda$.

Take a point $(m_\circ, \lambda_\circ)$ in $S^2 \times \Lambda$ and local coordinates $(x, y)$ in a neighbourhood of it, $m_\circ = (x_\circ, y_\circ)$. In a neighbourhood of $(x_\circ, y_\circ, \lambda_\circ)$ we have:

$$X(x, y, \lambda) = X_\lambda(x, y) = F(x, y, \lambda)\partial/\partial x + G(x, y, \lambda)\partial/\partial y. \tag{4.1}$$

We develop $F$ and $G$, in a neighbourhood $W_{m_\circ} \times U_{\lambda_\circ}$ of $(x_\circ, y_\circ, \lambda_\circ)$, in series of $(x - x_\circ), (y - y_\circ)$:

$$\begin{aligned} F(x, y, \lambda) &= \sum_{l,r} a_{lr}(x - x_\circ)^l (y - y_\circ)^r \\ G(x, y, \lambda) &= \sum_{l,r} b_{lr}(x - x_\circ)^l (y - y_\circ)^r, \end{aligned} \tag{4.2}$$

where $a_{lr}, b_{lr}$ are analytic in $U_{\lambda_\circ}$.

We will denote by $J_{\lambda_\circ}^{m_\circ}$ the ideal generated by the germs of all $a_{lr}, b_{lr}$ in $\lambda_\circ$. This ideal does not depend on the choice of local coordinates.

Using an argument similar to that of the proof of Prop. 1 in [**R2**], (cf. also the book of Hervé [**H**]) we show that this ideal does not depend on $m_\circ$.

**(IV.2) Definition.** The ideal $J_{\lambda_\circ}^{m_\circ}$ will be called *the ideal of coefficients of X* and denoted $J_{\lambda_\circ}$.

**Remark.** Of course $J_{\lambda_\circ} = \mathcal{O}_{\lambda_\circ}$ when $\lambda_\circ$ is *not* an exceptional value.

Now, the ring $\mathcal{O}_{\lambda_\circ}$ being Noetherian, the ideal $J_{\lambda_\circ}$ is finitely generated. We will choose a system of generators $\bar{f}_1, \dots, \bar{f}_m$ of $J_{\lambda_\circ}$ and denote by $f_1, \dots, f_m$ their representants in an open set $G \subset \Lambda$.



It is always possible to choose the generators of $J_{\lambda_\circ}$ among the germs of the coefficients $a_{lr}, b_{lr}$. We do that.

One can easily deduce (cf. [**R2**], Prop. 1) that $S^2$ is a finite union of connected open sets $W_j$ such that there is a connected open neighbourhood $U$ of $\lambda_\circ, U \subset G$, for which we have:

$$X = f_1 X_1^j + \cdots + f_m X_m^j \text{ in } W_j \times U, \tag{4.3}$$

where $X_1^j, \dots, X_m^j$ are analytic vector field families in $W_j$, for each $W_j$, $j = 1, \dots, r$. Remark that $X$ vanishes identically for a certain $\lambda \in U$ iff $f_1(\lambda) = \cdots = f_m(\lambda) = 0$ and $X_i^j$ never vanishes identically in $W_j$. This is due to the choice of $f_1, \dots, f_m$. Now we apply Corollary 4.9 of [**BM**] to the functions $f_1, \dots, f_m$ and get a proper analytic surjection $\phi: \tilde{U} \to U$, where $\tilde{U}$ is a real analytic manifold, such that $\phi$ is a local diffeomorphism on an open dense set of $\tilde{U}$ and $\tilde{f}_1 = f_1 \circ \phi, \dots, \tilde{f}_m = f_m \circ \phi$ are locally normal crossings in $\tilde{U}$.

For each point $\tilde{\lambda}_* \in U$ take a system of local coordinates $\tilde{\lambda}_1, \dots, \tilde{\lambda}_k$ in a neighbourhood $\tilde{V} \subset \tilde{U}$ such that $\tilde{f}_1 = u_1 \cdot \tilde{\lambda}^{\alpha_1}, \dots, \tilde{f}_m = u_m \cdot \tilde{\lambda}^{\alpha_m}$ in $\tilde{V}$, with $u_1, \dots, u_m$ unities and $\alpha_1, \dots, \alpha_m \in \mathbb{N}^k$.

In the proof given in [**BM**] we can read that, up to a suitable permutation of $f_i$, the coordinates $\tilde{\lambda}$ can be chosen in such a way that $\alpha_1 \leq \cdots \leq \alpha_m$ (where $\gamma = (\gamma_1, \dots, \gamma_k) \leq \beta = (\beta_1, \dots, \beta_k)$ means $\gamma_i \leq \beta_i$ for $i = 1, \dots, k$).

Now, take the field $\tilde{X}(x, y, \tilde{\lambda}) = X(x, y, \phi(\tilde{\lambda}))$, defined in $S^2 \times \tilde{U}$. By (4.3) we know that:

$$\tilde{X} = \tilde{f}_1 \tilde{X}_1^j + \cdots + \tilde{f}_m \tilde{X}_m^j \quad \text{in each} \quad W_j \times \tilde{U}, j = 1, \dots, r. \tag{4.4}$$

Observe that $\tilde{X}_{\tilde{\lambda}}^j$ never vanish identically in $W_j$ and $\tilde{\lambda}$ is an exceptional value for $\tilde{X}$ iff $\tilde{f}_1(\tilde{\lambda}) = \cdots = \tilde{f}_m(\tilde{\lambda}) = 0$.

By the ordering of $\alpha_1, \dots, \alpha_m$, we know that $\tilde{f}_i/\tilde{f}_1, i = 1, \dots, m$, are analytic functions in $\tilde{V}$, so we can take the field $\bar{X} = \tilde{X}/\tilde{f}_1$ and it will be (cf. (4.4)) an analytic vector field in $S^2 \times \tilde{U}$. Take a finite subcovering $\tilde{V}_1, \dots, \tilde{V}_s$ such that $\phi(\tilde{V}_1), \dots, \phi(\tilde{V}_s)$ cover $U$ and denote by $\bar{X}^i$ the field $\bar{X}$ obtained in $\tilde{V}_i$. Take an $i \in \{1, \dots, s\}$.

The family $\bar{X}^i$, defined on $S^2 \times \tilde{V}_i$, is obtained from $X$ using the induction defined by $\phi|_{\tilde{V}_i}$, followed by a local division. Since $\phi$ is surjective, the finiteness property for $X|_{V_i}$ results from the same property for $\bar{X}^i$. One has to note that the argument about induction is not used in the sense of Chapter II, i.e. it does not go from $X|_{V_i}$ to $\bar{X}^i$, as in Chapter II, but in the inversed direction and the result follows from the surjective property of $\phi$.

Finally, we can summarize the result obtained as follows:

**(IV.3) Proposition.** *The finiteness property of analytic vector field families $X_\lambda$ on $S^2 \times \Lambda$, as defined in the introduction, is implied by the finiteness property for the families without exceptional values of parameters.*

Suppose now we have a local vector field $X$ on $E$ such that codim $\mathcal{Z}(X) = 1$. We will now proceed to eliminate this situation too. We will need the following:

**(IV.4) Lemma.** *Given a local vector field X on E such that $H^1(E, \mathbb{Z}_2) = 0$, defined by an open finite covering $\{U_i\}$ of E, suppose we have a collection $\{f_i, Y_i\}$ of analytic functions $f_i$ and analytic vector fields $Y_i$, defined in $U_i$, verifying:*

$$X_i = f_i \cdot Y_i \text{ in } U_i, i = 1, \dots, r \quad \text{and} \tag{4.5}$$

*for each pair $i, j$ of indices such that $U_i \cap U_j \neq \emptyset$, there is an analytic nonvanishing function $g_{ij}$ defined in $U_i \cap U_j$ such that*

$$f_i = g_{ij} \cdot f_j \text{ in } U_i \cap U_j. \tag{4.6}$$

*Then, after an appropriate change of signs of $Y_i$ when necessary, the collection $\{U_i, \pm Y_i\}$ defines a local vector field on E, which is the result of local division of X.*

**Proof.** Let $\psi_{ij}: U_i \cap U_j \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the multiplicative group $\{1, -1\}$, be the function $\psi_{ij} = sgn g_{ij}$. As $sgn g_{ij} \cdot sgn g_{jk} = sgn g_{ik}$ because of (4.6), the collection $\{\psi_{ij}\}$ is a Cech cocycle. Since $H^1(E, \mathbb{Z}_2) = 0$, there exist continuous functions $\phi_i: U_i \to \mathbb{Z}_2$ such that $\psi_{ij} = \phi_i \cdot \phi_j^{-1}$, where $\phi_j^{-1}$ stands for $1/\phi_j$.

Let us take $\hat{Y}_i = \phi_i \cdot Y_i$. These are still analytic fields in $U_i$. The equality (4.5) implies $g_{ij} \cdot Y_i = Y_j$ in $U_i \cap U_j$. Let us multiply both sides of the last equality by $\phi_i \cdot \phi_j$. We get:

$$\phi_i \cdot \phi_j \cdot g_{ij} \cdot Y_i = \phi_i \cdot \phi_j \cdot Y_j, \quad \text{hence} \quad \phi_j \cdot g_{ij} \cdot \hat{Y}_i = \phi_i \cdot \hat{Y}_j.$$

Multiplying by $\phi_i^{-1}$ we obtain:

$$\phi_j \cdot \phi_i^{-1} \cdot g_{ij} \cdot \hat{Y}_i = \hat{Y}_j, \quad \text{therefore} \quad \psi_{ij} \cdot g_{ij} \cdot \hat{Y}_i = \hat{Y}_j.$$



Since $\psi_{ij} = \psi_{ji}$, we have $\hat{g}_{ij} = \psi_{ji} \cdot g_{ij} > 0$ in $U_i \cap U_j$.

We have thus obtained a family $\{U_i, \hat{Y}_i\}$ which defines a local vector field, with transition functions $\hat{g}_{ij}$. □

**(IV.5) Proposition.** *Suppose X is a local vector field on E, such that $H^1(E, \mathbb{Z}_2) = 0$ and that codim $\mathcal{Z}(X) = 1$. Then there exists a local vector field $\tilde{X}$ on E, induced by local division of X and such that codim $\mathcal{Z}(\tilde{X}) \geq 2$.*

**Proof.** Take a point $a \in \mathcal{Z}(X)$ and choose a local coordinate system $(U_a, \psi)$, $\psi(a) = 0$, in a neighbourhood $U_a$ of $a$, such that $X$ is written $X = \sum F_i \partial/\partial x_i$ with $F_i$ analytic in $\psi(U_a)$.

Take the germs $(F_i)_0, i = 1, \dots, n$. The ring $\mathcal{O}$ is a ring with unique factorization (cf. [T]), so there is the greatest common divisor of $(F_1)_0, \dots, (F_n)_0$, defined up to a unit. Take one and name it $f_a$. Take $f_a \equiv 1$ if $(F_i)_0, i = 1, \dots, n$ are relatively prime.

Now we have $(F_i)_0 = f_a \cdot (\tilde{F}_i)_0, i = 1, \dots, n$ and $(\tilde{F}_1)_0, \dots, (\tilde{F}_n)_0$ relatively prime.

The fact that the germs $(\tilde{F}_i)_0, i = 1, \dots, n$ are relatively prime is equivalent to the fact that the set $\{\tilde{F}_1 = \dots = \tilde{F}_n = 0\}$ is of codimension strictly greater than one in 0.

We can choose a neighbourhood $V$ of $O, V \subset \psi(U_a)$, in which the germs above have representants $F_i, f_a, \tilde{F}_i$ such that, at each point of $\mathcal{Z}(X) \cap V = \{F_1 = \dots = F_n = 0\}$, the germs of $\tilde{F}_i$ are relatively prime. In $V$ we have $F_i = f_a \tilde{F}_i, i = 1, \dots, n$. Put $V_a = \psi^{-1}(V)$. The open sets $V_a$ form a covering of $E$; choose a finite subcovering $V_1, \dots, V_r$.

Let $X = \sum F_i^j \partial/\partial x_i$ in $V_j$. Take $Y_j = \psi_*^{-1}(\sum \tilde{F}_i^j \partial/\partial x_i)$, where $\tilde{F}_i^j$ are the relatively prime functions defined above, taken for $V_j$ and let $f_j$ denote the greatest common divisor of $F_i^j, i = 1, \dots, n$. Put $h_j = f_j \circ \psi$. The collection $\{h_j, Y_i^j\}, j = 1, \dots, r$, verifies the conditions of Lemma (IV.4) because we have $F_i^j = f_j \cdot \tilde{F}_i^j$ in $\psi(V_j)$ and $F_i^j = f_k \cdot \tilde{F}_i^k$ in $\psi(V_k)$.

Thus, again by the uniqueness of factorization in $\mathcal{O}$, we conclude that locally, in $V_j \cap V_k$, the greatest common divisors $h_j, h_k$ differ by a nonvanishing function $g_{ij}$. Since the conditions of lemma (IV.4) are satisfied, there exists a local vector field $Y$ obtained from $X$ by local division and such that codim $\mathcal{Z}(Y) \geq 2$. □

## Appendix

In [**R1**] it was proved that in any analytic compact family, the finite cyclicity property implies the finiteness property. In chapter III we have used the same implication for foliated local vector fields. We give now a precise formulation and a proof of this result:

**Proposition.** *Let $\mathcal{E} = (E, \Lambda, \pi, X)$ be a foliated local vector field and $\mathcal{F}$ its associated foliation as in chapter II. Suppose that any periodic limit set in $\mathcal{E}$ has the finite cyclicity property. Then $\mathcal{E}$ has the finiteness property (see the definitions in chapter III).*

**Proof.** Let $d$ be any metric defining the topology of $E$. Let $\mathcal{C}(E)$ be the set of all compact subsets in $E$. We recall that $\mathcal{C}(E)$ endowed with the Hausdorff distance $d_h$ is also a compact metric space [**K**]. Now let $\mathcal{C}(\mathcal{F})$ be the compact subset in $\mathcal{C}(E)$, defined as the closure of the set

$$\{\bar{L} | L \text{ is a leaf of } \mathcal{F}\}.$$

Let also $\mathcal{C}(LC) \subset \mathcal{C}(E)$ be the closure of the subset of all limit cycles of $\mathcal{E}$. This last subset $\mathcal{C}(LC)$ contains every periodic limit set of $\mathcal{E}$. The assumption about the finite cyclicity property can be read as follows:

*For any $\Gamma \in \mathcal{C}(LC)$, there exist $\epsilon_\Gamma > 0$ and $K(\Gamma) \in \mathbb{N}$ such that, for every leaf $L$ of $\mathcal{F}$, the number of limit cycles $\gamma$ in $L$, with $dist_h(\gamma, \Gamma) < \epsilon_\Gamma$, is less than $K(\Gamma)$.*

Let $W(\Gamma)$ be the open $\epsilon_\Gamma$-neighbourhood of $\Gamma$ in $\mathcal{C}(E)$. Now, for any $M \in \mathcal{C}(\mathcal{F})$ we consider the following subset of $\mathcal{C}(E)$:

$$\mathcal{M}_M = \{\Gamma \in \mathcal{C}(CL) | \Gamma \subset M\}$$

Clearly $\mathcal{M}_M$ is a closed subset and so is a compact subset in $\mathcal{C}(E)$. By compactness, there exists a finite sequence $\Gamma_1, \ldots, \Gamma_s$ in $\mathcal{M}_M$ such that $W = W(\Gamma_1) \cup \ldots \cup W(\Gamma_s)$ is an open neighbourhood of $\mathcal{M}_M$ in $\mathcal{C}(E)$. We have the following property (*):

*There exists a neighbourhood $\mathcal{W}$ of $M$ in $\mathcal{C}(E)$ such that, for all $L$ with $\bar{L} \in \mathcal{W}$ and every limit cycle $\gamma \subset L$, we have $\gamma \in W$.*



Suppose that the property (*) is false. Then we can find a sequence $(L_i)$ with $(\bar{L}_i)$ converging to $M$ in $\mathcal{C}(E)$ such that for each $i$ there exists a limit cycle $\gamma_i \subset L_i$ with $\gamma_i \notin W$. Now, because $\mathcal{C}(E)$ is compact, we can choose the sequences $(L_i)$ and $(\gamma_i)$ such that $(\gamma_i)$ converges to some $\Gamma \subset M$. By definition this $\Gamma$ belongs to $\mathcal{M}_M$, which contradicts the property $\gamma_i \notin W$.

Now consider $\mathcal{W}$ with the property (*). For any $L$ with $\bar{L} \in \mathcal{W}$, every limit cycle of $L$ belongs to $W = W(\Gamma_1) \cup \ldots \cup W(\Gamma_s)$. So the number of such limit cycles is less than $K = K(\Gamma_1) + \cdots + K(\Gamma_s)$. Finally we have proved the following:

*For any $M \in \mathcal{C}(\mathcal{F})$ there exist a neighbourhood $\mathcal{W}_M$ in $\mathcal{C}(E)$ and a number $K_M$ such that any leaf of $\mathcal{F}$ with $\bar{L} \in \mathcal{W}_M$ contains less than $K_M$ limit cycles.*

The finiteness property follows now from the compactness of $\mathcal{C}(\mathcal{F})$: we can extract a finite subcovering of $\{\mathcal{W}_M\}_M$, say $\{\mathcal{W}_{M1}, \ldots, \mathcal{W}_{M1}\}$ such that $\mathcal{C}(\mathcal{F}) \subset \mathcal{W}_{M1} \cup \ldots \cup \mathcal{W}_{M1}$. A uniform bound for the number of limit cycles in each leaf of $\mathcal{F}$ is given by $K = \text{Sup}\,\{K_{M1}, \ldots, K_{M1}\}$. □

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