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Abundance of generic homoclinic tangencies in real-analytic families of diffeomorphisms

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Abstract. We consider one-parameter families of two-dimensional diffeomorphisms with homoclinic tangencies. Various authors considered the dynamical complexities due to such tangencies satisfying certain nondegeneracy conditions. In this paper we provide methods to actually verify, for real analytic families, that there are homoclinic tangencies which satisfy these (generic) nondegeneracy generic conditions.

1. Introduction

We consider one-parameter families φ_μ of 2-dimensional diffeomorphisms with saddle point p_μ, and stable and unstable separatrices W^s_μ and W^u_μ. A homoclinic tangency is a tangency of $W^s_{\bar{\mu}}$ and $W^u_{\bar{\mu}}$ for some $\mu = \bar{\mu}$. Without loss of generality we assume that the eigenvalues of $d\varphi_\mu$ at p_μ are positive (otherwise we consider φ^2_μ instead of φ_μ). Also it will be convenient to restrict to just one branch of each of the separatrices which, from now on, are denoted by W^s_μ and W^u_μ.

Under *generic assumptions* such a tangency leads to interesting dynamical complications for μ near $\bar{\mu}$:

- formation of infinitely many periodic orbits [B,1935];
- period doubling sequences [YA,1983];
- persistent tangencies and infinitely many coexisting periodic attractors or repellers [N,1979], [PT,1992];
- Hénon-like strange attractors or repellers [BC,1991],[MV,1991].

Although these generic conditions can sometimes be verified, usually this is very hard, see [GH,1983]. In this paper we show that for real analytic systems these generic conditions follow from seemingly much weaker conditions which can be easily verified in concrete examples. In order to state our results we first

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formulate the usual generic conditions and than our alternative conditions.

The Generic Conditions

There are three generic conditions for homoclinic tangencies, which are informally denoted by:

- C^3-linearizability;
- first order tangency (parabolic);
- positive speed.

They can be described more formally as follows. First, for a tangency of W^s_μ and W^u_μ for $\mu = \bar{\mu}$ we require that φ_μ can be C^3-linearized near p_μ for μ near $\bar{\mu}$. This linearization must be C^3 in both the two 'space variables' and μ. If $\lambda(\mu)$ and $\sigma(\mu)$ are the eigenvalues of $d\varphi_\mu$ at p_μ, this means that we should avoid

$$\alpha(\mu) = -\ln\sigma(\mu)/\ln\lambda(\mu)$$

taking at $\bar{\mu}$ values in some locally finite subset of $\mathbb{R}$ [S,1958].

At the tangency we construct μ-dependent coordinates x and y such that

$$W^u_\mu = \{y = 0\},$$
$$W^s_\mu = \{y = M(x,\mu)\},$$

for some function M. Assuming the tangency to be at $x = y = 0$, we have that $M(0,\bar{\mu}) = 0$ and $M'(0,\bar{\mu}) = 0$ (here and in what follows, M' means the derivative of M with respect to its first variable). The condition of first order tangency means that $M''(0,\bar{\mu}) \neq 0$ and positive speed means that $\partial M/\partial\mu(0,\bar{\mu}) \neq 0$, i.e. the speed of W^s with respect to W^u, when moving μ, is nonzero.

Alternative Conditions

Here we also have three conditions:

- $\varphi_\mu(x,y)$ is a real analytic function of x, y, and μ;
- the function $\alpha(\mu) = -\ln\sigma(\mu)/\ln\lambda(\mu)$, as defined before, is not constant;
- there is an *inevitable tangency* in the sense defined below.

In order to define this notion of inevitable tangency, we consider an open disc $U \subset \mathbb{R}^2$ and values $\mu_1 < \mu_2$ (or $\mu_2 < \mu_1$). We say that $\{\varphi_\mu\}$ has an inevitable tangency in U between μ_1 and μ_2 if:

- there are arcs $\gamma^{u/s}_\mu$, bounding in U open sets $a^{u/s}_\mu$, which are part of $W^{u/s}_\mu$;
- $\Gamma^{u/s} = \bigcup_{\mu_1 \le \mu \le \mu_2} \gamma^{u/s}_\mu \times \{\mu\}$ is a real analytic surface;

- $\partial\gamma^u_\mu \cap \overline{a^s_\mu}$ and $\partial\gamma^s_\mu \cap \overline{a^u_\mu}$ are empty;
- $\overline{a^s_{\mu_1}} \cap \overline{a^u_{\mu_1}} = \varnothing$ and $a^s_{\mu_2} \cap a^u_{\mu_2} \neq \varnothing$.

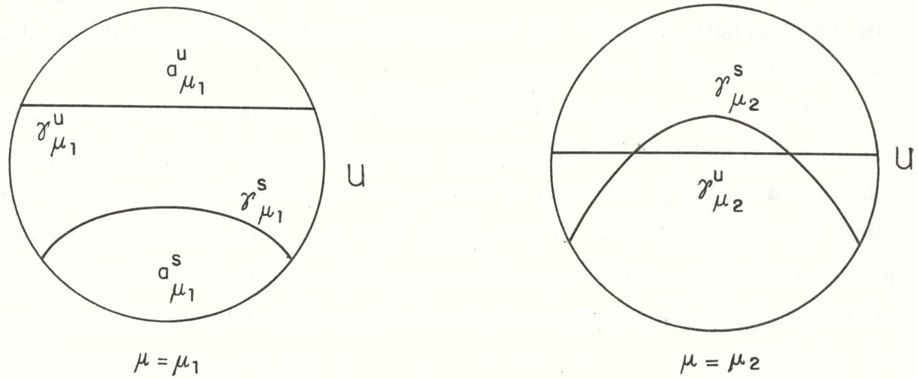

Figure 1. Inevitable tangency.

It should be clear that if the above conditions are satisfied, then there has to be some homoclinic tangency in U for some value of μ between μ_1 and μ_2 — this explains the term 'inevitable tangency'.

The first two conditions, analyticity and α not constant, are usually satisfied for concrete systems (and easy to verify). The third condition may be harder to verify, but if one cannot, one will in general not be able to show that there is *any tangency at all.* Our main result now is the following:

Theorem. *If the three alternative conditions are satisfied, then there are tangencies where the three generic conditions are satisfied.*

In the above notation, such generic tangencies will even occur in U for values of μ between μ_1 and μ_2. The following sections are devoted to the proof of the above theorem; in the remaining part of this introduction we give an example how our result can be applied to a concrete equation.

We consider the Duffing equation

$$u' = v$$
$$v' = u - u^3 + \varepsilon(\gamma\cos(\omega t) - \delta v)$$

in combination with the analysis of it given in [GH,1983]. This analysis is based on the Melnikov method; for references on this method we also refer to [GH,1983]. For $\gamma = 0$ and ε small, the quotient of the eigenvalues of the singularity at $u = 0$, $v = 0$ is not constant as a function of δ.

For $\gamma \neq 0$ we consider the Poincaré, or period, map. This map is real analytic in u, v, and the parameters, and it has a saddle p near $u = v = 0$. For $\varepsilon \neq 0$, γ, and $\omega \neq 0$ fixed, there is a function $\alpha(\delta)$, associated with this saddle, as in the second alternative condition. For $\gamma = 0$, $-\alpha(\delta)$ is the quotient of the eigenvalues of the singularity at $(0, 0)$ and hence not constant as function of δ. So also for small $\gamma \neq 0$, $\alpha(\delta)$ is not constant.

It follows from the Melnikov analysis, mentioned before, that for $\omega \neq 0$ and small $\varepsilon \neq 0$ and $\gamma \neq 0$ fixed, there is a value $\bar{\delta}$ for which there is a tangency of the separatrices of p with order of contact one and which unfolds in an intersecting way, see subsection (4.2). This implies that there is an inevitable tangency, so that by our main result there are also tangencies satisfying the generic conditions, implying all the above mentioned dynamical complications.

2. Local Properties of Tangencies

In this section we recall some notions associated with tangencies of real analytic curves in the plane.

2.1 Order of tangency

Let l and m be real analytic curves in $\mathbb{R}^2$ which have a point of tangency. Then there are local real analytic coordinates such that $l = \{y = 0\}$ and $m = \{y = M(x)\}$ with $M(0) = M'(0) = 0$, assuming that $x = y = 0$ is the point of tangency. We say hat the order of tangency is k if $M^{(i)}(0) = 0$ for $i = 0, \ldots, k$ and $M^{(k+1)}(0) \neq 0$. In that situation we have

$$M(x) = cx^{k+1} + \text{h.o.t., so}$$
$$M'(x) = c \cdot (k+1) \cdot x^k + \text{h.o.t..}$$

Hence we have that

$$|M'(x)|/|M(x)|^{k/(k+1)}$$

has a non-zero limit for $x \to 0$. This leads to the following 'coordinate free' property:

For a point $q\epsilon m$ let $d(q)$ be the distance from q to l and let $T(q)$ be the angle between $T_q(m)$ and $T_{q'}(l)$, where $q'\epsilon l$ is the point in l nearest to q. Then

$$T(q)/(d(q))^{k/(k+1)}$$

has a non-zero limit as q approaches the point of tangency (distances and angles are defined with respect to the coordinate system). If we use different (but at least

C^2) coordinates, we get corresponding functions $\tilde{d}(q)$ and $\tilde{T}(q)$ such that, both

$$\tilde{d}(q)/d(q) \text{ and } \tilde{T}(q)/T(q)$$

have a non-zero limit. This implies that also

$$\tilde{T}(q)/(\tilde{d}(q))^{k/(k+1)}$$

has a non-zero limit as q approaches the tangency.

2.2 Unfolding a tangency

We assume l_μ and m_μ to be real analytic curves in $\mathbb{R}^2$, depending analytically on μ and such that, for $\mu = 0$, they have a tangency. We can choose μ-dependent coordinates x and y such that $l_\mu = \{y = 0\}$ and $m_\mu = \{y = M(x, \mu)\}$, for some function M. We may assume the tangency to be at $x = y = 0$. Also we assume the order of the tangency to be *odd* so that locally m_0 is on one side of l_0 and vice versa. So we may assume that $M(x, 0) = -x^{k+1} + \text{h.o.t.}$ with k odd. We shall consider $\max_x M(x, \mu)$, or rather the local maxima for x near zero, as a function of μ, which we assume not de be constant. In order to analyze these local maxima we define the set $C = \{(x, \mu) | M'(x, \mu) = 0\}$. This, being an analytic set, consists, near zero, of a finite number of analytic curves (see the curve selection lemma [BC,1957]) which we denote by $c_1^-, \ldots, c_s^-$ and $c_1^+, \ldots, c_t^+$ with c_i^-, c_i^+ in $\{\mu \leq 0\}$, $\{\mu \geq 0\}$ respectively. Each of these curves can be parameterized analytically by a root of μ, so along each of these curves we have

$$M|c_i^\pm = d_i^\pm \cdot |\mu|^{\alpha_i^\pm} + \text{h.o.t.}$$

with $\alpha_i^\pm$ rational. The local maximum of M, as a function of x for fixed μ must be on one of these arcs. Let c_1^- and c_1^+ be dominating arcs in the following sense:

- if $d_1^+ < 0$ then all d_i^+ are negative and for all i we have $\alpha_i^+ \leq \alpha_1^+$ and if $\alpha_i^+ = \alpha_1^+$ then $d_i^+ \leq d_1^+$;
- if $d_1^+ > 0$ then for all i such that $d_i^+ > 0$, $\alpha_i^+ \geq \alpha_1^+$ and if $\alpha_i^+ = \alpha_1^+$ then $d_i^+ \leq d_1^+$.

(For c_1^- the conditions are analogous.) In this case we call α_1^+ the *positive exponent of motion* (of m_μ with respect to l_μ). The negative exponent of motion is defined similarly as α_1^-. In this definition the analyticity is important. However, once this exponent is defined, it makes sense for arbitrary coordinates $\tilde{x}$ and $\tilde{y}$ (which are at least C^1) and has the same interpretation: if $l_\mu = \{\tilde{y} = 0\}$ and

$m_\mu = \{\tilde{y} = \tilde{M}(\tilde{x}, \mu)\}$ are our analytic curves, but now expressed with respect to arbitrary C^1-coordinates for which the tangency, at $\mu = 0$, is still in the origin, and where $\tilde{M}(\tilde{x}, 0)$ has a local maximum in zero, the positive exponent of motion is the unique number α such that

$$\max_x \tilde{M}(\tilde{x}, \mu)/|\mu|^\alpha$$

is bounded and bounded away from zero as μ approaches zero from above (if there are no coordinates with respect to which l_μ and m_μ are analytic, this exponent may not be defined).

We finally note that it can happen that $\max_x M(x, \mu)$ is constant for $\mu \geq 0$ only, e.g. $M(x, \mu) = (\mu^3 - x^2)^2$. In that case the positive exponent is ∞.

3. Linearizations of saddle points and λ-lemma type estimates.

3.1 Linearizations

We consider a C^∞ or real analytic diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\varphi(x, y) = (\sigma x, \lambda y) + \text{h.o.t.}$ with $0 < \lambda < 1 < \sigma$. Linearizing φ means finding coordinates $\tilde{x}$ and $\tilde{y}$ (near the origin) with respect to which the higher order terms 'h.o.t.' of φ vanish. It is important for our constructions that $\tilde{x}$ and $\tilde{y}$ have a high degree of differentiability. According to [S,1958] there is, for each L, a locally finite subset of $\mathbb{R} - \{0\}$ such that, whenever $\ln\sigma/\ln\lambda$ does not belong to this subset, there are linearizing coordinates which are C^L.

In order to find such $\tilde{x}$ and $\tilde{y}$, one first considers the formal problem: find Taylor expansions for $\tilde{x}$ and $\tilde{y}$ which make the higher order terms zero up to a certain order. This is achieved by induction: if in the higher order terms all summands up to order $i+j-1$ are removed, then, in order to remove a term $x^i y^j$ in the first, respectively second, component of φ, we need $(i-1)\cdot\ln\sigma + j\cdot\ln\lambda$, respectively $i\cdot\ln\sigma + (j-1)\cdot\ln\lambda$, to be nonzero. In order to obtain a C^L-linearization one needs to do the formal linearization up to order $\tilde{L}(L, \ln\sigma/\ln\lambda)$, where $\tilde{L}$ is continuous in the second variable. So the set of ratio's $\ln\sigma/\ln\lambda$ which have to be excluded is indeed locally finite.

For our constructions we need a slightly stronger result: we need a C^L-linearization whose L-jets are C^∞ along the unstable separatrix (or along the stable separatrix or along both, in which cases there are similar arguments). On the formal level this means that we have to remove the terms $x^i y^j$, with $j \leq \tilde{L}$

and all i, in φ. For this we need $i\cdot\ln\sigma + j\cdot\ln\lambda \neq 0$ for $j \leq \tilde{L}$ and all i; this still corresponds with a locally finite subset of $\mathbb{R} - \{0\}$. Combining this with the methods of proof in [S,1958] and [T,1971], we obtain the following:

Proposition. *For each L there is a locally finite subset $S_L \subset \mathbb{R} - \{0\}$ such that for $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ as above and $\ln\sigma/\ln\lambda \notin S_L$, there are C^L-linearizing coordinates whose L-jets are C^∞ along the unstable separatrix. Moreover, when φ, λ, and σ depend on a parameter μ and $\ln\sigma(\bar{\mu})/\ln\lambda(\bar{\mu}) \notin S_L$, then the linearizing coordinates are C^L-functions of x, y, and μ whose L-jets are C^∞ along $\bigcup_\mu W^u_\mu \times \{\mu\}$ for μ near $\bar{\mu}$.*

3.2 λ-lemma type estimates

We consider a linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(x, y) = (\sigma x, \lambda y)$, with $0 < \lambda < 1 < \sigma$; we define as before $\alpha = -\ln\sigma/\ln\lambda \in \mathbb{R}_+$. Let $W = \{y = f(x)\}$ be a smooth curve intersecting W^s at $(0, f(0))$, with $f(0) > 0$. We shall consider the curves $\varphi^n(W) = \{y = (P^n f)(x)\}$ where P is the 'graph transform'. According to the λ-lemma [P,1969] these curves converge, in the C^1-sense to the unstable separatrix, i.e. to the x-axis. Here we also describe the convergence of the higher derivatives.

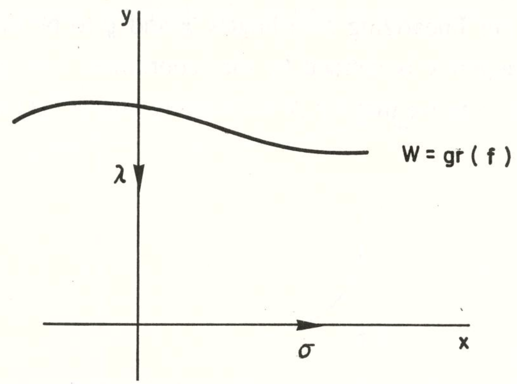

Figure 2. Curve W intersecting W^s.

We want to estimate the derivatives of $P^n f$ for $n \to \infty$. It is easy to see that if $\varphi^{-n}(x, 0) = (\bar{x}, 0)$, then

$$(P^n f)^{(i)}(x) = \lambda^n \cdot \sigma^{-ni} \cdot f^{(i)}(\bar{x}) = \lambda^{n(1+\alpha i)} \cdot f^{(i)}(\bar{x}),$$

where we used $\sigma^{-1} = \lambda^\alpha$. For $i = 0$ this gives $(P^n f)(x) = \lambda^n \cdot f(\bar{x})$. Using that for $n \to \infty$, $\bar{x}$ converges to zero, we find

$$\lim_{n\to\infty} (P^n f)^{(i)}(x)/((P^n f)(x))^{1+\alpha i} = f^{(i)}(0)/(f(0))^{1+\alpha i}.$$

Let now $\tilde{x}$ and $\tilde{y}$ be another coordinate system, with respect to which φ is not linear, and let $\tilde{f}_n$ be the functions such that $\varphi^n(W) = \{\tilde{y} = \tilde{f}_n(\tilde{x})\}$. If we assume that for some K, the K-jets of $(x - \tilde{x})$ and of $(y - \tilde{y})$ are zero along the x-axis, i.e., along the local unstable separatrix, then it is clear that the l-jets of $\tilde{f}_n$ and $P^n f$ in x differ at most by a term of the order $((P^n f)(x))^{K-l}$, uniformly in n. So for $K > 1 + i(1+\alpha)$, the above estimate implies that

$$\lim_{n\to\infty} \tilde{f}_n^{(i)}(x)/(\tilde{f}_n(x))^{1+\alpha i} = f^{(i)}(0)/(f(0))^{1+\alpha i}.$$

We can see from this that the higher derivatives of $\tilde{f}_n$ converge faster to zero than the lower derivatives. This will play an important role in later arguments.

3.3 λ-lemma type estimates – singular case

We assume φ to be as before, i.e. $\varphi(x, y) = (\sigma x, \lambda y)$. Let l and m be curves, tangent to W^s and W^u respectively in $(0, \bar{y})$ and $(\bar{x}, 0)$ with $\bar{x}, \bar{y} > 0$, and let L and M be the respective orders of tangency as defined before. The order of tangency is only defined for *analytic* curves, so we assume l and m to be analytic, but we allow our linearizing coordinates x and y to be only C^2, in which case the order of tangency is defined by the 'coordinate free' property in subsection (2,1). We want to prove that for N sufficiently big, $\varphi^N(l)$ and m have transversal intersections.

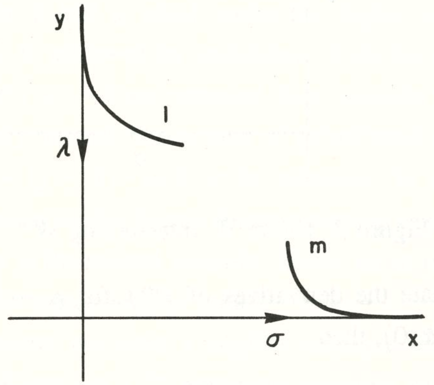

Figure 3. The curves l and m.

We represent l as the graph of a continuous function $f : [0, \varepsilon) \to \mathbb{R}_+$ which is

C^2 on $(0, \varepsilon)$. Due to the order of tangency, $f'(x)$, for positive x near zero, is of the order $x^{-L/(L+1)}$. Let $P^n f$, as before, be such that $\varphi^n(\mathrm{gr}(f)) = \mathrm{gr}(P^n f)$. Near $\bar{x}$ we then have:

$P^n f$ is of the order λ^n;

$(P^n f)'$ is of the order $\lambda^n \cdot \sigma^{-n} \cdot (\sigma^{-n})^{-L/(L+1)}$.

Using $\lambda^\alpha = \sigma^{-1}$, this last expression becomes $(\lambda^n)^{1+\alpha/(L+1)}$. This means that $(P^n(f))'/P^n(f)$, for x near $\bar{x}$, converges to zero as N tends to infinity. On the other hand, if we write m as the graph $m = \{y = g(x)\}$ of g, then, for $x \to \bar{x}$, $g'(x)/g(x)$ tends to infinite, because the tangency of m and W^u is of finite and positive order, see subsection (2,1). This implies that for n sufficiently big, the graphs of g and $P^n f$ intersect transversally near $\bar{x}$.

3.4 Angle of crossing W^s

We consider again a C^∞ diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) \mapsto (\sigma x, \lambda y) + \text{h.o.t.}$, with $0 < \lambda < 1 < \sigma$. When applying the λ-lemma type estimates in combination with linearizations, we need a consistent way of measuring, or rather comparing, the *angles of crossing* W^s for curves W intersecting W^s transversally. We want this definition of angle of crossing to be such that W and $\varphi(W)$ have the same angle of crossing W^s (this means that for curves intersecting W^s at the origin, the angle of crossing is not defined).

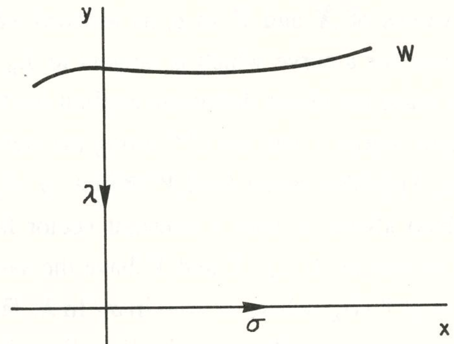

Figure 4. W intersecting W^s.

Let $\mathcal{L} = \{(p, L) | p \epsilon W^s,\ L \subset T_p(\mathbb{R}^2)$ a one dimensional subspace $\}$ and let $\Phi : \mathcal{L} \to \mathcal{L}$ be the map induced by $d\varphi$. At the fixed point $(0, T_0(W^u))$, $d\Phi$ has eigenvalues λ and $\lambda \cdot \sigma^{-1}$. Assuming that $\lambda \cdot \sigma^{-1}$ is not a power of λ, or that $-\ln\sigma/\ln\lambda$ is not an integer, there is a unique Φ-invariant curve, which is C^∞, and which is tangent to the eigenvector with eigenvalue λ. This invariant curve defines a unique C^∞-field of transverse directions to W^s. We denote the transverse direction at $p\epsilon W^s$ by L_p; if W intersects W^s at p and is tangent to L_p, we shall define its angle of crossing to be zero. Note that the present conditions on λ and σ are satisfied whenever there are C^1 linearizing coordinates whose 1-jets are C^∞ along the stable separatrix, see (3.1).

Next we consider the space $\mathcal{X} = \{(p, X)|X\epsilon L_p\}$ and the map on $\mathcal{X}$ induced by $\sigma^{-1}\cdot d\varphi$. This map has a line of fixed points, consisting of the vectors X in $L_{(0,0)}$, each with a 1-dimensional smooth stable separatrix. Each of these stable separatrices corresponds to a smooth vector field along W^s, which is multiplied by σ when applying $d\varphi$ and which is contained in the transverse directions L_p. For each $p\epsilon W^s$, each $X\epsilon L_p$ can be uniquely extended to a smooth vector field along W^s, corresponding to such a separatrix; we denote such vector field by $\tilde{X}$. If $\tilde{y}$ is a coordinate along W^s which linearizes $\varphi|W^s$, then we get a $d\varphi$-invariant vector field along W^s by putting $\bar{X}(\tilde{y}) = \tilde{y}^{-\alpha}\cdot\tilde{X}(\tilde{y})$. Also we can construct a vector field $\bar{Y}$ along W^s, which is tangent to W^s and which is $d\varphi$-invariant. Again we can fix $\bar{Y}$ in one point but in this case there is a canonical choice: we take $\bar{Y}$ so that the time one map of $\bar{Y}$ is the restriction of φ to W^s.

For $\bar{X}$ and $\bar{Y}$ as above we define the angle of crossing of a curve W tangent vector $Z = a\cdot\bar{X}_p + b\cdot\bar{Y}_p$, where $(0,0)\neq p\epsilon W^s$, $Z\epsilon T_p(W^s)$, and where $\bar{X}_p$ and $\bar{Y}_p$ denote the values of $\bar{X}$ and $\bar{Y}$ in p, as $\arctan(-b/a)$. (So the angle of crossing is indeed zero for a curve which is tangent to L_p.)

For later use we make the above definition explicit for the case where we have linearizing coordinates whose 1-jets are C^∞ along the stable separatrix. We then have $\varphi(x,y) = (\sigma x, \lambda y)$. The vector field $\bar{Y}$ is $\ln\lambda\cdot y\cdot\partial_y$. A vector field along W^s like $\tilde{X}$, as defined above, is then a constant vector field in the x-direction. Now we choose $\bar{X}$ so that in $(0,1)$, $\bar{X}$ and $\bar{Y}$ have the same length. This means that $\bar{X}(0,y) = \ln\lambda\cdot y^{-\alpha}\cdot\partial_x$, where $\alpha = -\ln\sigma/\ln\lambda$. From this it follows that the angle of crossing for a curve W, given in these linearizing coordinates as the graph of f, is $f^{(1)}(0)/(f(0))^{1+\alpha}$. This should be compared with the formulas in (3.2).

4. Approximating Tangencies

It is known that in a one-parameter family of diffeomorphisms a homoclinic tangency is usually approximated by other homoclinic tangencies, e.g. see [PT,1992]. This will play an important role in the proof of our main theorem. In the present section we analyze this phenomenon of approximating tangencies. For this we first need to distinguish various types of homoclinic tangencies and their unfoldings.

4.1 Homoclinic Tangencies of Type A, B, C, and D

Let φ_μ be a one-parameter family of diffeomorphisms in $\mathbb{R}^2$ as in the introduction, i.e. real analytic, with a saddle point p_μ and a homoclinic tangency for $\mu = \bar{\mu}$. We assume the point of tangency to be isolated and the tangency to be of odd order, see (2.2). This means that near the tangency W^s is locally on one side of W^u and vice versa. Now we consider two points on the orbit of tangency near the saddle point $p_{\bar{\mu}}$, one on the local stable and one on the local unstable separatrix, which we denote by q and r. We have a tangency of type A, B, C, or D depending on whether W^u, near q, is on the side of r or not and depending on whether W^s, near r, is on the side of q or not. The complete convention is indicated in figure 5.

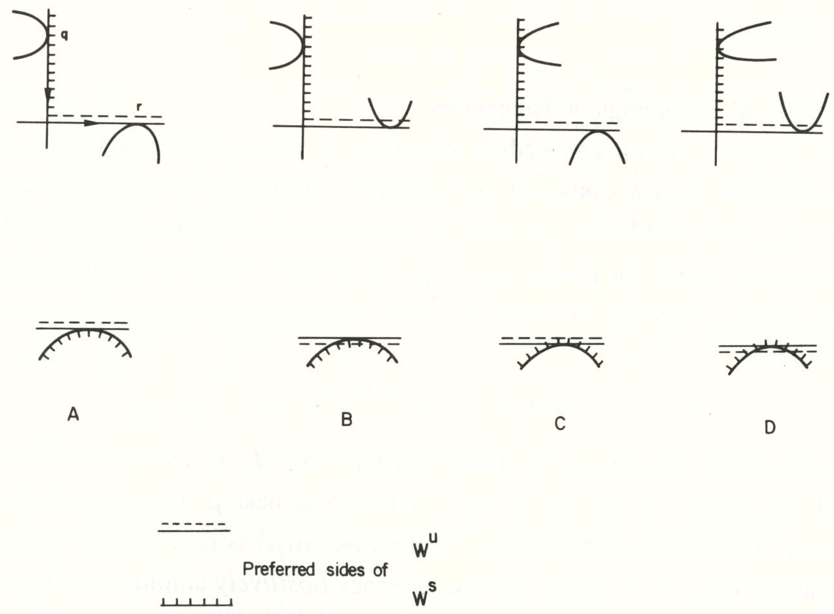

Figure 5. The types A, B, C, and D as defined near $p_{\bar{\mu}}$ and as defined with preferred sides.

In some situations it is more convenient to use a different but equivalent definition in terms of *preferred sides*. As indicated in the figure the preferred side of W^s is the side which is, near p, in the direction of W^u and vice versa (remember, W^s and W^u are each only one branch of the stable and unstable separatrices). Having these preferred sides, the types A, B, C, and D can also be distinguished according to whether, at the tangency, the preferred side of W^s is in the direction of W^u or not and vice versa. This is also indicated in figure 5.

In this whole description we assume that we have a tangency of odd order. We now indicate why there are such tangencies whenever there is an inevitable tangency. Using the notation of the introduction, indicated in figure 1, and assuming $\mu_1 < \mu_2$, we consider the lowest value of μ such that $\overline{a^s_\mu}$ and $\overline{a^u_\mu}$ have a point in common. It is clear that for this value of μ we have a tangency and that W^s and W^u don't cross each other. The order of tangency has to be finite because otherwise γ^s_μ and γ^u_μ would coincide, which would contradict the condition that $\partial\gamma^s_\mu \bigcap \overline{a^u_\mu}$ is empty, so it is a tangency of odd order. We note that we have also such a tangency of odd order if we take the infimum of the μ values for which the intersection of the interiors or a^s_μ and a^u_μ is not empty. These two tangencies may coincide.

4.2 Types of Unfoldings of Tangencies

For a tangency as in the preceding subsection in a one parameter family of diffeomorphisms we can construct real analytic μ-dependent coordinates such that $W^u = \{y = 0\}$ and $W^s = \{y = M(x, \mu)\}$ for some function M. Assuming that the tangency occurs for $\mu = \bar{\mu}$ at $x = y = 0$, we have $M(0, \bar{\mu}) = M'(0, \bar{\mu}) = 0$. Since we are assuming the tangency to be one-sided we may assume that, for some odd k

$$M(x, \bar{\mu}) = -x^{k+1} + \text{h.o.t.}.$$

See also (2.2). We denote the local maximum of M, as a function of x for μ fixed, by $m(\mu)$. As we saw in (2.2), for $\mu \geq \bar{\mu}$ near $\bar{\mu}$ we have $m(\mu)$ either constant, increasing, or decreasing. In the case $m(\mu)$ is decreasing, respectively increasing, for $\mu \geq \bar{\mu}$, we say that the tangency *positively* unfolds in a *detaching*, respectively *intersecting*, way. In the same way one defines the way it unfolds *negatively* which depends on $m(\mu)$ for $\mu \leq \bar{\mu}$.

At the end of the previous subsection we indicated how an inevitable tangency leads to two tangencies (which might coincide). From the above description and

the present definitions it is clear that the first unfolds negatively in a detaching way and that the second unfolds positively in an intersecting way.

4.3 Approximating Tangencies – First Approach

We assume first that $\varphi_{\bar{\mu}}$ has a homoclinic tangency of type A which positively unfolds in an intersecting way. It is easy to see that there are other tangencies for nearby values of μ: the piece γ of W^s near r, see figure 6, comes, for $\mu > \bar{\mu}$ above the local unstable manifold. The iterates of γ under φ_μ^{-1} accumulate on the stable separatrix and intersect near q the local piece of W^u, which penetrates the region to the right of W^s. Let μ_n denote the μ-value for which we have the first contact of the piece of W^u near q and the image of γ under φ_μ^{-n}. Also let $\hat{\mu}_n$ denote the infimum of the μ-values for which there is a crossing of the above two pieces of W^u and W^s. Then clearly we have that $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \hat{\mu}_n = \bar{\mu}$; for $\mu = \mu_n$ we have a homoclinic tangency which negatively unfolds in a detaching way, and for $\mu = \hat{\mu}_n$ we have a homoclinic tangency which positively unfolds in an intersecting way. In both cases the new tangencies are again of type A; this follows from a simple analysis of preferred sides as indicated in figure 6.

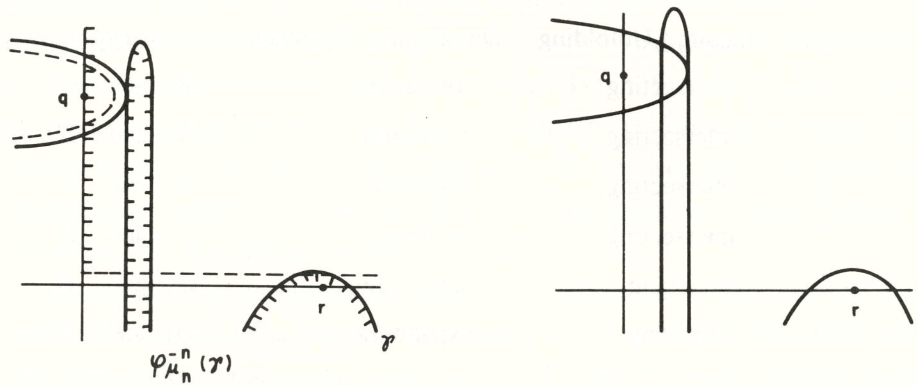

Figure 6. Approximating tangencies near a type A tangency.

Taking into account which of the eigenvalues of $d\varphi$ at the saddle point is dominant, we can show the existence of more approximating tangencies. We assume for the moment that for μ near $\bar{\mu}$ the contracting eigenvalue λ is dominating in the sense that $\lambda \cdot \sigma < 1$. Under this assumption we have, as in figure 6, that the maximal value of x on $\varphi_{\mu_n}(\gamma)$ is much higher than the x-coordinate of q. This means that the above arguments can be applied to the part of γ to the left of the 'maximum' and to the part of γ to the right of the 'maximum' (to make this point more precise we refer to (2.2): for a description of W^s and W^u near r with a function $M(x,\mu)$ we choose a dominating arc of local maxima, as in (2.2), and use this as an approximation of the real maximum). In this way we obtain not only tangencies of type A but also of type C; this follows from the positions of the preferred sides as indicated in figure 6. Among these new tangencies there are again ones unfolding negatively in a detaching way and also ones (not necessarily different ones) unfolding positively in an intersecting way.

If, in the beginning of this subsection, we would have taken a type A tangency negatively unfolding in an intersecting way, then we would have to interchange in the conclusions positive and negative and also $< \bar{\mu}$ and $> \bar{\mu}$. All the above arguments can also be applied to tangencies of the other types. In each case we get for the new tangencies one who is unfolding in a detaching way and one who is unfolding in an intersecting way (we omit the information on whether we have positively or negatively unfolding tangencies). The results for all the different types of tangencies is given in the table below.

Type:	Required Unfolding:	Dominating Eigenvalue:	New Types:
A	intersecting	contracting	A and C
A	intersecting	expanding	A and B
B	intersecting	irrelevant	B and D
C	intersecting	irrelevant	C and D
D	detaching	contracting	C and D
D	detaching	expanding	B and D

4.4 Transversal Intersections

Making approximating tangencies as in the preceding subsection we also obtain new (transversal) intersections of W^s and W^u. In this subsection we investigate transversality, and angles of crossing of these intersections and also discuss the compatibility of preferred sides. We restrict ourselves to the cases where we have tangencies of type A, B, or C: the case D is different and in the final arguments it is not needed.

We first consider in detail the case where we have for $\mu = \bar{\mu}$ a tangency of type A positively unfolding in an intersecting way. For $\mu - \bar{\mu} > 0$ small we have intersections of W^s and W^u near r and q. We want to show that for any such value of μ there are transversal intersections of W^s and W^u. Suppose that for some value of μ all the intersections of W^s and W^u near q and r are non-transversal. We shall derive a contradiction from this using (3.3): 'λ-lemma type estimates — singular case'. Indeed, since W^s and W^u cross each other near both q and r, W^u, respectively W^s, contains an arc near q, respectively r, like the curve l, respectively m, in figure 3. As explained in that section this immediately implies a transversal intersections of W^s and W^u near q and r. It is easy to see that in the cases B and C the same arguments apply.

We know that whenever there is a transversal intersection of W^s and W^u, both W^s and W^u accumulate on themselves. In the previous section we defined preferred sides; in this accumulation it may happen that these preferred sides agree in the sense that, restricting to W^s, if $\{W_i\}$ are arcs in W^s converging to an arc W in W^s, then the limit of the preferred sides of W_i is the preferred side of W. While it is easy to give examples where the preferred sides are not always compatible with this accumulation, we can choose, for any arc W in W^s, accumulating arcs W_i in W^s such that the limit of the preferred sides of W_i is the preferred side of W. For this we only need to observe that if not all accumulation is consistent with the preferred sides, then any arc W' in W^s can arbitrarily closely be approximated by an arc W'' in W^s with opposite preferred side; in this way, namely replacing arcs in W^s by nearby arcs with opposite preferred sides, any accumulating sequence can be modified to an accumulating sequence which is consistent with the preferred sides.

Finally we have to discuss the angles of crossing. In the proof of our main result we need to have a homoclinic tangency of W^s and W^u with, at the same time, two orbits of transversal intersection of W^s and W^u with *different* angles of crossing as defined in (3,4). For this purpose we consider here a homoclinic tangency of type A, B, or C for $\mu = \bar{\mu}$ which is unfolding in an intersecting way and such that for $\mu = \bar{\mu}$ there are also transversal intersections of W^s and W^u. Let, as before, q and r be points on the orbit of tangency near p. Due to the transversal intersection, the local unstable separatrix near r is approximated by arcs in the unstable separatrix; as we saw before, these approximating arcs can be chosen to have their preferred side in the same direction as the local unstable separatrix. We use these approximating arcs to make, by slightly increasing μ, approximating tangencies with the piece of W^s near r which we again denote by γ. Due to the consistency of preferred sides, these new tangencies have the same type as the tangency with which we started. Now we claim that choosing such an approximating tangency for a value of μ which is very close to $\bar{\mu}$, we get a new orbit of transversal intersection for which the angle of crossing is arbitrarily close to a right angle, i.e. which is almost a tangency (in this way we produce two orbits with different angles of crossing). To see this, we consider the intersection of γ with the local unstable separatrix. For μ very close to $\bar{\mu}$ these intersections are tangencies or almost tangencies. In the case of an almost tangency we are done. If we only have tangencies we proceed as follows. Let r' be such a tangency where W^s crosses W^u (there must be such tangencies since γ has points on both sides of the local unstable separatrix). Now we use the fact that W^u is accumulating on itself so that there are arcs W_i in W^u tending to the local unstable separatrix in the C^1 sense. From the formulas in (2.1), relating the tangent directions of γ and the local unstable separatrix near the tangency r', and (3.2), relating the tangent directions of the local unstable separatrix and W_i, it follows that for i sufficiently big, there is a transversal intersection of W_i with γ as indicated in the figure. (The validity of the estimates in (3.2) follows from the fact that there are always C^1 linearizing coordinates near a two-dimensional saddle, see [H,1960]).

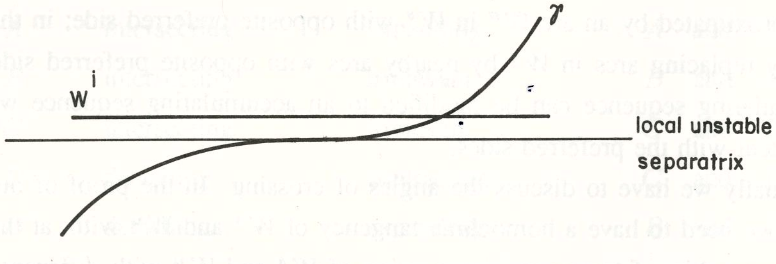

Figure 7. Almost tangencies near a tangency with crossing.

5. Proof of the Main Theorem

Under the assumptions of the main theorem we show first that there are many values of μ for which there is a homoclinic tangency of W^s and W^u having a series of extra properties, the so-called special tangencies. Then we show that among these special tangencies there is a dense subset of tangencies which satisfy

the three generic conditions. The condition of linearizability is automatically satisfied, but the conditions of first order contact and positive speed need special arguments.

5.1 Special Tangencies

Let H_1 denote the set of μ-values between μ_1 and μ_2 for which W^s and W^u have a tangency of type A, B, or C which unfolds in an intersecting way or which have a tangency of type D which unfolds in a detaching way (either positively or negatively). As we observed in the previous section, H_1 is not empty and even every point of it is an accumulation point (due to the approximating tangencies). For the following arguments it is important that every dense subset of H_1 has again the same properties: not empty and each of its points is an accumulation point. Besides H_1 we also consider the set $\mathcal{H}_1$ of pairs (μ, x) with μ in H_1 and x on a corresponding orbit of tangency.

The first dense subset of H_1 which we consider is the set of those values of μ in H_1 for which the product of the eigenvalues of $d\varphi$ at the saddle point is different from 1. This subset is denoted by H_2, the corresponding subset of $\mathcal{H}_1$ is denoted by $\mathcal{H}_2$. This subset is dense because, by assumption the ratio of the logarithms of the eigenvalues is not constant and hence, by analyticity, nowhere locally constant. For tangencies belonging to this set the dominating eigenvalue is well defined.

The next dense subset which we consider is the set of those values $\mu \epsilon H_2$ for which the tangency is of type A, B, or C and unfolds in an intersecting way. This subset is denoted by H_3. It is dense in H_2 because, as we saw in (4.3), any tangency of type D which unfolds in a detaching way is approximated by tangencies of type B or C which unfold in an intersecting way.

Next the dense subset H_4 in H_3 is obtained by requiring that for the values of μ in H_4 the angle of crossing of W^s can be defined as in (3.4) and for which there is a C^3-linearization in a neighbourhood of the saddle. Here we only have to exclude $\alpha(\mu) = -\ln\sigma(\mu)/\ln\lambda(\mu)$ to belong to a locally finite subset of $\mathbb{R} - \{0\}$. So the fact that H_4 is dense in H_3 follows as above for H_2.

Finally the dense subset H_5 of H_4 consists of those values of μ in H_4 for which there are, besides the orbit of tangency, at least two orbits of transversal intersection of W^s and W^u so that the angles with which W^u is crossing (in the sense of (3.4)) W^s are different. The denseness follows here by combining the various arguments in the previous section.

The definition of the corresponding spaces $\mathcal{H}_3$ to $\mathcal{H}_5$ is now obvious. It is easy to see from the results in the previous sections that they are all dense in $\mathcal{H}_1$.

5.2 Tangencies of Order One

We continue with the set H_5 as constructed in the previous subsection. In this set, H_5^k denotes the set of μ-values for which the tangency has order of contact at most k. We first observe that each point of H_5^k is an accumulation point of this set. This follows from the arguments in (4.4). Indeed, for tangencies corresponding to μ-values in H_5 there are also transversal intersections of W^s and W^u, so that W^u accumulates on itself. The way in which arcs in W^u converge to the local unstable separatrix also implies convergence of the $(k+1)$-jets; this follows from the C^r-section theorem in [HPS,1977]. Now, for a tangency with order of contact at most k, the $(k+1)$-jets of W^s and W^u, at the point of tangency are different. This means that if we make approximating tangencies as in (4.4), using the accumulation of W^u on itself, also for the new tangencies the $(k+1)$-jet of W^s and W^u must be different. So these approximating tangencies correspond also to elements of H_5^k. Now we fix k as the smallest integer for which H_5^k is not empty. We assume this value of k to be bigger than 1 and derive a contradiction. This contradiction also shows that H_5^1, or $\mathcal{H}_5^1$ is dense: just assume that there is a neighbourhood in $\mathcal{H}_5$ which contains no points of $\mathcal{H}_5^1$ and the same reasoning gives a contradiction.

First we take another dense subset H of H_5^k by restricting to those μ-values for which there is a local C^K linearization in a neighbourhood of the saddle point whose K-jets are C^∞ along W^s, where $K = 2+k\cdot(1+\alpha)$ and $\alpha = -\ln\sigma/\ln\lambda$, see (3.2). Let now $\bar\mu$ be an element of H. We shall construct the required contradiction by showing that near this tangency there have to be tangencies of order lower than k.

Let $\tilde x$, $\tilde y$ be μ-dependent C^K linearizing coordinates, whose K-jets are C^∞ along W^u, defined for μ near $\bar\mu$. Let x, y be C^∞-coordinates which have, along W^u, the same K-jets as $\tilde x$, and $\tilde y$. We assume, as usual, that the local unstable separatrix is the $\tilde x$-axis and that the local stable separatrix is the $\tilde y$-axis. We also assume that $(\tilde x = x_0, \tilde y = 0)$ is a point of the orbit of tangency. Then we can describe, locally near the tangency, the stable and unstable separatrix as follows:

$$W^u = \{\tilde y = 0\} = \{y = 0\}$$

$$W^s = \{\tilde x = x_0 + s,\ \tilde y = \tilde M(s,\mu)\}$$
$$\{x = x_0 + s,\ y = M(s,\mu)\}.$$

Without loss of generality we may assume that

$$M(s,\bar\mu) = -s^{k+1} + \text{h.o.t. and}$$

$$\max_s M(s,\mu) \simeq (\mu-\bar\mu)^l \text{ for } \mu > \bar\mu.$$

By the construction of $\mathcal{H}_5$ there are two arcs γ and γ' in W^u, intersecting W^s with different angles of crossing and such that $\varphi^n(\gamma)$ and $\varphi^n(\gamma')$ accumulate on the local unstable separatrix in such a way that the preferred sides are all on the same side. Let μ_i be such that $\varphi^i_{\mu_i}(\gamma)$ and the part of W^s near $(x = x_0, y = 0)$ have a tangency belonging to $\mathcal{H}_5$ and such that W^s is locally below $\varphi^i_{\mu_i}(\gamma)$.

Let $f_i(s,\mu)$ and $\tilde f_i(s,\mu)$ be such that

$$\varphi^i_\mu(\gamma) = \{x = x_0 + s, y = f_i(s,\mu)\} = \{\tilde x = x_0 + s, \tilde y = \tilde f_i(s,\mu)\}.$$

Let s_i and $\tilde s_i$ be such that the point of tangency is

$$(x = x_0 + s_i, y = f_i(s_i,\mu_i) = M(s_i,\mu_i)) =$$
$$(\tilde x = x_0 + \tilde s_i, \tilde y = \tilde f_i(\tilde s_i,\mu_i) = \tilde M(\tilde s_i,\mu_i)).$$

Since the exponent of motion, see (2,2), is l, both $M(s_i,\mu_i)/(\mu_i-\bar\mu)^l$ and $\tilde M(\tilde s_i,\mu_i)/(\mu_i-\bar\mu)^l$ go to the same non-zero limit L. At the tangency, the k-jets of $\tilde M$ and $\tilde f_i$ have to be equal. So from (3.2) we know that

$$|\tilde M^{(k)}(\tilde s_i,\mu_i)| \le C_1((\mu_i-\bar\mu)^l)^{1+\alpha k}$$

(as before we use here (k) for the k-fold differentiation with respect to the first variable). Since $(\mu_i-\bar\mu)^l$ is proportional to the distance of the tangency to the local unstable separatrix, since the K-jets of x, y and $\tilde x$, $\tilde y$ are the same along the local unstable separatrix and since $K = 2 + k\cdot(1+\alpha)$, the above estimate also holds for M:

$$|M^{(k)}(s_i,\mu_i)| \le C_1((\mu_i-\bar\mu)^l)^{1+\alpha k}$$

(maybe after adapting the constant C_1).

Since the order of tangency is exactly k, there is a constant C_2 such that, for (s,μ) near $(0,\bar\mu)$, $|M^{(k+1)}(s,\mu)| \ge C_2$. Then it follows that there are $\bar s_i$ such that $M^{(k)}(\bar s_i,\mu_i) = 0$ and

$$|s_i - \bar s_i| \le C_3((\mu_i-\bar\mu)^l)^{1+\alpha k}.$$

For the first derivative of f_i, and hence M, at (s_i, μ_i) we have

$$\lim_{i\to\infty} M^{(1)}(s_i,\mu_i)/(M(s_i,\mu_i))^{1+\alpha} =$$

$$\lim_{i\to\infty} f_0^{(1)}(0,\mu_i)/(f_0(0,\mu_i))^{1+\alpha} = \lim_{i\to\infty} D_\gamma(\mu_i),$$

where $D_\gamma(\mu_i)$ is the angle of crossing W^s of the curve γ for $\mu = \mu_i$, see (3.4). This means that

$$M^{(1)}(s_i,\mu_i) = D_\gamma(\mu_i)\cdot(M(s_i,\mu_i))^{1+\alpha} + o((M(s_i,\mu_i))^{1+\alpha}) =$$

$$D_\gamma(\bar\mu)\cdot(L\cdot(\mu_i-\bar\mu)^l)^{1+\alpha} + \text{h.o.t.},$$

where h.o.t. stands for terms which are, as functions of $(\mu_i - \bar\mu)$, of order bigger than $l(1+\alpha)$. Because $|s_i - \bar s_i| \le C_3((\mu_i-\bar\mu)^l)^{1+\alpha k}$ and because $M^{(2)}(s,\mu)$ is bounded near $(0,\bar\mu)$, we also have

$$M^{(1)}(\bar s_i,\mu_i) = D_\gamma(\bar\mu)\cdot(L\cdot(\mu_i-\bar\mu)^l)^{(1+\alpha)} + \text{h.o.t.}.$$

The above considerations, referring to γ, also apply to γ', which yields $D_{\gamma'} \ne D_\gamma$, s_i', $\bar s_i'$ and μ_i' (the constant L remains the same), so that

$$M^{(k)}(\bar s_i',\mu_i') = 0 \text{ and}$$

$$M^{(1)}(\bar s_i',\mu_i') = D_{\gamma'}\cdot(L(\mu_i'-\bar\mu)^l)^{1+\alpha} + \text{ h.o.t.}.$$

Now we observe that

$$\Gamma = \{(s,\mu)|M^{(k)}(s,\mu) = 0\}$$

is a C^∞-curve: this follows from the implicit function theorem, the fact that we have a tangency of order exactly k (so that $M^{(k+1)}(s,\mu)$ is bounded away from zero near $(0,\bar\mu)$) and the fact that we use C^∞-coordinates. This curve contains the points $(\bar s_i,\mu_i)$ and $(\bar s_i',\mu_i')$. We consider the function $M^{(1)}$ along the curve Γ as a function of μ. We derive a contradiction by showing that this function cannot be C^∞.

Without loss of generality we may assume that $D_{\gamma'} < D_\gamma$. We choose D_- and D_+ so that $D_{\gamma'} < D_- < D_+ < D_\gamma$. Then, for i sufficiently big:

$$M^{(1)}(\bar s_i',\mu_i') < D_-(L(\mu_i'-\bar\mu)^l)^{1+\alpha}$$

$$M^{(1)}(\bar s_i,\mu_i) > D_+(L(\mu_i-\bar\mu)^l)^{1+\alpha}.$$

We observe that both μ_i and μ_i' converge to $\bar\mu$ and that, for i sufficiently big, between each μ_i and μ_{i+1} there is some μ_{i+s}'. For $M^{(1)}$, restricted to Γ, this means that:

- as a function of $(\mu-\bar\mu)$, $M^{(1)}|\Gamma$ is of the order $l(1+\alpha)$, so if $M^{(1)}$ is sufficiently differentiable then $l(1+\alpha)$ is an integer;
- the $l(1+\alpha)$th derivative of $M^{(1)}|\Gamma$ cannot be continuous at zero: it oscillates at least between D_- and D_+ in any neighbourhood of $\bar\mu$.

This proves that $M^{(1)}|\Gamma$ cannot be C^∞ and we have the required contradiction.

The overall conclusion of this subsection is that k, the smallest integer for which H_5^k is not empty, is one. It even follows that H_5^1 is dense in H_1. Consequently also H and the corresponding set $\mathcal{H}$, which were obtained by requiring C^K-linearizability, are also dense in H_1, respectively $\mathcal{H}_1$. These tangencies in $\mathcal{H}$ form the starting point for the considerations in the next subsection, except that we now have to require the linearization to be at least $C^{3+[\alpha]}$ (this does of course not influence the denseness).

5.3 Reducing the exponent of motion

Finally we have to prove in this subsection that there are tangencies in H, or in $\mathcal{H}$, with exponent of motion equal to one, see (2.2). We shall assume all exponents of motion to be bigger than one and derive a contradiction from this. As before, the same arguments can be used to show that the tangencies in $\mathcal{H}$ with exponent of motion one are dense.

Let $\bar\mu$ be in H and let, as in the previous section, $\tilde x$ and $\tilde y$ be C^K linearizing coordinates near the saddle of $\varphi_{\bar\mu}$ whose K-jets are C^∞ along the local unstable separatrix, where K is now $3+[\alpha]$. As before, the $\tilde x$, respectively $\tilde y$, axis is the local unstable, respectively local stable, separatrix, the eigenvalues of $d\varphi$ are $0<\sigma<1<\lambda$, and $\alpha = -\ln\sigma/\ln\lambda$. We assume $(\tilde x = x_0, \tilde y = 0)$ to be a point on the orbit of tangency. Near this point we have the usual representation:

$$W^u = \{\tilde y = 0\}$$

$$W^s = \{\tilde x = x_0 + s,\ \tilde y = \tilde M(s,\mu)\}.$$

Without loss of generality we may assume that $\tilde M(s,\mu) = -s^2 + \text{h.o.t.}$ and that $\max_s \tilde M(s,\mu) > 0$ for $\mu > \bar\mu$: in the other cases one can proceed similarly.

Let γ be an arc in W^u intersecting W^s transversally and such that $\varphi^n(\gamma)$ accumulates on the local unstable separatrix in such a way that the preferred sides are all in the same direction. We write:

$$\gamma = \{\tilde{x}, \tilde{y} = f_0(\tilde{x}, \mu)\} \text{ and}$$
$$\varphi^n_\mu(\gamma) = \gamma_n = \{\tilde{x}, \tilde{y} = f_n(\tilde{x}, \mu)\}.$$

Let μ_n be the μ-value where γ_n and W^s have a tangency in $\mathcal{H}$ and let this tangency be at

$$(\tilde{x} = x_0 + s_n, \tilde{y} = \tilde{M}(s_n, \mu_n) = f_n(x_0 + s_n, \mu_n)).$$

In order to have at these tangencies the exponent of motion bigger than one, we must have

$$\frac{\partial \tilde{M}}{\partial \mu}(s_n, \mu_n) = \frac{\partial f_n}{\partial \mu}(x_0 + s_n, \mu_n).$$

We have $f_n(\tilde{x}, \mu) = (\lambda(\mu))^n \cdot f_0((\sigma(\mu))^{-n} \cdot \tilde{x}, \mu)$ so that

$$\frac{\partial f_n}{\partial \mu}(\tilde{x}, \mu) = n \cdot (\lambda(\mu))^{n-1} \cdot \lambda'(\mu) \cdot f_0((\sigma(\mu))^{-n} \cdot \tilde{x}, \mu) +$$
$$(\lambda(\mu))^n \cdot f_0^{(1)}((\sigma(\mu))^{-n} \cdot \tilde{x}, \mu) \cdot (-n) \cdot (\sigma(\mu))^{-n-1} \cdot \sigma'(\mu) +$$
$$(\lambda(\mu))^n \cdot \frac{\partial f_0}{\partial \mu}((\sigma(\mu))^{-n} \cdot \tilde{x}, \mu).$$

Defining $d_n = \tilde{M}(s_n, \mu_n)$, the above formula and:

$$\lambda^n \simeq d_n; \quad n \simeq |\ln d_n|; \quad \sigma^{-n} \simeq d_n^\alpha$$

imply that

$$\left|\frac{\partial f_n}{\partial \mu}(f_n(x_0 + s_n, \mu_n))\right| \leq C_1 \cdot d_n \cdot |\ln d_n|$$

for some constant C_1. Also, as in the argument in the previous subsection, we have

$$\tilde{M}^{(1)}(s_n, \mu_n) \leq C_2 \cdot d_n^{1+\alpha}$$

for some constant C_2.

Now we transfer these estimates to C^∞ coordinates x, y and the corresponding function M; remember that, as in the previous subsection, the $(3 + [\alpha])$-jets of $(x - \tilde{x})$ and $(y - \tilde{y})$ are zero along the local unstable separatrix. We define $\bar{s}_n$ to be the unique value near s_n such that $M^{(1)}(\bar{s}_n, \mu_n) = 0$. Then for some constant C_3 we have $|s_n - \bar{s}_n| \leq C_3 \cdot d_n^{1+\alpha}$, so we also have

$$\frac{\partial M}{\partial \mu}(\bar{s}_n, \mu_n) \leq C_4 \cdot d_n \cdot |\ln d_n|$$

for some constant C_4.

Next we define $\Gamma = \{(s, \mu) | M^{(1)}(s, \mu) = 0\}$. Near $(0, \bar{\mu})$ this is a C^∞-curve parameterized by μ: $\Gamma = \{(s = \gamma(\mu), \mu)\}$. If l is the exponent of motion of the tangency at $\mu = \bar{\mu}$ then we have $m(\mu) = M(\gamma(\mu), \mu) = D \cdot (\mu - \bar{\mu})^l +$ h.o.t.. From this we conclude that

$$m'(\mu)/(m(\mu))^{(l-1)/l}$$

is bounded and bounded away from zero for μ near $\bar{\mu}$. This however is incompatible with

$$\left|\frac{\partial M}{\partial \mu}(\bar{s}_n, \mu_n)\right| = |m'(\mu_n)| \leq C_4 \cdot d_n \cdot |\ln d_n|$$

and the fact that $m(\mu_n)/d_n$ is bounded and bounded away from zero. This is the required contradiction which completes the proof of the main theorem.

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