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# On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus

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**Abstract.** We study solutions $u$ of quasilinear elliptic equations $-\operatorname{div}(F_p(x,\nabla u)) = 0$ on $\mathbb{R}^n$, where $F(x,p)$ is periodic in $x = (x_1,\ldots,x_n)$ and satisfies suitable convexity and growth assumptions with respect to $p$. If $u$ has asymptotically linear growth, we show that $u$ is, in fact, a linear function up to a periodic perturbation. This partially generalizes recent results of Avallaneda–Lin from the linear to the nonlinear case, and we also achieve a simplified proof of their results. Our work is motivated by the study of minimals of variational problems on a torus and, moreover, has contact with homogenization theory.

## 1. Results, open problems

**a)** The well-known Liouville theorem asserts that every bounded harmonic function in $\mathbb{R}^n$ is a constant. A simple generalization shows that every harmonic function growing at most polynomially *is* a polynomial. In a recent note [**1**] Avallaneda and Lin generalized this theorem to linear elliptic differential equations in divergence form

$$\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) = 0 \tag{1.1}$$

with periodic coefficients. Using tools from homogenization theory [**2**] they characterized the solutions of polynomial growth and showed, in particular, that they are necessarily polynomials with periodic coefficients.

In this note we prove a similar result for nonlinear elliptic equations. The simple proof depends on the standard estimates of elliptic regularity theory. This approach also gives rise to a simple proof of the beautiful result of Avallaneda and Lin which we present in section 3.

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**b)** In the nonlinear setting we consider the scalar Euler equation

$$\sum_{i=1}^{n} \partial_{x_i}(F_{p_i}(x,u_x)) = 0\,, \tag{1.2}$$

where $F = F(x,p)$ is a strictly convex function of $p \in \mathbb{R}^n$ of quadratic growth; moreover, $F$ has period 1 in all components of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. More precisely, we assume that $F(x,\cdot) \in C^2(\mathbb{R}^n)$ and that there exist positive constants $\lambda, \mu, \gamma$ such that

$$\begin{aligned} &\text{i)} && \lambda|\xi|^2 \le \sum_{i,j=1}^{n} F_{p_ip_j}\xi_i\xi_j \le \mu|\xi|^2\,, \\ &\text{ii)} && |F_p(x,0)| \le \gamma\,, \\ &\text{iii)} && F(x+z,p) = F(x,p) \quad \text{for all} \quad z \in \mathbb{Z}^n\,. \end{aligned} \tag{1.3}$$

With respect to $x$ the function $F$ is assumed to be measurable, and need not be smooth.

We consider $u$ as a weak solution of (1.2) in $H^1_{\text{loc}}(\mathbb{R}^n) = W^{1,2}_{\text{loc}}(\mathbb{R}^n)$. From regularity theory [7; Chapter 4] it is well-known that such weak solutions are Hölder continuous. Assume $u$ has *linear growth:* $u = O(|x|)$. We give the condition the weaker form

$$|B_r|^{-1} \int_{B_r} u^2 dx \le cr^2 \quad \text{for} \quad r \ge 1\,, \tag{1.4}$$

where $B_r = \{x \in \mathbb{R}^n; |x| < r\}$ and $|B_r|$ denotes the volume of $B_r$.

**Theorem 1.** *If $u$ is a weak solution of (1.2) satisfying the linear growth condition (1.4) then it has the form*

$$u(x) = (\alpha, x) + \beta + p(x,\alpha) \tag{1.5}$$

*with some constant vector $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ and a Hölder continuous periodic function $p(x+z,\alpha) = p(x,\alpha)$, $z \in \mathbb{Z}^n$, having mean value zero. $p$ is uniquely determined by $\alpha$.*

The existence of solutions of the form (1.5) is readily verified. Indeed, $p = p(x,\alpha)$ can be found as a minimizer of the variational integral

$$\int_\Omega F(x, \alpha + p_x)\,dx\,, \quad \Omega = [0,1)^n\,,$$

in the class of periodic functions in $H^1_{\text{loc}}(\mathbb{R}^n)$. It is determined only up to a constant which can be fixed by the normalization

$$[p] := \int_\Omega p(x)\,dx = 0\,.$$



The point of theorem 1 is that the formula (1.5) gives the most general solution of (1.2) with linear growth.

**c)** The proof of theorem 1 will be presented in section 2. The argument also yields a simple proof of Avallaneda and Lin's theorem, avoiding largely homogenization theory — at least as far as the estimates are concerned.

To formulate their theorem in [1] we assume that the coefficients $a_{ij}$ are periodic: $a_{ij}(x+z) = a_{ij}(x), z \in \mathbb{Z}^n$, measurable(*), $a_{ij} = a_{ji}$ and satisfy for some positive constants $\lambda < \mu$

$$\lambda|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \tag{1.6}$$

for all $\xi \in \mathbb{R}^n$.

We consider weak $H^1_{\text{loc}}$ solutions of (1.1) which grow of order $O(|x|^N)$. More precisely, we denote by $S^{(N)}$ the space of weak solutions of (1.1) satisfying

$$|B_r|^{-1} \int_{B_r} u^2\,dx \le cr^{2N} \quad \text{for all} \quad r \ge 1\,. \tag{1.7}$$

The following theorem gives a characterization of $S^{(N)}$:

**Theorem 2. (Avallaneda–Lin, 1989.)**
i) *Any solution $u \in S^{(N)}$ has the form*

$$u(x) = \sum_{|\nu| \le N} p_\nu(x) x^\nu\,; \quad x^\nu = \prod_{i=1}^{n} x_i^{\nu_i} \tag{1.8}$$

*where $p_\nu(x)$ are $\mathbb{Z}^n$-periodic and Hölder–continuous; the coefficients $p_\nu$ of highest order $|\nu| = \sum_{i=1}^n \nu_i = N$ are constants.*

ii) *The homogeneous polynomial*

$$u^{(N)} = \sum_{|\nu|=N} p_\nu x^\nu$$

*solves an elliptic equation with constant coefficients*

$$Qu^{(N)} = \sum_{i,j} q_{ij}\partial_{x_i}\partial_{x_j} u^{(N)} = 0\,, \tag{1.9}$$

*($Q$ is called the homogenized operator of (1.1)). A solution of (1.9) is called $Q$-harmonic.*

(*) The Lipschitz continuity of the $a_{ij}$ required in [1], is not needed in our argument.

iii) *Denoting the space of Q-harmonic polynomials of degree* $\le N$ *by* $\mathcal{H}^{(N)}$ *there exists a linear isomorphism between* $\mathcal{H}^{(N)}$ *and* $\mathcal{S}^{(N)}$*; in particular*

$$\dim \mathcal{S}^{(N)} = \dim \mathcal{H}^{(N)} .$$

For $N = 1$ this theorem clearly is a special case of theorem 1 with

$$F(x,p) = \frac{1}{2}(ap,p) = \frac{1}{2}\sum_{i,j} a_{ij}p_ip_j$$

being a quadratic form in $p$. For $N = 1$ the homogenized operator is irrelevant. On the other hand, the case $N > 1$ does not seem to be meaningful for the nonlinear problem, since then $|u_x|$ would become unbounded.

**d)** This investigation was motivated by a study of minimals of a variational problem

$$\int F(x,u,u_x)\,dx \tag{1.10}$$

on a torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$, $d = n+1$, where in general $F$ also depends on the variable $u$, in fact, such that $F(x,\cdot,\cdot) \in C^2$ and $F(x,u+1,u_x) = F(x,u,u_x)$. Otherwise we impose the same assumption (1.3) on $F$ as before, uniformly in $u$.

The problem is to describe solutions of the Euler equation

$$\sum_{i=1}^{n} \partial_{x_i} F_{p_i}(x,u,u_x) = F_u(x,u,u_x)$$

with linear growth $u = O(|x|)$. One may expect that there exists a vector $\alpha \in \mathbb{R}^n$ such that $u - (\alpha, x)$ is bounded in $\mathbb{R}^n$. It turns out that such a statement is false even for $n = 1$. However, we conjecture that it is true for minimal solutions, i.e. functions $u \in H^1_{\mathrm{loc}}$ satisfying

$$\int_{\mathbb{R}^n} (F(x,u+\varphi,u_x+\varphi_x) - F(x,u,u_x))\,dx \ge 0 \tag{1.11}$$

for all $\varphi \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$. As a matter of fact, in [10] the first author proved such statements for minimals without self-intersections, i.e. minimals $u$ for which for any $j \in \mathbb{Z}^n$, $j_0 \in \mathbb{Z}$ the function $u(x+j) - j_0 - u(x)$ has a fixed sign or vanishes identically.

We conclude this section with two open problems:

1) If $u$ is a minimal in the sense of (1.11) satisfying $u = O(|x|)$, does there exist an $\alpha \in \mathbb{R}^n$ such that $u - (\alpha, x)$ is bounded?



2) Under the same assumptions on $u$, is it true that $u$ has no self-intersections? If the second question has a positive answer so does the first.

If $F = F(x,u_x)$ is independent of $u$, both these questions are answered positively by theorem 1. Hence in this case the set of minimals without self-intersections agrees with the set of minimals with linear growth.

It has to be pointed out that in the case $F_u \equiv 0$ every extremal is automatically minimal. This is due to the invariance under the translation $u \to u + \mathrm{const}$. For the same reason the case of Cantor sets of minimals described in **[10]** cannot occur (see **[11]**, p. 87) making this situation rather simple, and possibly not typical.

In Section 4 we show some connection between the theory of minimals and the homogenization theory, in order to give some wider perspective of these problems. In Section 5 we state an example showing that statement 1) is false for solutions of Euler equations for which $F = F(x,u,u_x)$ depends also periodically on $u$. This example shows the need to restrict ourselves to minimals. Finally, to make this note self-contained, we describe in the appendix the "two scale expansion" of homogenization theory. It is our purpose to separate the analytic estimates from the formal expansion which has more algebraic character.

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## 2. Proof of theorem 1

a) We begin with two standard inequalities from the theory of elliptic differential equations. For this purpose the periodicity in $x$ is irrelevant, and we assume that $F = F(x,p)$ satisfies the condition given in (1.3) except for (1.3.iii).

**Lemma 1.** *If* $u \in H^1_{\mathrm{loc}}(B_R)$ *is a weak solution of* (1.2), *then for* $0 < r < R$ *one has the inequality*

$$\int_{B_r} u_x^2 dx \le c_1\{(R-r)^{-2}\int_{B_R} u^2 dx + \gamma^2|B_R|\} \tag{2.1}$$

*where* $c_1$ *is a constant depending only on* $\lambda, \mu$ *in* (1.3.i).

From the work of de Giorgi we have the classical pointwise estimates for the solution of (1.1):

**Lemma 2.** *If $v \in H^1_{\text{loc}}(B_{2r})$ is a weak solution of the linear equation* (1.1) *whose coefficients are measurable and satisfy* (1.6), *then $v$ is Hölder continuous and satisfies*

$$\text{ess sup}_{B_r} v^2 \le c_2 |B_{2r}|^{-1} \int_{B_{2r}} v^2 dx\,, \tag{2.2}$$

*where $c_2$ depends on $\frac{\mu}{\lambda}$ only. Moreover, there exists a constant $\theta \in (0,1)$, also depending only on $\frac{\mu}{\lambda}$, such that*

$$\text{osc}_{B_r}\ v \le \theta\, \text{osc}_{B_{2r}}\ v$$

*where* $\text{osc}\ v = \sup v - \inf v$.

The last inequality, which originally was used locally to establish the Hölder continuity of solutions, yields for large $r$ a generalization of the Liouville Theorem for uniformly elliptic equations (1.1). Indeed, if $M = \text{osc}_{\mathbb{R}^n}\ v < \infty$ then we obtain $M \le \theta M$, implying $M = 0$, i.e. $v$ is a constant, if bounded.

**b)** To prove theorem 1 we note that with $u = u(x)$ also $u(x+z)$ for $z \in \mathbb{Z}^n$ is a solution. Let $e_k$ denote the $k$-th basis vector in $\mathbb{R}^n$ and consider

$$v_k(x) = u(x+e_k) - u(x)\,.$$

Taking the difference of the differential equations for $u(x)$ and $u(x+e_k)$, we see that $v_k$ satisfies a linear equations of the form (1.1) with

$$a_{ij}(x) = \int_0^1 F_{p_ip_j}(x, u_x + t v_{k\,x})\, dt\,.$$

From lemma 1 and lemma 2 we will establish that $v_k$ is bounded, hence a constant.

Indeed, by assumption (1.4) and lemma 1 we obtain

$$\int_{B_r} u_x^2 dx \le c_1 \Big(r^{-2} \int_{B_{2r}} u^2 dx + \gamma^2 |B_{2r}|\Big) = O(r^n)\,.$$

Writing

$$v_k = u(x+e_k) - u(x) = \int_0^1 u_{x_k}(x + te_k)\, dt$$

one obtains

$$\sum_{k=1}^n \int_{B_r} v_k^2\, dx \le \int_{B_{r+1}} |u_x|^2 dx = O(r^n)\,.$$

By lemma 2 this gives rise to a pointwise estimate:

$$\text{ess sup}_{B_r}\ v_k^2 \le c_2 |B_{2r}|^{-1} \int_{B_{2r}} v_k^2\, dx = O(1)$$



for all $r \ge 1$. Hence $v_k$ is a bounded solution, hence a constant which we denote by $\alpha_k$. From the equations

$$u(x+e_k) - u(x) = \alpha_k$$

we see that $u(x) - \sum_{k=1}^n \alpha_k x_k$ is a $\mathbb{Z}^n$-periodic Hölder continuous function. Denoting its mean value by $\beta$ we obtain theorem 1.

**c)** For completeness we add a simple proof of lemma 1. Because of

$$F_{p_i}(x,p) - F_{p_i}(x,0) = \sum_j \int_0^1 F_{p_ip_j}(x,tp) p_j\, dt$$

we obtain from (1.3.i) the inequalities

$$\begin{cases} (p, F_p(x,p) - F_p(x,0)) \ge \lambda |p|^2 \\ |F_p(x,p) - F_p(x,0)| \le \mu |p| \end{cases} \tag{2.3}$$

where $(\cdot,\cdot)$, $|\cdot|$ denote Euclidean inner product and norm, respectively.

Let $\int \ldots$ denote integration over $\mathbb{R}^n$. Inserting in the weak form

$$\int (\varphi_x, F_p(x,u_x))\, dx = 0$$

of (1.2) the test function $\varphi = \eta^2 u$, where $\eta \in C^1_{\text{comp}}(B_R)$, we obtain from (1.3)

$$\int ((\eta^2 u)_x, F_p(x,u_x) - F_p(x,0))\, dx \le \gamma \int \left|(\eta^2 u)_x\right| dx$$

and therefore with (2.3)

$$\lambda \int \eta^2 u_x^2 dx \le 2\mu \int |\eta u_x||\eta_x u|\, dx + \gamma \int \left|(\eta^2 u)_x\right| dx\,.$$

Choose the function $\eta$ so that $\eta = 1$ in $B_r$, $0 \le \eta \le 1$ in $B_R$, $|\eta_x| \le 2(R-r)^{-1}$. With the Schwarz inequality and the estimate $2|ab| \le \varepsilon a^2 + \varepsilon^{-1} b^2$ for any $\varepsilon > 0$, we obtain

$$\int \eta^2 u_x^2\, dx \le 4(\tfrac{\mu}{\lambda})^2 \int \eta_x^2 u^2 dx + \tfrac{2\gamma}{\lambda} \int \left|(\eta^2 u)_x\right| dx\,,$$

which proves the estimate for $\gamma = 0$. Treating the last term in the same manner we obtain

$$\int \eta^2 u_x^2 dx \le c\Big(\int \eta_x^2 u^2 dx + \gamma^2 \int \eta^2 dx\Big)\,,$$

with a constant depending on $\lambda, \mu$; i.e. (2.1) with $c_1 = 4c$.

We remark that (2.1) was extended by Giaquinta and Giusti [5] to functions $F = F(x, u, u_x)$ which may also depend on $u$; however, in that case it is essential that $u$ are minimals in the sense of (1.11) of the corresponding variational problem.

## 3. Proof of the theorem by Avallaneda and Lin

a) The argument of the previous section gives a simple proof of parts i) and ii) of theorem 2, as we show now.

In the linear case the function $v_k(x) = u(x + e_k) - u(x)$ is again a solution of (1.1) if $u$ is. Therefore, the above argument can be repeated: Defining the difference operator $\Delta_i$ by $\Delta_i\varphi = \varphi(x + e_i) - \varphi(x)$ and for a multi-index $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$, $\nu_i \geq 0$,

$$E^\nu = \prod_{i=1}^{n} \Delta_i^{\nu_i}, \tag{3.1}$$

then also $E^\nu u$ is a solution of (1.1) for all such $\nu$.

Recalling the definitions of space $\mathcal{S}^{(N)}$ of solutions $u$ of (1.1) satisfying (1.7), we show

**Lemma 3.** *If $u \in \mathcal{S}^{(N)}$ then $\Delta_i u \in \mathcal{S}^{(N-1)}$.*

This follows from the argument

$$\begin{aligned}\int_{B_r} (\Delta_i u)^2 dx &\leq \int_{B_{r+1}} u_x^2 dx \\ &\leq c_3\Big(r^{-2} \int_{B_{2r+1}} u^2 dx + \gamma^2 |B_{2r+1}|\Big) = O(r^{n+2N-2}).\end{aligned}$$

More generally, we obtain

$$E^\nu u \in \mathcal{S}^{(N-|\nu|)} \quad \text{for} \quad |\nu| \leq N,$$

and by lemma 2

$$\text{ess sup}_{B_R} |E^\nu u| = O(R^{N-|\nu|}). \tag{3.2}$$

Hence for $|\nu| = N$ the solution $E^\nu u$ is bounded, hence a constant, and $E^\nu u = 0$ for $|\nu| > N$.

It is easily seen by induction that the most general functions $u$ for which $E^\nu u = 0$ for all $|\nu| > N$ have the form (1.8). Indeed, if

$$\Delta_i v = 0 \quad \text{for any} \quad i = 1, \ldots, n,$$



then $v$ is a periodic function. Suppose now that for some number $N \in \mathbb{N}$ we know that

$$E^\nu v = 0 \quad \text{for all} \quad |\nu| \geq N$$

implies that $v$ has the form (1.8), and let $w$ satisfy

$$E^\nu w = 0 \quad \text{for} \quad |\nu| > N.$$

Then for any multi-index $\nu$, $|\nu| = N$, there holds

$$\Delta_i(E^\nu w) = 0, \quad i = 1, \ldots, n.$$

Hence, for any such $\nu$, $|\nu| = N$, we have

$$E^\nu w = \nu!\, p_\nu \ (*)$$

for some periodic function $p_\nu$. Consider now the function

$$v(x) = w(x) - \sum_{|\nu|=N} p_\nu(x) x^\nu.$$

By construction,

$$E^\nu v = 0 \quad \text{for all} \quad \nu, |\nu| \geq N.$$

By induction hypothesis

$$v(x) = \sum_{|\nu|<N} p_\nu(x) x^\nu,$$

and the induction step is complete.

For $u \in \mathcal{S}^{(N)}$ the coefficients $p_\nu(x)$ can be expressed as a finite linear combination of $u(x + z)$, $z \in \mathbb{Z}^n$, and therefore are Hölder continuous. The highest coefficients $p_\nu$ for $|\nu| = N$ are given by

$$p_\nu = (\nu!)^{-1} E^\nu u, \quad |\nu| = N$$

and thus are constants, proving i) of theorem 1.

b) The statement ii) of theorem 2 is trivial for $N = 0$ and $N = 1$, as any linear function is harmonic. For $N = 0$ the solutions are constants and for $N = 1$ of the form

$$u = \sum_{i=1}^{N} \alpha_j x_j + p(x) \tag{3.3}$$

(*) as usual we write $\nu! = \prod_{i=1}^{n}(\nu_i!)$

where $p = p(x)$ is $\mathbb{Z}^n$-periodic and is solution of

$$\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)(\alpha_j + p_{x_j})) = 0 .$$

The solution $u$ is clearly unique up to a constant and can be written in the form

$$u = \sum_j \alpha_j(x_j + \psi_j(x)) + \beta \tag{3.4}$$

where $\psi_k$ is the periodic solution of

$$\sum_{i,j} \partial_{x_i} a_{ij}(\delta_{jk} + \psi_{k\,x_j}) = 0 , \tag{3.5}$$

with mean value zero. In matrix notation we write

$$\partial_x(a(I + \psi_x^T)) = 0 , \quad [\psi] = 0 \tag{3.6}$$

where $\psi$ is the vector with components $\psi_1, \ldots , \psi_n$, and [ ] denotes the mean value over the unit cube $[0, 1]^n$.

In the case $N = 2$ the solution can be written in the form

$$u = \tfrac{1}{2}(Cx, x) + \sum_{j=1}^{n} x_j p_j(x) + p_0(x) \tag{3.7}$$

with a constant symmetric matrix $C$, and periodic functions $p_0, p_1, \ldots , p_n$.

**Proposition 1.** *If a function* (3.7) *solves the equation* $Lu = \sum \partial_{x_i}(a_{ij}\partial_{x_j} u) =$ 0, *then the vector* $p = (p_1, p_2, \ldots , p_n)^T$ *satisfies*

$$p - [p] = C\psi , \tag{3.8}$$

*with* $\psi$ *given by (3.5), and* $C$ *satisfies*

$$tr(qC^T) = 0 , \quad \textit{where} \quad q = [a(I + \psi_x^T)] . \tag{3.9}$$

**Proof.** Applying the operator $L$ to $u$, given by (3.7), after a routine calculation we obtain

$$Lu = \sum_{j=1}^{n} f_j(x)x_j + g(x)$$

with

$$f = \partial_x(a(C^T + p_x^T))$$
$$g = tr(a(C^T + p_x^T)) + \sum_{i,j} \partial_{x_i}(a_{ij}p_j) + Lp_0 .$$



For $Lu = 0$ we have $f = 0$, $g = 0$. One verifies from (3.6) that $p = C\psi$ is a solution of the equation $f = 0$, and since $p$ is determined uniquely up to a constant, this proves (3.8). From the equation $g = 0$, taking the mean value, one obtains the compatibility condition

$$[g] = tr[a(C^T + p_x^T)] = tr\Big([a(I + \psi_x^T)]C^T\Big) = tr(qC^T) = 0 ,$$

proving the proposition.

The matrix $q = (q_{ij})$ can — using (3.6) — be rewritten as

$$q = [(I + \psi_x)a(I + \psi_x^T)] , \tag{3.10}$$

showing that it is positive symmetric. The associated differential operator

$$Q = \sum_{i,j=1}^{n} q_{ij}\partial_{x_i}\partial_{x_j} \tag{3.11}$$

is called the homogenized operator of $L$. The relation (3.9) shows that $u^{(2)} = \frac{1}{2}(Cx, x)$ is a solution of the equation

$$Qu^{(2)} = 0 .$$

Thus the case $N = 2$ is settled.

For $N > 2$ we can proceed by induction. Assume that the claim ii) has been proven for $N - 1$ in place of $N$, and let $u \in \mathcal{S}^{(N)}$. Then $\Delta_i u \in \mathcal{S}^{(N-1)}$. The leading part of $\Delta_i u$ is given by

$$(\Delta_i u)^{(N-1)} = \partial_{x_i} u^{(N)}$$

and by induction hypothesis

$$\partial_{x_i}(Qu^{(N)}) = Q(\partial_{x_i} u^{(N)}) = 0 ,$$

hence $Qu^{(N)} =$ const. Since $Qu^{(N)}$ is homogeneous of degree $N - 2 > 0$ we get $Qu^{(N)} = 0$, proving ii).

**c)** The proof of part iii) requires the construction of a $u \in \mathcal{S}^{(N)}$ with given leading part in $\mathcal{H}^{(N)}$. This can be done via formal expansions extending the formulae used in b), which are developed in homogenization theory; for the convenience of the reader we supplied them in the appendix. Here we want to show that the above results show that $\mathcal{S}^{(N)}$ is finite dimensional. Indeed, the mapping $u \to u^{(N)}$ gives rise to an injective linear map

$$\mathcal{S}^{(N)} \,/\, \mathcal{S}^{(N-1)} \to \mathcal{H}^{(N)} \,/\, \mathcal{H}^{(N-1)}$$

for every $N = 1, 2, \ldots$. Therefore the dimension of the left space is dominated by that of the right space, and since $\mathcal{S}^{(0)} = \mathcal{H}^{(0)}$ we have

$$\dim \mathcal{S}^{(N)} \leq \dim \mathcal{H}^{(N)} .$$

In the appendix we show that actually we have equality.

## 4. Minimals and homogenization

a) We discuss briefly the connection between the theory of minimals of a variational problem (1.10) and homogenization. We assume that $F = F(x, u, u_x)$ depends on $u$ also, with period 1, and satisfies the conditions (1.3). Without loss of generality we may assume that $F(x, u, 0) = 0$ so that (1.3.i-ii) implies an inequality

$$c_0^{-1}|p|^2 - c_1 \leq F(x, u, p) \leq c_0|p|^2 + c_1 . \tag{4.1}$$

In [10] the class $\mathcal{M}$ of minimals, in the sense of (1.11), without self-intersections has been studied. These are minimals such that for any $j \in \mathbb{Z}^n$, $j_0 \in \mathbb{Z}$ the functions

$$u(x + j) - j_0 - u(x) \tag{4.2}$$

do not change sign. We list some of the results:

a.1) For every $u \in \mathcal{M}$ there exists an $\alpha \in \mathbb{R}^n$ such that $u(x) - (\alpha, x)$ is bounded. The set of these $u$ will be denoted by $\mathcal{M}_\alpha$.

a.2) For any $\alpha \in \mathbb{R}^n$ there exist $u \in \mathcal{M}$ for which $u(x) - (\alpha, x)$ is bounded, i.e. $\mathcal{M}_\alpha \neq \emptyset$.

a.3) For any $u \in \mathcal{M}_\alpha$ the average action

$$\Phi(\alpha) = \lim_{r \to \infty} |B_r|^{-1} \int_{B_r} F(x, u, u_x)\, dx$$

exists and is independent of the choice of $u \in \mathcal{M}_\alpha$. Moreover, $\Phi(\alpha)$ is strictly convex.

The statement a.3) was proven by Senn [14]. Incidentally, as observed by Giaquinta and Giusti [5], for minimals of variational problems satisfying (4.1) the inequality (2.1) holds.

b) In the case that $F = F(x, u_x)$ is independent of $u$, these statements take a much simpler form. In particular, we want to show





b.1) If $F_u \equiv 0$ then every solution of (1.2) is a minimal.

b.2) The set $\mathcal{M}$ agrees with the set of solution (1.5) of (1.2) with linear growth $u = O(|x|)$.

b.3) The average action has the form

$$\Phi(\alpha) = \int_\Omega F(x, \alpha + p_x(x, \alpha))\, dx , \quad \Omega = [0, 1)^n \tag{4.3}$$

where $p = p(x, \alpha)$ was defined in (1.5).

b.4) In the quadratic case $F(x, u_x) = \frac{1}{2}(a(x)u_x, u_x)$ one has

$$\Phi(\alpha) = \frac{1}{2}(q\alpha, \alpha)$$

where $q$ is the matrix (3.10) of the homogenized operator $Q$ corresponding to $L = \partial_x(a\partial_x)$.

The proofs are straightforward: Since

$$F(x, p + q) \geq F(x, p) + (F_p(x, p), q) + \tfrac{\lambda}{2}|q|^2 ,$$

for any $\varphi \in C^1_{\text{comp}}(\mathbb{R}^n)$ one finds

$$\int (F(x, u_x + \varphi_x) - F(x, u_x))\, dx \geq \tfrac{\lambda}{2} \int \varphi_x^2 dx ,$$

if $u$ is a weak solution of (1.2). This proves b.1); it would be sufficient to assume convexity of $F(x, \cdot)$.

If $u \in \mathcal{M}$ then the functions (4.2), in particular, $v_k = u(x + e_k) - u(x)$, have a fixed sign or vanish identically. On the other hand $v_k$ satisfies a linear equation (1.1) (see section 2 b)), and by the Harnack inequality for such equations (see [13]) it is constant, say $\alpha_k$. Therefore we conclude as before that $u$ has the form (1.5). Thus, in view of b.1) our claim b.2) is verified.

The statement b.3) is obvious if one makes use of the characterization of $\mathcal{M}$, and the formula (1.5). Finally, b.4) is a consequence of (3.4), i.e.

$$u_x = (I + \psi_x^T)\alpha .$$

Hence, by (3.10),

$$\Phi(\alpha) = \frac{1}{2}\left[(a(x)u_x, u_x)\right] = \frac{1}{2}(q\alpha, \alpha) .$$

c) The functional

$$\int \Phi(v_x)\, dx ,$$

obtained by replacing $\alpha$ by the gradient of the unknown function $v$, gives rise to the homogenized differential equation

$$\sum_i \partial_{x_i} \Phi_{\alpha_i}(v_x) = 0 .$$

Indeed, in the quadratic case, this equation agrees with $Qv = 0$. In the nonlinear case, it takes the form

$$\sum_i \partial_{x_i} \int_\Omega F_{u_{x_i}}(x', v_x + p_x(x', v_x))\, dx' = 0 . \tag{4.4}$$

This follows readily from the formula

$$\Phi_{\alpha_i}(\alpha) = [F_{u_{x_i}}(x, \alpha + p_x)] + \sum_j \int_\Omega F_{u_{x_j}}(x, \alpha + p_x) p_{x_j \alpha_i}\, dx .$$

Since $\alpha x + p$ is a solution and $p$ periodic, the last sum vanishes, which gives (4.4). To justify the preceding computation we need to assure that $p_x(x, \cdot)$ depends diferentiably on $\alpha$ — at least in a certain Sobolev space topology. First observe that by the uniqueness assertion in Theorem 1 we have a map

$$\alpha \longmapsto p(\cdot, \alpha) .$$

Taking difference quotients in the differential equation (1.2), there results for fixed $j$

$$\begin{aligned} 0 &= \sum_i \partial_{x_i} \left[ \frac{F_{u_{x_i}}(x, \alpha + he_j + p_x(x, \alpha + he_j)) - F_{u_{x_i}}(x, \alpha + p_x(x, \alpha))}{h} \right] \\ &= \sum_i \partial_{x_i} \int_0^1 F_{u_{x_i} u_{x_j}}(x, \alpha + \vartheta h e_j + p_x(x, \alpha + he_j)) d\vartheta \\ &\quad + \sum_{i,k} \partial_{x_i} \left( \left\{ \int_0^1 F_{u_{x_i} u_{x_k}}(x, \alpha + p_x(x, \alpha) + \vartheta h w_x) d\vartheta \right\} \partial_{x_k} w \right) , \end{aligned}$$

where

$$w(x) = w^{(h)}(x) = \frac{p(x, \alpha + he_j) - p(x, \alpha)}{h} .$$

That is, $w$ satisfies an elliptic equation

$$-\sum_{i,k} \partial_{x_i}(b_{ik} \partial_{x_k} w) = \sum_i \partial_{x_i} b_i , \tag{4.5}$$

where

$$b_{ik}(x) = b_{ik}^{(h)}(x) = \int_0^1 F_{u_{x_i} u_{x_k}}(x, \alpha + p_x(x, \alpha) + \vartheta h w_x) d\vartheta$$





and

$$b_i(x) = b_i^{(h)}(x) = \int_0^1 F_{u_{x_i} u_{x_j}}(x, \alpha + \vartheta h e_j + p_x(x, \alpha + he_j)) d\vartheta .$$

Note that by (1.3.i) the matrix $(b_{ij})$ satisfies the uniform ellipticity and boundedness condition (1.6). Moreover, $b_i \in L^\infty$ with

$$|b_i| \le \mu ,$$

uniformly in $h$. Multiplying (4.5) by $w$ and integrating by parts, we thus obtain

$$\lambda \|w_x\|_{L^2}^2 \le \sum_{i,k} \int_\Omega b_{ik} w_{x_i} w_{x_k} dx = -\sum_i \int_\Omega b_i w_{x_i} dx \le \mu \|w_x\|_{L^2} .$$

That is, $w = w^{(h)}$ is bounded in $H^1(\Omega)$. In particular, $p(x, \alpha + he_j) \to p(x, \alpha)$ in $H^1(\Omega)$ as $h \to 0$. Moreover, multiplying (4.5) by $w\varphi^2$, where $\varphi \in C_0^\infty$ is a smooth cut-off function, we easily verify that $\int \left| w_x^{(h)} \right|^2 dx$ is uniformly absolutely continuous. Now take the difference of equations (4.5) for $h, \ell > 0$ and multiply by $w^{(h)} - w^{(\ell)}$. Since $\left| b_{ik}^{(h)} - b_{ik}^{(\ell)} \right| \to 0$, $\left| b_i^{(h)} - b_i^{(\ell)} \right| \to 0$ almost everywhere, from Vitali's theorem we obtain that

$$\begin{aligned} \lambda \left\| (w_x^{(h)} - w_x^{(\ell)}) \right\|_{L^2}^2 &\le \sum_{i,k} \int_\Omega b_{ik}^{(h)} (w_{x_i}^{(h)} - w_{x_i}^{(\ell)})(w_{x_k}^{(h)} - w_{x_k}^{(\ell)})\, dx \\ &\le \sum_i \int_\Omega \left| b_i^{(h)} - b_i^{(\ell)} \right| \left| w_{x_i}^{(h)} - w_{x_i}^{(\ell)} \right| dx \\ &\quad + C \int_\Omega \sup_{i,k} \left| b_{ik}^{(h)} - b_{ik}^{(\ell)} \right| \left( \left| w_x^{(h)} \right|^2 + \left| w_x^{(\ell)} \right|^2 \right) dx \\ &\longrightarrow 0 \quad \text{as} \quad h, \ell \to 0 . \end{aligned}$$

That is, $w^{(h)}$ is a Cauchy–sequence in $H^1$, showing that

$$\alpha \longmapsto p(\cdot, \alpha)$$

is differentiable in the $H^1$–topology, as desired.

This implies that $\Phi(\alpha)$ is a $C^1$-function, and by **[14]** it is strictly convex. We remark that in the general case the derivatives of $\Phi$ may have a dense set of discontinuities, according to results by V. Bangert and J. Mather (see **[9]**).

The formula (4.4) is a special case of the homogenized equation for monotone operators derived by Dal Maso and Defranceschi **[3]**. Of course, the goal in their theory is quite different, namely to study a boundary value problem, say the

Dirichlet problem, while we imposed a growth condition.

## 5. A Counterexample

Here we want to point out that in the non-autonomous case equation (1.2) may very well have solutions for which one has $u = O(|x|)$ but for which $u - (\alpha, x)$ is not bounded for any $\alpha \in \mathbb{R}^n$. For this reason we formulated our problem in the introduction only for minimals.

For the counterexample we take $n = 1$ and

$$F(x, u, u_x) = \tfrac{1}{2} u_x^2 + V(x, u) \tag{5.1}$$

where $V \in C^2(T^2)$. In **[12]** (see section 6) a function $V$ is constructed, possessing a large bump in each fundamental domain, such that every minimal of (5.1) with $|\alpha| \leq 1$ avoids a certain disc $D \subset T^n$. Thus the set is constructed of minimals with $|\alpha| \leq 1$ is not dense on the torus. Using the work of Mather [8] one can for given $\alpha_+, \alpha_-$ with $-1 < \alpha_- < \alpha_+ < 1$, construct solutions of the Euler equation

$$u_{xx} = V_u(x, u)$$

for which

$$u(x)/x \longrightarrow \alpha_\pm \quad \text{for} \quad x \longrightarrow \pm\infty \,, \text{resp.}$$

For such a solution one has obviously $u = O(|x|)$ for $|x| \to \infty$ but $u(x) - (\alpha, x)$ is not bounded for any $\alpha$.

This example leans on the theory of monotone twist maps — and there are many smooth examples known for which the minimals are not dense, i.e. for which there are no invariant curves.

## Appendix

a) Using the formulae of homogenization theory [2] we want to establish an isomorphism

$$\Phi : \mathcal{H}^{(N)} \longrightarrow \mathcal{S}^{(N)} . \tag{A.1}$$

This amounts to constructing for a given $Q$-harmonic homogeneous polynomial $h$ of degree $N$ a solution $u \in \mathcal{S}^{(N)}$ with $u^{(N)} = h$. This is an existence problem in a finite-dimensional space; therefore it is more or less an algebraic question.

In this part we follow [1] but clarify a point left unattended in that paper.



To reduce this question to a perturbation problem it is customary to rescale $u \in \mathcal{S}^{(N)}$ and consider

$$\varepsilon^N u\left(\frac{x}{\varepsilon}\right) = \sum_{|\nu| \leq N} \varepsilon^{N-|\nu|} x^\nu p_\nu\left(\frac{x}{\varepsilon}\right),$$

which tends to $u^{(N)}$ for $\varepsilon \to 0$. Considering $x$ and $y = \varepsilon^{-1} x$ as independent variables and writing

$$\begin{aligned} U(x, y, \varepsilon) &= \sum_{|\nu| \leq N} \varepsilon^{N-|\nu|} x^\nu p_\nu(y) \\ &= U_0(x) + \varepsilon U_1(x, y) + \cdots + \varepsilon^N U_N(x, y) \end{aligned} \tag{A.2}$$

one has $U_0 = u^{(N)}$. The differential equation for $U$ takes the form

$$(\partial_y + \varepsilon \partial_x)(a(y)(\partial_y + \varepsilon \partial_x))U = (L_0 + \varepsilon L_1 + \varepsilon^2 L_2)U = 0 \,, \tag{A.3}$$

where

$$\begin{aligned} L_0 &= \partial_y (a(y) \partial_y) \,, \\ L_1 &= \partial_y (a \partial_x) + \partial_x (a \partial_y) \,, \\ L_2 &= \sum_{i,j} a_{ij}(y) \partial_{x_i} \partial_{x_j} \,. \end{aligned} \tag{A.4}$$

In a first attempt one tries to find a solution in the form

$$U(x, y, \varepsilon) = \Psi U_0(x) \,,$$

where

$$\Psi = I + \varepsilon \Psi_1 + \varepsilon^2 \Psi_2 + \cdots = \sum_\nu \varepsilon^{|\nu|} \psi_\nu(y) \partial_x^\nu \tag{A.5}$$

is a formal differential operator with $\mathbb{Z}^n$-periodic coefficients $\psi_\nu(y)$. We need not be concerned about convergence questions, since these series terminate when applied to a polynomial. This first attempt fails but the coefficients $\psi_\nu(y)$ can be so determined as to simplify the problem. One has the following

**Proposition 2.** *There exists a unique formal series* $\Psi$ *of the form* (A.5) *with* $\psi_0 \equiv 1$, $[\psi_\nu] = 0$ *for* $|\nu| \geq 1$ *such that*

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)\Psi = M + (\partial_y + \varepsilon \partial_x)(a(y) \partial_y) \,, \tag{A.6}$$

*where the formal operator*

$$M = \varepsilon^2 M_2 + \varepsilon^3 M_3 + \cdots = \sum_{|\nu| \geq 2} \varepsilon^{|\nu|} m_\nu \partial_x^\nu$$

*has constant coefficients. Moreover, $M_2$ agrees with the homogenized operator $Q$ of (3.11).*

**Remarks.** Note, if $M_3 = M_4 = \cdots = 0$ then our problem would be solved: If $U_0 = U_0(x)$ is any homogeneous polynomial of degree $N$ then by (A.6)

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)\Psi U_0 = MU_0 = \varepsilon^2 QU_0\,,$$

since the last term in (A.6) cancels when applied to a polynomial depending on $x$ only. Hence if $U_0$ is $Q$-harmonic the right hand side vanishes and

$$\Psi U_0 = U_0 + \varepsilon U_1 + \cdots + \varepsilon^N U_N$$

would be the desired solution.

This "Ansatz" agrees with formula (14) in [1]. The constants $m_\nu$ correspond to the $k_\alpha$ in [1] which, however, are dropped in the later estimates. We will show below how to take care of $M_3, M_4, \ldots$.

The proof of the proposition can be gleaned from [1] or [2]. It consists in a comparison of coefficients which yields the following equations for the coefficients of $\varepsilon^s$.

$$\text{(A.7)}\qquad \begin{aligned} &L_0 = \partial_y a(y)\partial_y\,, \quad s=0\,,\\ &L_0\Psi_1 + L_1 = \partial_x(a(y)\partial_y)\,, \quad s=1\,,\\ &L_0\Psi_s + L_1\Psi_{s-1} + L_2\Psi_{s-2} = M_s\,, \quad s\geq 2\,. \end{aligned}$$

The first equation is automatic; the second gives

$$L_0\Psi_1 + \partial_y(a(y)\partial_x) = 0\,,$$

which can be solved for the periodic coefficients $\psi_j$ of $\Psi_1$, uniquely if they are normalized by $[\psi_j] = 0$. Similarly, the coefficients of $\Psi_s$ can be solved, provided that the compatibility condition is satisfied; for this purpose the constant coefficients of $M_s$ are needed.

The second and third equations (for $s=2$) of (A.7) correspond to the equations $f=0$, $g=0$ of Proposition 3.1; from this we read off that $M_2$ agrees with the operator $Q$ introduced there. (See also [2], Chapter 1.)

By induction on $|\nu|$ one easily verifies $C^\alpha$-regularity of $\psi_\nu$; see for instance [6; Theorem 3.2, pp. 88–89].

b) It remains to get rid of the terms $M_3, M_4, \ldots$. For this purpose we need the following elementary proposition.



Let $\mathcal{P}$ denote the space of polynomials in $x_1, x_2, \ldots, x_n$ with real coefficients, and $\mathcal{P}^{(s)} \subset \mathcal{P}$ the space of homogeneous polynomials of degree $s$. Hence $Q$ maps $\mathcal{P}^{s+2}$ into $\mathcal{P}^s$ for $s \geq 0$.

**Proposition 2.** *There exists a right inverse $R : \mathcal{P} \to \mathcal{P}$ of the elliptic operator $Q : \mathcal{P} \to \mathcal{P}$, such that*

$$R : \mathcal{P}^{(s)} \longrightarrow \mathcal{P}^{(s+2)}\,, \quad QR = \mathrm{id}\,.$$

This proposition shows, in particular, that $Q : \mathcal{P}^{(s+2)} \to \mathcal{P}^{(s)}$ is surjective. Of course, there are many such right inverses. For our purposes it suffices to define $R$ through the Cauchy problem: If $g \in \mathcal{P}^{(s)}$ define $Rg = v$ as the solution of

$$Qv = g\,, \quad v = v_{x_1} = 0 \quad \text{for} \quad x_1 = 0\,.$$

By an expansion in powers of $x_1$ one finds readily a unique $v \in \mathcal{P}^{(s+2)}$. Of course, if one is interested in estimates this is a very poor choice and one will construct a better $R$, for example, as an integral operator. But for our purposes it suffices.

Defining the formal series

$$A = I + \varepsilon RM_3 + \varepsilon^2 RM_4 + \cdots$$

we have

$$\text{(A.8)}\qquad \varepsilon^2 M_2 A = \varepsilon^2 M_2 + \varepsilon^3 M_3 + \cdots = M\,.$$

We form the unique formal inverse

$$A^{-1} = I - \varepsilon RM_3 + \cdots\,.$$

Then for any $Q$-harmonic polynomial $U_0 = U_0(x)$ define the polynomial

$$V(x) = A^{-1}U_0(x) = U_0 - \varepsilon(RM_3)U_0 + \cdots$$

with leading term $U_0$. By (A.8) this polynomial satisfies

$$MV = \varepsilon^2 M_2 AV = \varepsilon^2 M_2 U_0 = 0,$$

since $M_2 = Q$. Hence for $U_0 \in \mathcal{H}^{(N)}$ the desired solution $U = U_0 + \varepsilon U_1 + \cdots$ of (A.3) is given by

$$U = \Psi A^{-1} U_0$$

and the isomorphism is given by $\Phi = \Psi A^{-1}$.

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