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# The set of axiom A diffeomorphisms with no cycles

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**Abstract.** Let $M$ be a $C^\infty$ closed manifold and $\mathrm{Diff}^1(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$ topology. This paper contains an affirmative answer to the following conjecture raised by Mañé, which is an extension of the stability and $\Omega$-stability conjectures of Palis and Smale, as follows: the $C^1$ interior of the subset of diffeomorphism such that all the periodic points are hyperbolic is characterized as the set of diffeomorphisms satisfying Axiom A and the no-cycles condition. Moreover, it is showed that the $C^1$ interior of the set of all Kupka-Smale diffeomorphisms coincides with the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

## 0. Introduction

Let $M$ be a $C^\infty$ closed manifold and $\mathrm{Diff}^1(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$ topology. Let $\mathcal{P}^1(M)$ be the set of $f \in \mathrm{Diff}^1(M)$ such that all the periodic points of $f$ are hyperbolic. Then $\mathcal{P}^1(M)$ is a residual subset of $\mathrm{Diff}^1(M)$ by the Kupka-Smale theorem. Let us define

$$\mathcal{F}^1(M) = \left\{ f \in \mathrm{Diff}^1(M) \;\middle|\; \begin{array}{l} \text{there exists a } C^1 \text{ neighborhood } \mathcal{U}(f) \\ \text{such that all periodic points of every} \\ g \in \mathcal{U}(f) \text{ are hyperbolic} \end{array} \right\}.$$

Then $\mathcal{F}^1(M)$ coincides with the interior of $\mathcal{P}^1(M)$ in $\mathrm{Diff}^1(M)$ and for $f \in \mathcal{F}^1(M)$ all periodic points of $f$ are dense in $\Omega(f)$ (see [**Ma 6**]). As we shall explain later $\mathcal{F}^1(M)$ contains all structurally stable diffeomorphisms and by the same reason it contains all $\Omega$-stable diffeomorphisms .

Our aim is to prove the following theorems announced in [**Ao**].

**Theorem 1.** *Every diffeomorphism belonging to $\mathcal{F}^1(M)$ satisfies Axiom A and the no-cycles condition.*

We denote as $KS(M)$ the $C^1$ interior of the set of all Kupka-Smale diffeo-

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morphisms belonging to $\mathrm{Diff}^1(M)$. Obviously $KS(M) \subset \mathcal{F}^1(M)$ and $KS(M)$ contains Morse-Smale diffeomorphisms which are open in $\mathrm{Diff}^1(M)$ (Palis [**Pal 1**]). If we establish Theorem 1, then it will be checked that Kupka-Smale diffeomorphisms belonging to $KS(M)$ yields strong transversality. Thus we obtain the following

**Theorem 2.** *Every diffeomorphism belonging to $KS(M)$ satisfies Axiom A and the strong transversality condition.*

Recall that $\Lambda$ is a *hyperbolic set* of $f \in \mathrm{Diff}^1(M)$ if $\Lambda$ is compact, invariant $(f(\Lambda) = \Lambda)$ and there exist a unique continuous splitting $TM|_\Lambda = E^s \oplus E^u$ that is invariant $(Df(E^s) = E^s$ and $Df(E^u) = E^u)$ and constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n \mid E^s\| \le c\lambda^n, \quad \|Df^{-n} \mid E^u\| \le c\lambda^n$$

for all $n \ge 0$. Thus $E^s$ is contracted by $Df$ and $E^u$ is expanded by $Df$. If $\Lambda$ is a periodic orbit of period $p$, then $\Lambda$ is hyperbolic if and only if $D_x f^p$ has no eigenvalues of absolute value one for $x \in \Lambda$, in which case $x$ is called a *hyperbolic point*. Moreover for $x \in \Lambda$ the *stable* and *unstable* manifolds are defined by

$$W_f^s(x) = \{y \mid d(f^n(y), f^n(x)) \to 0 \text{ as } n \to \infty\},$$
$$W_f^u(x) = \{y \mid d(f^{-n}(y), f^{-n}(x)) \to 0 \text{ as } n \to \infty\}.$$

We sometimes denote $W_f^s(x)$ and $W_f^u(x)$ by $W^s(x)$ and $W^u(x)$ respectively.

Let $f \in \mathrm{Diff}^1(M)$. Then $f$ is *structurally stable* if there exists a neighborhood $\mathcal{U}(f)$ of $f$ in $\mathrm{Diff}^1(M)$ such that for every $g \in \mathcal{U}(f)$ there exists a homeomorphism $h\colon M \to M$ such that $h \circ f(x) = g \circ h(x)$ for all $x \in M$. We denote by $\Omega(f)$ or $\Omega$ the set of nonwandering points of $f$. When there exists a homeomorphism $h\colon \Omega(f) \to \Omega(g)$ such that $h \circ f(x) = g \circ h(x)$ $(x \in \Omega(f))$ for every $g$ in a certain neighborhood $\mathcal{U}(f)$, $f$ is $\Omega$-stable. Structural stability implies $\Omega$-stability. We recall that $f$ satisfies *Axiom A* if $\Omega$ is hyperbolic and the periodic points are dense in $\Omega$.

One can check (see [**Fr1**]) that if $f \in \mathrm{Diff}^1(M)$ is structurally stable then each periodic point of $f$ is hyperbolic by Franks's lemma and the Kupka-Smale's theorem. The result of Kupka and Smale is that the set of all diffeomorphisms such that

(i) all periodic point of $f$ are hyperbolic, and

(ii) given any pair $(x, y)$ of periodic points, $W^s(x)$ and $W^u(y)$ meet transversally,

is a residual subset of $\mathrm{Diff}^1(M)$ (notice that the theorem holds in the $C^r$ topology, $r \ge 1$). A diffeomorphism satisfying (i) and (ii) is called *Kupka-Smale*.



Let $\Lambda$ be a hyperbolic set for $f$, $\Lambda = \overline{\mathrm{Per}(f) \cap \Lambda}$. Then $\Lambda$ can be written as the finite disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_s$ of closed invariant sets $\Lambda_i$, such that each $f|_{\Lambda_i}$ is topologically transitive. Such a set $\Lambda_i$ is called a *basic set* with respect to $\Lambda$. The *stable* and *unstable* sets of a basic set $\Lambda_i$ are defined by

$$W^s(\Lambda_i) = \{y \mid d(f^n(y), \Lambda_i) \to 0 \text{ as } n \to \infty\},$$
$$W^u(\Lambda_i) = \{y \mid d(f^{-n}(y), \Lambda_i) \text{ as } n \to \infty\}.$$

Then $W^\sigma(\Lambda_i) = \cup\{W^\sigma(x) \mid x \in \Lambda_i\}$ for $\sigma = s, u$. If $\varepsilon > 0$ is small, then for $x \in \Lambda$ the *local stable* and *unstable* sets

$$W_\varepsilon^s(x) = \{y \mid d(f^n(x), f^n(y)) \le \varepsilon, \ \forall n \ge 0\},$$
$$W_\varepsilon^u(x) = \{y \mid d(f^n(x), f^n(y)) \le \varepsilon, \ \forall n \le 0\}$$

are $C^1$ disks tangent at $x$ to the subspaces $E^s(x)$ and $E^u(x)$, respectively, such that $T_x M = E^s(x) \oplus E^u(x)$. Moreover there exist $0 < \lambda < 1$ and $c > 0$ such that

$$d(f^n(x), f^n(y)) \le c\lambda^n d(x, y) \quad \text{for all} \quad y \in W_\varepsilon^s(x) \text{ and } n \ge 0,$$
$$d(f^{-n}(x), f^{-n}(y)) \le c\lambda^n d(x, y) \quad \text{for all} \quad y \in W_\varepsilon^u(x) \text{ and } n \ge 0$$

(see Hirsch and Pugh [**HiPu**]). This tells us that the set of hyperbolic periodic points with period $n$ is finite and $W_\varepsilon^\sigma(x) \subset W^\sigma(x)$ for $x \in \Lambda$ $(\sigma = s, u)$, which shows that

$$W^s(x) = \cup\{f^{-n} W_\varepsilon^s(f^n(x)) \mid n \ge 0\},$$
$$W^u(x) = \cup\{f^n W_\varepsilon^u(f^{-n}(x)) \mid n \ge 0\}.$$

If, in particular, $f$ is Axiom A then $M = \cup\{W^\sigma(x) \mid x \in \Omega(f)\}$ for $\sigma = s, u$.

Let $f \in \mathrm{Diff}^1(M)$ be structurally stable. Then it follows (see [**Ma4**]) that all periodic points are dense in the nonwandering set $\Omega$ by Pugh's closing lemma [**PuRo**], that is, let $f \in \mathrm{Diff}^1(M)$ and $x \in \alpha(x) \cup \omega(x)$, then there is a diffeomorphism $g$, arbitrarily $C^1$ near to $f$, such that $x$ is a periodic point of $g$. Thus we can conclude that $f$ is Axiom A if the hyperbolicity of $\Omega$ is shown.

An Axiom A diffeomorphism $f$ satisfies the *strong transversality condition* if and only if the stable manifold $W_f^s(x)$ and the unstable manifold $W_f^u(x)$ are transversal for all $x \in M$ (i.e. $T_xW_f^s(x) + T_xW_f^u(x) = T_xM$). For $f \in \mathrm{Diff}^1(M)$, if $\Omega(f)$ is finite and $f$ satisfies Axiom A and the strong transversality condition, then $f$ is called *Morse-Smale*.

Recently Mañé [**Ma6**] gave an answer to the $C^1$ Stability Conjecture of Palis and Smale.

**Theorem A.** *If $f \in \mathrm{Diff}^1(M)$ is structurally stable, then $f$ satisfies Axiom A and the strong transversality condition.*

It was already proved by Robinson [**Ro1**] that if an Axiom A diffeomorphism is structurally stable then it satisfies the strong transversality condition.

The converse of Theorem A had been proved by Robbin [**R1**] for $C^2$ diffeomorphisms and Robinson [**Ro4**] for $C^1$ diffeomorphisms.

A relation on the basic sets $\Lambda_j$, $1 \le j \le s$ of a hyperbolic set $\Lambda$ is defined as $\Lambda_i > \Lambda_j$ if $(W^s(\Lambda_i) - \Lambda_i) \cap W^u(\Lambda_j) \ne \emptyset$; i.e. there is a point $x$ which comes from $\Lambda_i$ and goes to $\Lambda_j$ under negative iterates of $f$. We say that $f$ has *no cycles* with respect to $\Lambda$ (say simply no cycles) if $\Lambda_{i_0} > \Lambda_{i_1} > \cdots > \Lambda_{i_j} > \Lambda_{i_0}$ is impossible among the basic sets. If $f$ is Axiom A, then strong transversality implies no cycles (with respect to $\Omega$).

To obtain a result toward the proof of the $\Omega$-Stability Conjecture, Palis [**Pal3**] showed, by using the result obtained in proving Theorem A, the following facts.

**Theorem B.** *Every diffeomorphism belonging to $\mathcal{F}^1(M)$ can be approximated by Axiom A diffeomorphisms with no cycles.*

This result implies the following theorem.

**Theorem C.** *If $f \in \mathrm{Diff}^1(M)$ is $\Omega$-stable, then $f$ is an Axiom A diffeomorphism with no cycles.*

Long ago the converse of Theorem C had been proved by Smale [**Sm4**].

As a problem related to Theorem B, Mañé conjectured that $\mathcal{F}^1(M)$ is characterized as the set of Axiom A diffeomorphisms with no cycles. Theorem 1 gives an answer for this conjecture.

Let $P(f)$ be the set of all periodic points of $f \in \mathrm{Diff}^1(M)$ and denote by $\overline{P}(f)$ the closure of $P(f)$. For $x \in M$ define

$$\alpha(x,f) = \{y \in M \mid \text{there exists an increasing sequence } \{n_i\} \text{ such that } f^{-n_i}(x) \to y \text{ as } i \to \infty\},$$

$$L_\alpha(f) = \{x \in M \mid \text{there exists } y \in M \text{ such that } x \in \alpha(y,f)\}.$$





Let $\omega(x,f) = \alpha(x,f^{-1})$ and $L_\omega(f) = L_\alpha(f^{-1})$. Also write $\alpha(x) = \alpha(x,f)$ and $\omega(x) = \omega(x,f)$ and denote as $L^-$ and $L^+$ the closures of $L_\alpha(f)$ and $L_\omega(f)$ respectively. Obviously $\overline{P}(f) \subset L^- \cap L^+ \subset L^- \subset L^- \cup L^+ \subset \Omega(f)$. In [**Ne2**] Newhouse gave examples such that (i) $L^-$ is hyperbolic but $L^+$ is not hyperbolic, (ii) $\overline{P}(f)$ is hyperbolic and finite but $L^-$ and $L^+$ are neither and (iii) $L^- \cup L^+$ is finite and hyperbolic but $\Omega(f)$ is neither. However, when we are concerned with the stability, then we see by Theorem 1 that if $f \in \mathrm{Diff}^1(M)$ is $\overline{P}(f)$-stable then $f$ satisfies Axiom A and $\overline{P}(f) = L^- \cap L^+ = L^- = L^- \cup L^+ = \Omega(f)$. A diffeomorphism $f$ is called to be *$\overline{P}(f)$-stable* if there exists a neighborhood $\mathcal{U}(f)$ of $f$ in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{U}(f)$ then there is a homeomorphism $h: \overline{P}(f) \to \overline{P}(g)$ satisfying $h \circ f(x) = g \circ h(x)$ for $x \in \overline{P}(f)$.

A diffeomorphism is *quasi-Anosov* if the fact that $\|Df^n(v)\|$ is bounded for all $n \in \mathbb{Z}$ implies that $v = 0$. The set of all quasi-Anosov diffeomorphisms belonging to $\mathrm{Diff}^1(M)$, $QA(M)$, is open and it coincides with the $C^1$ interior of all expansive diffeomorphisms in $\mathrm{Diff}^1(M)$, and moreover every diffeomorphism belonging to $QA(M) \cap KS(M)$ is Anosov (Mañé [**Ma2**]). However an example of a diffeomorphism on the connected sum $M'$ of two 3-tori that is quasi-Anosov but not Anosov was given by Franks and Robinson [**FrRo**]. It is clear that $M'$ is a manifold which is not the 3-torus. In this case we have that $QA(M')$ does not intersect $KS(M')$. This follows from the fact that every closed manifold which admits an Anosov diffeomorphism of co-dimension one is the torus (Newhouse [**Ne3**]).

For the proof of Theorem 1 it remains only to prove that $\Omega(f)$ is hyperbolic since the periodic points are dense in $\Omega(f)$. To see this let $\overline{P}_i(f)$ be the closure of periodic points with the unstable splitting of dimension $i$. Since $\Omega(f) = \cup\overline{P}_i(f)$ and specially $\overline{P}_0(f)$ is hyperbolic (by a result of Pliss), we suppose that $\Lambda(j_0) = \cup\{\overline{P}_i(f) \mid 0 \le i \le j_0\}$ is hyperbolic by induction on indices $i$. As explained above $\Lambda(j_0)$ is expressed as the union $\Lambda(j_0) = \Lambda_1 \cup \cdots \cup \Lambda_s$ of disjoint basic sets for $f$. Since the hyperbolicity of $\overline{P}_{j_0+1}(f)$ is obtained by proving that $\overline{P}_{j_0+1}(f) \cap \Lambda(j_0) = \emptyset$ (see Mañé [**Ma6**]), on the contrary we suppose that $\overline{P}_{j_0+1}(f) \cap \Lambda(j_0) \ne \emptyset$ and then $\overline{P}_{j_0+1}(f) \cap \Lambda_a \ne \emptyset$ for some basic set $\Lambda_a$. We use here two of the perturbation techniques introduced by Mañé. Under the assumption $\Lambda_a \cap \overline{P}_{j_0+1}(f) \ne \emptyset$, we shall prove that there exists a basic set $\Lambda_b \ne \Lambda_a$ such that $W^u(\Lambda_a) \cap W^s(\Lambda_b) \ne \emptyset$.

Therefore $\Lambda_b \cap \overline{P}_{j_0+1}(f) \ne \emptyset$ in this case; by the same reason we have

$W^u(\Lambda_b) \cap W^s(\Lambda_c) \neq \emptyset$ for some basic set $\Lambda_c \neq \Lambda_b$. In this repetition we shall be able to find a cycle among basic sets of $\Lambda(j_0)$ and reach a contradiction.

## 1. The results concerning the stability conjecture

Here we describe some results concerning the stability of diffeomorphisms. In [**Pe**] Peixoto proved that Morse-Smale diffeomorphisms are $C^1$ open and dense in diffeomorphisms of a circle $S^1$. In higher dimensions, Newhouse [**Ne1**] exhibited an open set in $\mathrm{Diff}^1(S^2)$ of nonhyperbolic diffeomorphisms. Smale [**Sm3**] proved that the structurally stable systems on compact 4-manifolds are not dense, and Williams [**Wi**] obtained the same result on compact 3-manifolds. After that Newhouse and Palis [**NPal**] showed that for every two-closed manifold if $\Omega(f)$ is hyperbolic then the set of periodic points is dense in $\Omega(f)$ (i.e. $f$ satisfies Axiom A) and an Axiom A diffeomorphism can be $C^0$ approximated by Axiom A diffeomorphisms with no cycles. Patterson [**Pa**] showed the existence of an Axiom A diffeomorphism which can not be $C^2$ approximated by Axiom A diffeomorphisms with no cycles when $\dim M \geq 3$. The corresponding $C^1$ question is unknown. In general the properties of diffeomorphisms belonging to the boundary of $\mathcal{F}^1(M)$ are still unknown. It seems likely that these questions are concerned with the study of the phenomena which occur in the bifurcation theory of one-parameter families of diffeomorphisms. The bifurcation theory of diffeomorphisms and vector fields has been studied in many works by Newhouse, Palis, Takens and other mathematicians.

It is unknown whether Theorems 1 and 2 presented in this paper hold for the set of diffeomorphisms with the $C^r$ topology ($r \geq 2$). This is concerned with the problems mentioned above.

We can define $\mathcal{F}^1(M)$ for differentiable maps by analogous forms to those used for diffeomorphisms. It is not known if the conclusion of Theorem 1 holds for differentiable maps. In [**Sh3**] Shub stated that strong transversality can be introduced for regular maps. When it is established, it seems likely that Theorem 2 can be discussed for regular maps.

In the rest of this section we briefly describe the historical notes on the stability of diffeomorphisms and flows.

For at least 14 years (1962-1976) there were some results to the stability conjecture. First, structural stability was proved in [**Pe**] as stated above for generic flows on oriented two-manifolds. Anosov [**An**] proved the structural stability of diffeomorphisms and vector fields when $M$ is hyperbolic. For Morse-Smale diffeomorphisms and flows (they exist on every manifold) the stability was proved by Palis and Smale [**PalSm**], and in [**Sm4**] Smale proved the $\Omega$-stability for Axiom A diffeomorphisms with no cycles. Moser [**Mos**] described that the stability of the case when $M$ is hyperbolic can be obtained by solving a functional equation by means of the implicit function theorem. After that Robbin [**R1**] proved, using Moser's idea on one hand and Palis-Smale ideas on the other, the structural stability of $C^2$-diffeomorphisms satisfying Axiom A and strong transversality. De Melo [**De**] showed Robbin's result for $C^1$ diffeomorphisms on a two-manifold. Robinson [**Ro3**] adapted Robbin's techniques to $C^1$ vector fields satisfying Axiom A and strong transversality. Moreover, in [**Ro4**] he gave the proof of the structural stability for $C^1$ diffeomorphisms satisfying Axiom A and strong transversality. The $\Omega$-stability for flows was proved by Pugh and Shub [**PuSh**].





The characterization of structural stability was stated in [**PalSm**] as the Stability and $\Omega$-Stability Conjectures as mentioned before. Besides the results of Mañé and Palis described above, at present there are some results concerned with the conjectures. When $\dim M = 2$, Liao [**Li**], Mañé [**Ma3**] and Sannami [**Sa**] proved independently that if $f$ is structurally stable, or $\Omega$-stable, then it satisfies Axiom A and strong transversality, or Axiom A and no cycle condition respectively. After that Mañé [**Ma4**] showed, in proving the Ergodic closing lemma, that if $f$ is structurally stable and the closures of the sets $P_i$ of the periodic points with stable splitting of dimension $i$ are mutually disjoint then it satisfies Axiom A. See Hurley [**Hu**] to a related result.

The conjecture for flows is whether structural stability or $\Omega$-stability implies that the closure of the periodic points is hyperbolic. Only recently Hu [**H**] announced that the conjecture for flows is true in the case $\dim M = 3$. The assertion is based on Mañé [**Ma6**], Doering [**Do**] and Liao [**Li**]. In the case of a compact manifold with boundary, Labarca and Pacífico [**LP**] gave an exaple of a structural stable flow, tangent to the boundary, that does not satisfy Axiom A. Define $\mathcal{F}^1(M)$ for flows by analogous forms to those used for diffeomorphisms; i.e. $x \in \mathcal{X}^1(M)$ if there exists a $C^1$ neighborhood $\mathcal{U}$ ( $\mathcal{U} \in \mathcal{X}^1(M)$), $X \in \mathcal{U}$, such that for every $Y \in \mathcal{U}$ all the singularities and periodic orbits of the flow generated by $Y$ are hyperbolic. It is not in general true that the periodic orbits of a flow in $\mathcal{F}^1(M)$ are dense in $\Omega$. An example can be found in Guckenheimer and Williams [**GuWi**] and Guckenheimer [**Gu**]. If a flow in $\mathcal{F}^1(M)$ satisfies the structural or $\Omega$-stability, then the density of the periodic orbits in $\Omega$ is obtained by the $C^1$ closing lemma. In view of Liao's result **[Li]**, to see that the flow satisfies Axiom A, a key difficulty is to separate singularities from periodic orbits. The $C^1$ Stability Conjecture for flows is still unknown when $n \geq 4$.

## 2. The preparation of proof of Theorem 1

For the proof of Theorem 1 we need the argument developed in Mañé **[Ma5,6]** and Palis **[Pal2]**. We begin by briefly recalling Mañé's remarkable proof of the $C^1$ Stability Conjecture and facts concerning this conjecture.

As before let $P(f)$ denote the set of periodic points of $f \in \mathrm{Diff}^1(M)$. If $x \in P(f)$ is hyperbolic, then $T_xM = E^s(x) \oplus E^u(x)$, $Df(E^s(x)) = E^s(f(x))$ and $Df(E^u(x)) = E^u(f(x))$. For $f \in \mathcal{F}^1(M)$ define as $P_i(f)$ the set of points $x \in P(f)$ such that $\dim E^u(x) = i$ (compare with the definition $P_i(f)$ in Mañé **[Ma6]**). Since $f \in \mathcal{F}^1(M)$, we have $\Omega(F) = \overline{P}(f)$ as explained before and so $\Omega(f) = \cup\{\overline{P}_i(f) \,|\, 0 \leq i \leq \dim M\}$. It is easily checked that $\overline{P}_i(f) \cap \overline{P}_j(f) = \varnothing$ when $\overline{P}_i(f)$ and $\overline{P}_j(f)$ $(i \neq j)$ are hyperbolic.

To obtain Theorem 1 it is enough to show that for every $f \in \mathcal{F}^1(M)$, $\overline{P}_i(f)$ is a hyperbolic set for all $0 \leq i \leq \dim M$ because $P(f)$ is dense in $\Omega(f)$. For the cases $i = 0$ and $i = \dim M$ it follows from a result due to Pliss that $P_0(f)$ and $P_{\dim M}(f)$ are finite. Thus these sets are hyperbolic and

$$(P_0(f) \cup P_{\dim M}(f)) \cap (\cup\{\overline{P}_j(f) \,|\, 0 < j < \dim M\}) = \varnothing.$$

To show that the splitting of $TM|_{P_i(f)}$ extends to a splitting of $TM|_{\overline{P}_i(f)}$ satisfying the definition of hyperbolicity we introduce the following notions:

Given a compact invariant set $\Lambda$ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ we say that a splitting $TM|_\Lambda = E \oplus F$ is a *dominated splitting* if it is continuous, invariant and there exist $c > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n \,|\, E(x)\| \cdot \|Df^{-n} \,|\, F(f^n(x))\| \leq c\lambda^n$$

for all $x \in \Lambda$ and $n > 0$.

Then we have (See Pliss **[Pl]** and Liao **[Li]** for a related result).

**(I). ([Ma4].)** *If $f \in \mathcal{F}^1(M)$, then there exist $c > 0$, $0 < \lambda < 1$, $m > 0$ and a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ such that for all $g \in \mathcal{U}(f)$, $0 < i < \dim M$ and $m_0 \geq m$ there exists a splitting $TM|_{\overline{P}_i(g)} = E_i \oplus F_i$ satisfying*

(a) $\|Dg^{m_0} \,|\, E_i(x)\| \cdot \|Dg^{-m_0} \,|\, F_i(g^{m_0}(x))\| \leq \lambda \quad (x \in \overline{P}_i(g))$,

(b) $E_i(x) = E^s(x)$, $F_i(x) = E^u(x) \quad (x \in P_i(g))$,

(c) *if $x \in P_i(g)$ has period $n > m_0$ then*

$$\prod_{j=0}^{[n/m_0]-1} \left\| Dg^{m_0} \,|\, E_i(g^{m_0 j}(x)) \right\| \leq c\lambda^{[n/m_0]},$$

$$\prod_{j=1}^{[n/m_0]} \left\| Dg^{-m_0} \,|\, F_i(g^{m_0 j}(x)) \right\| \leq c\lambda^{[n/m_0]},$$

*(here [ ] denotes the Gauss symbol) and*

(d) *for all $x \in P_i(g)$*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Dg^{m_0} \,|\, E_i(g^{m_0 j}(x)) \right\| \leq \log \lambda,$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Dg^{-m_0} \,|\, F_i(g^{m_0 j}(x)) \right\| \leq \log \lambda.$$

After this result, to show that the splitting $TM|_{\overline{P}_i(f)} = E_i \oplus F_i$ is hyperbolic for all $1 \leq i < \dim M$ we need the following result based on Liao's idea:

**(II). (Theorem I.4, [Ma6].)** *If $F_i$ is expanding $(1 \leq i < \dim M)$ then $E_i$ is contracting.*

Therefore to show Theorem 1 it suffices to prove that $f \in \mathcal{F}^1(M)$ implies that $F_i$ is expanding. Now Mañé translated, to solve the $C^1$ Stability Conjecture, the contracting property into averages of ergodic measures as follows.

**(III). (Lemma I.5, [Ma6].)** *Let $m_0$ be as in (I). If*

$$\int \log \|Df^{-m_0} \,|\, F_i\| \, d\mu < 0$$

*for every ergodic $\mu \in \mathcal{M}(f^{m_0} \,|\, \overline{P}_i(f))$ then $F_i$ is expanding.*

Here $\mathcal{M}(f^{m_0} \,|\, \overline{P}_i(f))$ denotes the set of all $f^{m_0}$- invariant probabilities on the Borel $\sigma$-algebra of $\overline{P}_i(f)$. As we explained above $P_0(f)$ is hyperbolic. If $P_1(f)$ is nonempty, then the following (IV) for $j_0 = 0$ means that

$$\int \log \|Df^{-m_0} \,|\, F_1\| \, d\mu < 0$$

for all ergodic $\mu \in \mathcal{M}(f^{m_0} \,|\, \overline{P}_1(f))$ because $\Lambda(0) \cap \overline{P}_1(f) = \varnothing$ and $\mu(\Lambda(0)) = 0$ for all $\mu \in \mathcal{M}(f^{m_0} \,|\, \overline{P}_1(f))$. Thus $F_1$ is expanding by (III), and $E_1$ is contracting by (II). Therefore $\overline{P}_1(f)$ is hyperbolic. When $P_0(f)$ is empty and $P_1(f) \neq \varnothing$, then we consider that the empty set $P_0(f)$ is hyperbolic, and apply the following (IV). Then we have that $\overline{P}_1(f)$ is hyperbolic.

Our proof will be done by induction on indices $i$ and the rest of this section and the next section will be devoted to complete the induction step. Suppose that there exists $j_0$ such that

$$\Lambda(j_0) = \cup\{\overline{P}_k(f) \mid 0 \le k \le j_0\} \tag{2.1}$$

is hyperbolic. Then we have

**(IV). (Theorem I.6, [Ma6].)** *Let $\lambda > 0$ be as (I). If*

$$\int \log \|Df^{-m_0} \mid F\| \, d\mu \ge \log \lambda$$

*for some $\mu \in \mathcal{M}(f^{m_0} \mid \overline{P}_{j_0+1}(f))$, then $\mu(\Lambda(j_0)) > 0$.*

(III) and (IV) mentioned above are based on the following

**(V). (Ergodic closing lemma, [Ma4].)** *Let $\Sigma(f)$ be the set of points $x \in M$ such that for all $\varepsilon > 0$, every compact set $K$ disjoint from the closure of the orbit of $x$ and every $C^1$ neighborhood $\mathcal{U}(f)$ of $f$, there exists $g \in \mathcal{U}(f)$ which coincides with $f$ on $K$ and has a periodic point $y$ such that if $n$ is its period, then $d(f^j(x), g^j(y)) \le \varepsilon$ for all $0 \le j \le n$. Then $\Sigma(f)$ is a $f$- invariant Borel set and $\mu(\Sigma(f)) = 1$ for all $\mu \in \mathcal{M}(f)$.*

To complete the induction step, it suffices to show that $\mu(\Lambda(j_0)) = 0$ for all $\mu \in \mathcal{M}(f^{m_0} \mid \overline{P}_{j_0+1}(f))$. This means that there are no measures $\mu \in \mathcal{M}(f^{m_0} \mid \overline{P}_{j_0+1}(f))$ such that

$$\int \log \|Df^{-m_0} \mid F\| \, d\mu \ge \log \lambda.$$

Using only the hypothesis $f \in \mathcal{F}^1(M)$ to obtain that $\mu(\Lambda(j_0)) = 0$, for all $\mu \in \mathcal{M}(f^{m_0} \mid \overline{P}_{j_0+1}(f))$, it will be enough to show that $\Lambda(j_0) \cap \overline{P}_{j_0+1}(f) = \emptyset$. In fact $\overline{P}_{j_0+1}(f)$ is hyperbolic if and only if $\Lambda(j_0) \cap \overline{P}_{j_0+1}(f) = \emptyset$. To see the hyperbolicity of $\overline{P}_{j_0+1}(f)$ we use the fact that the hyperbolic set $\Lambda(j_0)$ can be written as a finite union of basic sets $\Lambda_a (\Lambda(j_0) = \Lambda_1 \cup \cdots \cup \Lambda_s)$ (Smale [Sm3]), and the fact that if $p$ is a periodic point and $q$ is a transversal homoclinic point of $p$ then in every neighborhood of $q$ there are infinitely many periodic points which are $h$-related to $p$. Here $x$ is called *h-related* to $p$ if $W^u(O(x))$ has a point of transversal intersection with $W^s(O(p))$ and $W^u(O(p))$ has a point of transversal intersection with $W^s(O(x))$, where $O(x)$ denotes the orbit of $x$ (see





Smale [Sm1] and for a simple proof see Newhouse [Ne2]). It is easily checked that each basic set $\Lambda$ of $\Lambda(j_0)$ has the local product structure, i.e. for some $\varepsilon > 0$

$$W^u_\varepsilon(p) \cap W^s_\varepsilon(q) \subset \Lambda$$

for $p, q \in \Lambda$. Thus there exists a compact neighborhood $U$ of $\Lambda$ such that $\cap\{f^n(U) \mid -\infty < n < \infty\} = \Lambda$. Such a neighborhood $U$ is called an *isolating block* of $\Lambda$. In [Pal3] Palis made use of these facts to prove the following result, on which depends heavily the proof of Theorem 1.

**(VI).** *If $f \in \mathcal{F}^1(M)$, then there can be no cycle among basic sets of the hyperbolic set $\Lambda(j_0)$.*

Therefore to obtain the hyperbolicity of $\overline{P}_{j_0+1}(f)$ we shall prove that if $\Lambda(j_0) \cap \overline{P}_{j_0+1}(f) \ne \emptyset$ then there exist basic sets $\Lambda_a \ne \Lambda_b$ such that $W^s(\Lambda_a) \cap W^u(\Lambda_b) \ne \emptyset$.

To do this we must take advantage of two powerful perturbation techniques of Mañé [Ma5,6]. First we explain the creation of a homoclinic orbit for the case when an orbit accumulates on the stable and unstable manifold of a hyperbolic set (see [Ma5] for details). The technique will play an important role to show Theorem 1.

Let $\Lambda$ be a hyperbolic set of $f \in \mathrm{Diff}^1(M)$ and suppose that there exists a compact neighborhood $U$ of $\Lambda$ such that $\cap f^n(U) = \Lambda$. Choose $\varepsilon > 0$ so small that $W^s_\varepsilon(x) \cup W^u_\varepsilon(x) \subset U$ for all $x \in \Lambda$ and define

$$V^-_\Lambda = \cup\{W^u_\varepsilon(x) \mid x \in \Lambda\}, \quad V^+_\Lambda = \cup\{W^s_\varepsilon(x) \mid x \in \Lambda\}.$$

Obviously $V^-_\Lambda$ and $V^+_\Lambda$ are compact and satisfy

$$f^{-1}(V^-_\Lambda) \subset V^-_\Lambda, \quad f(V^+_\Lambda) \subset V^+_\Lambda, \quad \Lambda = V^-_\Lambda \cap V^+_\Lambda.$$

Define a closed neighborhood of $\Lambda$ by

$$V(r, \Lambda) = \{x \mid d(x, V^-_\Lambda) \le r, \quad d(x, V^+_\Lambda) \le r\}$$

and choose $\varepsilon_1 > 0$ so small that $V(\varepsilon_1, \Lambda) \subset U$ (notice that $\varepsilon_1$ is arbitrarily). By taking $0 < \delta_0 < 1$ and $0 < r_0 < \varepsilon_1$ we set an strictly decreasing sequence

$$r_0 > r_1 > r_2 > \cdots$$

satisfying $r_{n+1} = r_n^{1+\delta_0}$ for all $n \ge 0$.

Define $S^\Lambda_n$ as the closed set of points $x \in V(r_0, \Lambda)$ that can be written as $x = f^m(y_n)$ $(m \in \mathbb{Z})$ with $y_n \in V(r_n, \Lambda)$ and $f^i(y_n) \in V(r_0, \Lambda)$ for all $0 \le$

$i \le m$ if $m \ge 0$, or for all $m \le i \le 0$ if $m \le 0$. We say here that $S_n^\Lambda$ is a *star shaped neighborhood* of $\Lambda$. It is clear that each $S_n^\Lambda$ is the smallest neighborhood of $\Lambda$ containing $V(r_n,\Lambda)$ and $\cap\{S_n^\Lambda \mid 0 \le n < \infty\} = (V_\Lambda^+ \cup V_\Lambda^-) \cap V(r_0,\Lambda)$, (see Figure 1).

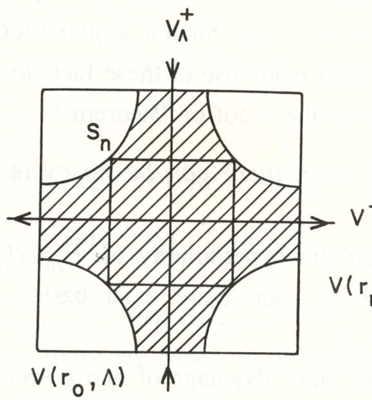


Figure 1

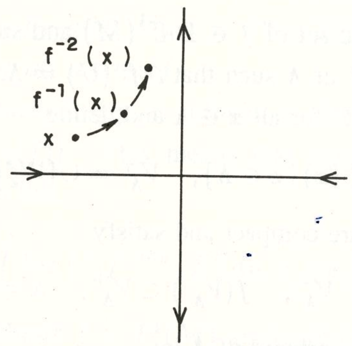


Figure 2

**(VII). (Lemmas 3 and 2, [Ma5].)**

(1) *Let $\varepsilon_1 > 0$ be as above. Then there exist $0 < \varepsilon_0 < \varepsilon_1$ and $0 < \gamma_1 < \lambda_1 < 1$ such that if $d(x, V_\Lambda^+) < \varepsilon_0$ and $d(x, V_\Lambda^-) < \varepsilon_0$ then*

$$\gamma_1 d(x, V_\Lambda^-) \le d(f(x), V_\Lambda^-) \le \lambda_1 d(x, V_\Lambda^-),$$
$$\gamma_1 d(x, V_\Lambda^+) \le d(f^{-1}(x), V_\Lambda^+) \le \lambda_1 d(x, V_\Lambda^+).$$





(2) *There exists $C_2 > C_1 > 0$ such that for $x \in V(r_0, \Lambda_a)$, letting*

$$T(x) = \sup\{j \mid f^l(x) \in V(r_0,\Lambda_a) \text{ for all } 0 \le l \le j\} + \sup\{j \mid f^{-l}(x) \in V(r_0,\Lambda_a) \text{ for all } 0 \le l \le j\},$$

*one has*

$$T(x) \le C_2(1+\delta_0)^i \text{ if } x \in V(r_0,\Lambda_a) - S_i^{\Lambda_a},$$
$$T(x) \ge C_1(1+\delta_0)^i \text{ if } x \in S_i^{\Lambda_a}$$

*for sufficiently large $i$ (Figure 2).*

This shows the behaviour of $f$-iterates of points $x$ close to $\Lambda$. If for a point $x$ a set of the form $\sigma = \{f^{-j}(x), \dots, f^{-(j+l)}(x)\}$ $(0 \le j < j+l)$ satisfies $\sigma \subset S_n^\Lambda$, $f^{-j+1}(x) \notin S_n^\Lambda$ and $f^{-j-l-1}(x) \notin S_n^\Lambda$, then we say that the set $\sigma$ is a *$(f^{-1}, x; n)$-string*. To simplify the notation $(f^{-1}, x; n)$-strings will be called *$(f^{-1}, x)$-strings* if there are no confusions. It is clear that there is a natural ordered relation among $(f^{-1}, x)$-strings, defined by $\sigma_1 < \sigma_2$ if the first element of $\sigma_2$ is a strictly negative iterate of the last member of $\sigma_1$.

Let $m_0$ be as in (I) and, as above, let $U$ be an isolating block of $\Lambda$. Let $\overline{x} \notin \Lambda$ be a point such that there exist a sequence $\{x_k\} \subset M$ converging to $\overline{x}$ and a sequence of integers $0 < n_1 < n_2 < \cdots$ such that the sequence of probabilities

$$\mu_k = \frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{f^{-m_0 j}(x_k)}$$

converges to $\mu \in \mathcal{M}(f^{m_0})$. If $A \subset M$ is a Borel set and $\mu(A) > 0$, then it is not difficult to see that the limit point $\nu$ of the sequence

$$\frac{1}{m_0 n_k} \sum_{j=1}^{m_0 n_k} \delta_{f^{-j}(x_k)}$$

satisfies $\nu(A) > 0$.

We suppose that the probability $\mu$ satisfies $\mu(\Lambda) > 0$. Then for all $N > 0$ one of the following (VIII) and (IX) holds (Lemma 4, [Ma5]).

**(VIII).** *There exist $n \ge N$, $k > 0$ and two $(f^{-1}, k)$-strings $\sigma_1, \sigma_2 \subset S_{n+1}^\Lambda$ such that*

$$\sigma \cap (S_n^\Lambda - S_{n+1}^\Lambda) = \varnothing$$

*for every $(f^{-1}, k)$-string $\sigma_1 < \sigma < \sigma_2$.*

(IX). *There exist $n \geq N$, $k > 0$ and a $(f^{-1}, k)$-string $\sigma_1 \subset S^\Lambda_{n+1}$ such that*

$$\sigma \cap S^\Lambda_n = \varnothing$$

*for every $(f^{-1}, k)$-string $\sigma \neq \sigma_1$.*

When (VIII) is satisfied, we denote as $q_1$ the last point of $\sigma_1 \cap V(r_n, \Lambda)$. Moreover let $q_2$ be the first point of $\sigma_2 \cap V(r_n, \Lambda)$. In the simplified case the situation is described in Figure 3.

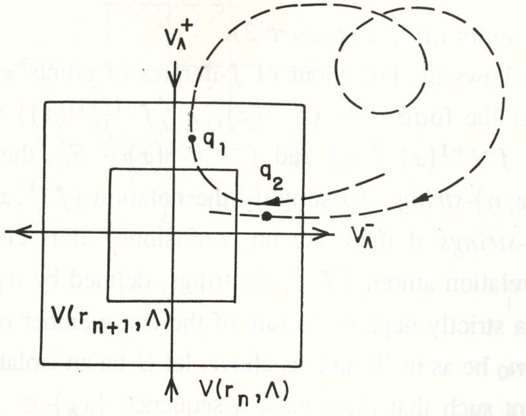


Figure 3

Since $q_1$ is the last point of $\sigma_1 \cap V(r_n, \Lambda)$, $f^{-1}(q_1)$ does not belong to $V(r_n, \Lambda)$. Let $\gamma_1$ and $\lambda_1$ be as in (VII), then we can write $\lambda_1 = \gamma_1^\alpha$ for some $0 < \alpha < 1$. As the only restriction we used about $\delta_0$ was $0 < \delta_0 < 1$, we can take it so near to 1 that $\delta_0(1+\alpha) > 1$, and choose $\beta$ such that $0 < 2\beta < \delta_0(1+\alpha) - 1$. Then the last inequality yields $d(q_1, V^+_\Lambda) < r_n^{2+\beta}$ if $n$ is large enough, and so there exists $y_1 \in V^+_\Lambda$ such that $q_1 \in B(r_n^{2+\beta}, y_1)$ (Figure 4, see p.150, [Ma5]).

Moreover we write $q_1 = f^{t_1}(x_k)$, $q_2 = f^{t_2}(x_k)$ and define $N_0 = t_2 - t_1$. Then it is checked that $B(r_n^{1+\beta/3}, y_1) \subset S^\Lambda_n$, from which we have $f^{-j}(q_1) \notin B(r_n^{1+\beta/3}, y_1)$ for $0 < j < N_0$ when $n$ is large enough.

Similarly it is proved that if $n$ is large then there exists $y_2 \in V^-_\Lambda$ such that $q_2 \in B(r_n^{2+\beta}, y_2)$ and $f^j(q_2) \notin B(r_n^{1+\beta/3}, y_2)$ for all $N_0 > j > 0$, and moreover $f^{-j}(y_2) \notin B(r_n^{1+\beta/3}, y_1)$ for all $j > 0$.

Now for $\mathcal{U}(f)$, a neighborhood of $f$ in $\mathrm{Diff}^1(M)$ apply the following well known result to $f^{-1} \circ \mathcal{U}(f)$ and $c = [(2+\beta)/(1+\beta/3)] - 1$.

(X). *Given $c > 0$ and a neighborhood $\mathcal{N}$ of the identity in* $\mathrm{Diff}^1(M)$ *there exists $R > 0$ such that for every $0 < r < R$ and every pair of points $a, b \in M$ satisfying $d(a,b) \leq r^{1+c}$, there exists $h \in \mathcal{N}$ such that $h(a) = b$ and $h(x) = x$ for all $x \notin B(r, a)$.*



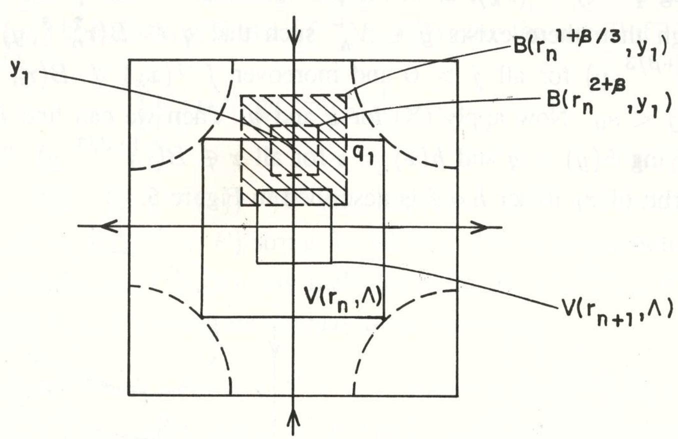


Figure 4

Since the neighborhood $f^{-1} \circ \mathcal{U}(f)$ and the number $c$ give $R > 0$, we can suppose, taking $n$ large enough, that $r_n < R$. Then there exists $h \in f^{-1} \circ \mathcal{U}(f)$ such that $h(y_2) = q_2$, $h(q_1) = y_1$ and $h(z) = z$ for all $z \notin B(r_n^{1+\beta/3}, y_1) \cup B(r_n^{1+\beta/3}, y_2)$. Define $g = f \circ h \in \mathcal{U}(f)$; then we have a homoclinic orbit of $g$ associated with $\Lambda$ (Figure 5).

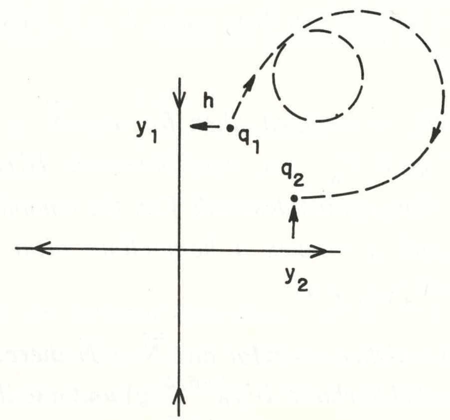


Figure 5

For the case when (IX) is satisfied let $\sigma_1$ have the form

$$\sigma_1 = \{f^{-s_0}(x_k), \dots, f^{-(s_0+l)}(x_k)\}.$$

Setting $q = f^{-s_0}(x_k)$, as in the previous case we can prove that if $n$ is large enough then there exists $y \in V_\Lambda^-$ such that $q \in B(r_n^{2+\beta}, y)$ and $f^{-j}(y) \notin B(r_n^{1+\beta/3}, y)$ for all $j > 0$ and moreover $f^{-j}(x_k) \notin B(r_n^{1+\beta/3, y}, y)$ for all $0 \leq j < s_0$. Now apply (X) for $q$ and $y$. Then we can find $h \in f^{-1} \circ \mathcal{U}(f)$ satisfying $h(y) = q$ and $h(z) = z$ for all $z \notin B(r_n^{1+\beta/3}, y)$. The behaviour of the orbit of $x_k$ under $h \circ f$ is described in Figure 6.

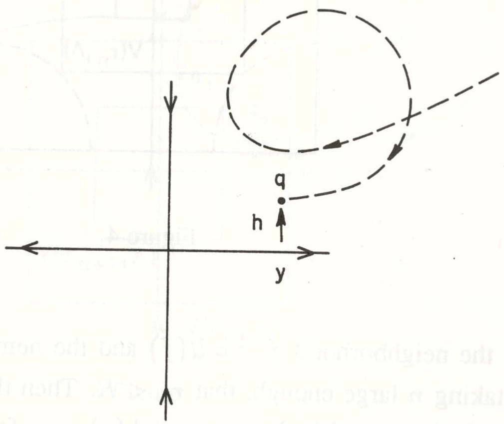


Figure 6

Let $r_n$ $(n \geq 0)$ and $\beta$ be as above. Since $\mu(\Lambda) > 0$ by hypothesis, from (VIII) and (IX) it follows that there exists $\overline{N} > 0$ satisfying one of the following (VIII)$^o$ and (IX)$^o$.

**(VIII)$^o$.** *If (VIII) is satisfied, then for any* $\overline{\overline{N}} \geq \overline{N}$ *there exist* $n \geq \overline{\overline{N}}$, $y_1 \in V_\Lambda^+ - \Lambda$, $y_2 \in V_\Lambda^- - \Lambda$, *neighborhoods* $B(r_n^{1+\beta/3}, y_i)$, $i = 1, 2$, *and* $g \in \mathcal{U}(f)$ *such that* $g$ *coincides with* $f$ *in the complement of* $B(r_n^{1+\beta/3}, y_1) \cup B(r_n^{1+\beta/3}, y_2)$ *and* $g$ *creates a homoclinic orbit associated with* $\Lambda$; *i.e.* $[W_g^s(\Lambda) - \Lambda] \cup W_g^u(\Lambda) \neq \emptyset$.

**(IX)$^o$.** *If (IX) is satisfied, then for any* $\overline{\overline{N}} \geq \overline{N}$ *there exist* $n \geq \overline{\overline{N}}$, $x_k \in \{x_n\}$, $y \in V_\Lambda^- - \Lambda$, *a neighborhood* $B(r_n^{1+\beta/3}, y)$ *and* $g \in \mathcal{U}(f)$ *such that* $g$ *coincides with* $f$ *in the complement of* $B(r_n^{1+\beta/3}, y)$ *and* $x_k$ *is contained in* $W_g^u(\Lambda)$.

For the proof of (VIII)$^o$ and (IX)$^o$ see pp. 149-155, [Ma5].





To create a homoclinic orbit associated with $\Lambda$ we are going to explain another perturbation technique of Mañé.

We use the following definitions. Let $\Sigma$ be a compact invariant set of $f \in \mathrm{Diff}^1(M)$ having a dominated splitting $TM|_\Sigma = E \oplus F$. We say that a pair of points $(x, f^{-n}(x))$ contained in $\Sigma$ is a $(f^{-1}, \gamma, E)$*-string* for $n > 0$ if

$$\prod_{j=1}^{n} \left\| Df \mid E(f^{-j}(x)) \right\| < \gamma^n$$

and we say that $(x, f^{-n}(x))$ is an *uniform* $(f^{-1}, \gamma, E)$*-string* if $(f^{-j}(x), f^{-n}(x))$ is a $(f^{-1}, \gamma, E)$-string for all $0 \leq j < n$ (these notions are essentially used in the proof of (II)). By replacing $E$ by $F$, a $(f^{-1}, \gamma, F)$-string and a uniform $(f^{-1}, \gamma, F)$-string are defined in the same way.

**(XI). ([Pl], [Li] and [Ma6].)** *For* $0 < \lambda' < \gamma' < 1$ *there exist* $N = N(\lambda', \gamma') > 0$ *and* $0 < c(\lambda', \gamma') < 1$ *such that if* $(x, f^{-n}(x))$ *is a* $(f^{-1}, \lambda', E)$*-string for* $n \geq N$, *then there exist* $0 < n_1 < n_2 < \cdots < n_k \leq n$ *such that* $k \geq nc(\lambda', \gamma')$ *and* $(x, f^{-n_i}(x))$ *is a uniform* $(f^{-1}, \gamma', E)$*-string for all* $1 \leq i \leq k$.

If $n_k$ is the largest integer satisfying the statement of (XI) and if $n_k \neq n$, then we can easily check that, setting $x' = f^{-n_k}(x)$ and $n' = n - n_k$,

$$\prod_{j=1}^{n} \left\| Df \mid E(f^{-j}(x')) \right\| \geq \gamma'^l$$

for all $1 \leq l \leq n'$. In this case $(x', f^{-n'}(x'))$ is called a *uniform* $(f^{-1}, \gamma', E)$*-obstruction.* By replacing $E$ by $F$ a uniform $(f^{-1}, \gamma, F)$-obstruction is defined.

Let $x \in M$ and $f \in \mathrm{Diff}^1(M)$. Define the probability $\mu_n(f, x)$ by

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{f^{-m_0 j}(x)}$$

and denote as $\mathcal{M}(f, x)$ the set of accumulation points of the sequence $\{\mu_n(f, x) \mid n > 0\}$.

**(XII). ([Ma5].)** *If* $\Sigma$ *is an isolated hyperbolic set of* $f \in \mathrm{Diff}^1(M)$, *with* $\Omega(f \mid \Sigma) = \Sigma$, *and there exists* $x \notin W^u(\Sigma)$ *such that* $\mu(\Sigma) > 0$ *for all* $\mu \in \mathcal{M}(f, x)$, *then there exists* $g \in \mathrm{Diff}^1(M)$ *arbitrarily* $C^1$ *near to* $f$ *and coinciding with* $f$ *in a neighborhood of* $\Sigma$, *having a homoclinic point associated to a basic set of* $\Sigma$.

Denote as $\mathcal{M}(f^{-1},x)$ the set of accumulation points of the sequence

$$\{\frac{1}{n_k}\sum_{j=1}^{n_k}\delta_{f^{m_0 j}(x_k)}\}.$$

Then we have that if $\Sigma$ is as in (XII) and there exists $x \notin W^s(\Sigma)$ such that $\mu(\Sigma) > 0$ for all $\mu \in \mathcal{M}(f^{-1},x)$, then the conclusion of (XII) holds. In this case notice that $W^u(\Sigma)$ is replaced by $W^s(\Sigma)$.

Let $\{y_n\}$ ($\subset M$) be a sequence converging to a point $p$. We recall that a set $\Sigma_1$ is *attainable* from $\{y_n\}$ if for $\delta > 0$, a neighborhood $U$ of $p$ and a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$, there exists a sufficiently large $k_0$ such that for all $k > k_0$ there exist $g \in \mathcal{U}(f)$ and $l > 0$ satisfying

$$\begin{cases} y_k \in U \quad \text{and} \quad g^{-l}(y_k) \in \Sigma_1, \\ g^{-1}(y) = f^{-1}(y), \quad \text{for all} \quad y \notin U, \\ d(f^{-n}(y_k), g^{-n}(y_k)) < \delta, \quad \text{for all} \quad 0 \le n \le l. \end{cases}$$

Take the set $\overline{P}_{j_0+1}(f)$ as $\Sigma$. By the properties of the dominated splitting $TM|_{\overline{P}_{j_0+1}}(f) = E \oplus F$, there exists a family of embedded $C^1$ disks $D(y,F)$ ($y \in \overline{P}_{j_0+1}(f)$) such that

(2.2) $$y \in D(y,F) \text{ and } T_y D(y,F) = F(y),$$

(2.3) $f(D(y,F))$ contains a neighborhood of $f(y)$ in the disk $D(f(y),F)$),

(2.4) $D(y,F)$ depends continuously on $y$.

Similarly there exists a family of embedded $C^1$ disks $D(y,F)$ ($y \in \overline{P}_{j_0+1}(f)$) such that (2.2), (2.3) and (2.4) hold, with $F$ replaced by $E$.

Let $D_r(y,F)$ be the closed disk of points in $D(y,F)$ whose distance in $D(y,F)$ to $y$ is less than or equal to $r$. To create a linking between stable and unstable manifolds of basic sets, Mañé [**Ma6**] prepared the following

**(XIII). (Attainability Theorem.)** *Let $\{y_n\}$ be a sequence in $\overline{P}_{j_0+1}(f)$ converging to $p \notin \Lambda(j_0)$ and $\nu\colon \{y_n\} \to \mathbb{Z}^+$ is a function satisfying $\nu(y_n) \to \infty$ as $n \to \infty$. Let $m_0$ be as in (I). Given $r > 0$ and $0 < \gamma < 1$ there exists $\varepsilon = \varepsilon(m_0, r, \gamma) > 0$ such that if the following conditions are satisfied,*

(1) *for $n$, $(y_n, f^{-m_0\nu(y_n)}(y_n))$ is a uniform $(f^{-m_0 j}, \gamma, E)$-string,*

(2) *there exist $n_0 > 0$ and $\hat{n} > 0$ such that $(y_n, f^{-m_0 j}(y_n))$ is a $(f^{-m_0}, \gamma, E)$-string for all $\hat{n} < j \le \nu(y_n)$ and all $n \ge n_0$,*

(3) *$p$ is a non periodic point,*

(4) *$y \in \overline{P}_{j_0+1}(f)$ is $\varepsilon$-near to an accumulation point of the sequence $\{f^{-m_0\nu(y_n)}(y_n)\}$,*

*then the closed disk $D_r(y,F)$ is attainable from $\{y_n\}$.*

Now we prepare a corollary which is obtained from the attainability theorem.

**(XIV).** *Let $\{y_n\}$, $\nu$ and $m_0$ be as in (XIII). For $r > 0$ and $0 < \gamma < 1$ there exists $\varepsilon = \varepsilon(m_0, r, \gamma) > 0$ such that if the conditions (1) and (4) of (XIII) are satisfied, then for $\delta > 0$ there exist a sufficiently large $k > 0$ and $z \in D_r(y,F)$ such that*

$$d(f^{-j}(y_k), f^{-j}(z)) < \delta$$

*for all $0 \le j \le m_0\nu(y_k)$.*

Replacing $F$ by $E$ in (XIV), we have that there exists $\varepsilon > 0$ such that if $(y_n, f^{m_0\nu(y_n)}(y_n))$ is a uniform $(f^{m_0}, \gamma, F)$-string for $n > 0$ and if $y \in \overline{P}_{j_0+1}(f)$ is $\varepsilon$-near to an accumulation point of $\{y_n\}$, then for $\delta > 0$ there exist $k > 0$ and $z \in D_r(y,E)$ such that $d(f^j(y_k), f^j(z)) < \delta$ for all $0 \le j \le m_0\nu(y_k)$.

## 3. Proof of Theorem 1

Let $\Lambda(j_0)$ be as in (2.1). Then $\Lambda(j_0)$ is written as the finite union $\Lambda(j_0) = \Lambda_1 \cup \cdots \cup \Lambda_s$ of basic sets $\Lambda_j$. To obtain the hyperbolicity of $\overline{P}_{j_0+1}(f)$ suppose $\overline{P}_{j_0+1}(f) \cap \Lambda_a \ne \varnothing$ for some basic set $\Lambda_a$ (otherwise $\overline{P}_{j_0+1}(f)$ is hyperbolic). For $U$ an isolating block of $\Lambda_a$ put $\Lambda_a^s = \cap\{f^{-n}(U) \mid n \ge 0\}$. Then it follows (see p.203, [**Ma6**]) that

$$(\Lambda_a^s - \Lambda_a) \cap \overline{P}_{j_0+1}(f) \ne \varnothing$$

and denote it by $\Gamma_a$. Notice that $\Gamma_a \subset W^s(\Lambda_a)$ and $\Gamma_a \cap P(f) = \varnothing$.

Let $m_0$ be as in (I) and $S_k^{\Lambda_a}$ be a star neighborhood of $\Lambda_a$ (see section 2). Fix $p \in \Gamma_a \cap S_0^{\Lambda_a}$ with $f^{-1}(p) \notin S_0^{\Lambda_a}$ and take $x_k \in (S_k^{\Lambda_a} - S_{k+1}^{\Lambda_a}) \cap P_{j_0+1}(f)$ satisfying $x_k \to p$ as $k \to \infty$. For $k > 0$ denote as $\overline{n}_k$ the positive integer such that $f^{-\overline{n}_k}(x_k)$ is the first point of the orbit of $x_k$ belonging to $S_k^{\Lambda_a} - S_{k+1}^{\Lambda_a}$. Clearly $\overline{n}_k \to \infty$ as $k \to \infty$. Put $n_k = [(\overline{n}_k - 1)/m_0] + 1$ for $k > 0$, then each $n_k$ is the integer such that the point of the form $f^{-m_0 n_k}(x_k)$ is the first point belonging to $S_k^{\Lambda_a} - S_{k+1}^{\Lambda_a}$.

For some time write $S_k = S_k^{\Lambda_a}$ for all $k > 0$ if there are no confusions. It is clear that all accumulation points of $\{f^{-m_o n_k}(x_k)\}$ is a subset of $V^-_{\Lambda_a} - \Lambda_a$ and for all $k > 0$

$$[\mathcal{O}_{n_k}(x_k) - \{x_k, f^{-m_o n_k}(x_k)\}] \cap [S_k - S_{k+1}] = \varnothing$$

where

$$\mathcal{O}_{n_k}(x_k) = \{x_k, f^{-1}(x_k), \ldots, f^{-m_o n_k}(x_k)\}$$

denotes the finite orbit from $x_k$ to $f^{-m_o n_k}(x_k)$ (Figure 7).

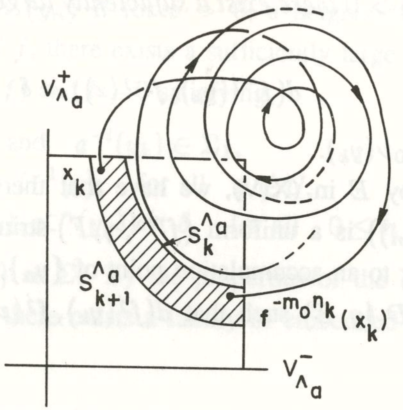


Figure 7

Let $\overline{N}$ be as in (VIII)$^o$ and (IX)$^o$ and put

$$S(i) = S_i - S_{i+1}$$

for all $i \geq \overline{N}$ (sometimes we write $S^{\Lambda_a}(i) = S(i)$). For the pair $(\{x_k\}, \{n_k\})$ let $x_k^i$ be the last point of the form $f^{-m_o q_i}(x_k)$ in $\mathcal{O}_{n_k}(x_k) \cap S(i)$ that runs lastly away $S(i)$ for $\overline{N} \leq i \leq k$. Put $n_{k,i} = n_k - q_i$. Obviously

$$f^{-m_o n_{k,i}}(x_k^i) = f^{-m_o n_k}(x_k)$$

and

$$S(i) \cap \{f^{-m_o}(x_k^i), \ldots, f^{-m_o n_{k,i}}(x_k^i)\} = \varnothing.$$

To avoid complication write $\overline{x}_k = f^{-m_o n_k}(x_k)$ for $k > 0$ (Figure 8). Because $f$ is a diffeomorphism without homoclinic orbits under small $C^1$ perturbations of $f$, we have

$$S_{i+1} \cap [\mathcal{O}_{n_{k,i}}(x_k^i) - \{\overline{x}_k\}] = \varnothing$$





for all $\overline{N} \leq i \leq k$ and all $k > 0$.

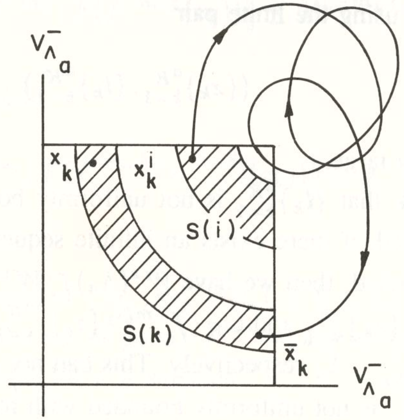


Figure 8

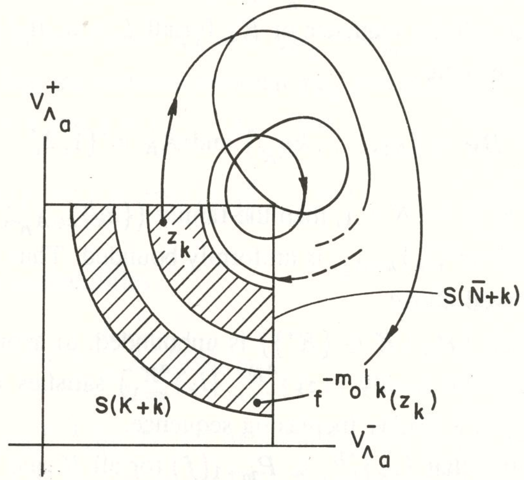


Figure 9

Let $l > 0$ be an integer (this will be determined later on). Fix $K > l + \overline{N} + 1$ and define

$$z_k = x_{K+k}^{\overline{N}+k}, \quad l_k = n_{K+k, \overline{N}+k}$$

for all $k > 0$. We set $a_K = K - (\overline{N} + l + 1)$. Since $S_{K-l} - S_K \subset S_{\overline{N}+k+1}$ and $[S_{K-l} - S_K] \cap S_{K+k} = \varnothing$ for all $1 \leq k \leq a_K$, we have

(3.1) $$\mathcal{O}_{l_k}(z_k) \cap (S_{K-l} - S_K) = \varnothing$$

for all $1 \le k \le a_K$ (Figure 9).

From now on we shall show that there exists a basic set $\Lambda_j$ such that $W^u(\Lambda_a) \cap W^s(\Lambda_j) \neq \emptyset$ by using the finite pair

$$(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$$

for $K$ sufficiently large.

We can check that $\{l_k\}_{k=1}^{a_K}$ is not uniformly bounded with respect to $K > l + \overline{N} + 1$. Indeed, if there exists an infinite sequence of $K$ such that $\{l_k\}_{k=1}^{a_K}$ is uniformly bounded, then we have $W^s(\Lambda_a) \cap W^u(\Lambda_a) \neq 0$ since certain subsequences of $\cup_K \{z_k\}_{k=1}^{a_K}$ and $\cup_K \{f^{-m_0 l_k}(z_k)\}_{k=1}^{a_K}$ converge to some points of $V_{\Lambda_a}^+ - \Lambda_a$ and $V_{\Lambda_a}^- - \Lambda_a$ respectively. This can not happen because of (VI).

Since $\{l_k\}_{k=1}^{a_K}$ is not uniformly bounded with respect to large $K$, for $L > 0$ there exists $K_L > l + \overline{N} + 1$ such that for $K \ge K_L$ there exists a sequence $1 \le k_1 < k_2 < \cdots < k_{R_K} \le a_K$ so that $l_{k_q} \ge L$ for $1 \le q \le R_K$. Denote as $\{K'\}$ an infinite sequence of $K_L$ for all $L > 0$. If $\{R_K \mid K \in \{K'\}\}$ is bounded and if we write

$$B_K = \{k_1, \dots, k_{R_K}\} \text{ and } A_K = \{1, 2, \dots, a_K\} - \cup_{\{K'\}} B_{K'}$$

for all $K > l + \overline{N} + 1$, then the family $(\{z_k\}_{k \in A_K}, \{l_k\}_{k \in A_K})$ satisfies the condition where $\{l_k\}_{k \in A_K}$ is uniformly bounded. Thus we can derive a contradiction as observed above.

When $\{R_K \mid K \in \{K'\}\}$ is unbounded, to avoid complication we may suppose that the family $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies the condition such that each $\{l_k\}_{k=1}^{a_K}$ is a strictly increasing sequence.

Notice that $\{z_k\}_{k=1}^{a_K} \subset P_{j_0+1}(f)$ for all $K$ and $\overline{P}_{j_0+1}(f)$ has the dominated splitting

$$TM|_{\overline{P}_{j_0+1}(f)} = E \oplus F$$

satisfying (I) mentioned in section 2. Let $\lambda > 0$ be as in (I).

First we consider the case when for all large $K$ the finite pair $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies

$$\prod_{j=1}^{l_k} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| > \lambda^{l_k} \tag{3.2}$$





for all $0 < k \le a_K$. If there exists the smallest integer $m'_k > 0$ such that

$$\prod_{j=m'_k}^{l_k} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| \le \lambda^{l_k - m'_k + 1} \tag{3.3}$$

for $1 \le k \le a_K$, then we have

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(m'_k - 1 - j)}(z_k)) \right\| \ge \lambda^{n+1} \tag{3.4}$$

for all $0 \le n \le m'_k - 1$.

Suppose that the cardinality of $\{m'_k \mid m'_k \neq 0, 1 \le k \le a_K\}$ is bounded with respect to $K$. Let $C_K$ be the set of points $f^{-m_0 l_k}(z_k)$ of indices $k$ satisfying $m'_k = 0$. Then each $f^{-m_0 l_k}(z_k) \in C_K$ satisfies

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(l_k - j)}(z_k)) \right\| \ge \lambda^{n+1}$$

for all $1 \le n \le l_k$. For $\delta > 0$ there exist $K_\delta > 0$, $q_{K_\delta} \in \overline{P}_{j_0+1}(f)$ and finitely many points $q \in C_{K_\delta}$ such that the distance between $q_{K_\delta}$ and $q$ is very small. If $\delta \to 0$, then we have $q_{K_\delta} \to \hat{q}$ and $\hat{q} \in V_{\Lambda_a}^- - \Lambda_a$. Thus

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{m_0 j}(\hat{q})) \right\| \ge \lambda^{n+1}$$

for every $n \ge 0$. Define the probabilities

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{f^{m_0 j}(\hat{q})}$$

for $n \ge 0$ and denote as $\mathcal{M}(f^{-1}, \hat{q})$ the set of accumulation points of $\{\mu_n\}$. Then $\mu(\Lambda(j_0)) > 0$ for all $\mu \in \mathcal{M}(f^{-1}, \hat{q})$ by (IV). By applying (XII) we have $\hat{q} \in W^s(\Lambda_b)$ for some $\Lambda_b$ and thus

$$\hat{q} \in W^u(\Lambda_a) \cap W^s(\Lambda_b),$$

which is our requirement.

Therefore it is enough to check the case when the cardinality of $\{m'_k \mid m'_k \neq 0, 1 \le k \le a_K\}$ is unbounded with respect to $K$. To avoid complication we may suppose $m'_k \neq 0$ for all $1 \le k \le a_K$ and large $K$.

For the case when $\{l_k - m'_k\}_{k=1}^{a_K}$ is uniformly bounded with respect to $K > l + \overline{N} + 1$ (i.e. there exists $L > 0$ such that $l_k - m'_k \le L$ for all large $K$ and

$1 \leq k \leq a_K$), we have

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F^{-m_0(m'_k-j)}(z_k)) \right\| \geq \lambda^{n+1}$$

for all $1 \leq n \leq m'_k$ and all large $K$, from which

$$W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \varnothing$$

for some $\Lambda_b$. Our requirement was obtained in this case.

For the case when $\{l_k - m'_k\}_{k=1}^{a_K}$ is not uniformly bounded with respect to $K > l + \overline{N} + 1$. Given $L > 0$ there exists $K_L > l + \overline{N} + 1$ such that for $K \geq K_L$ there exists a sequence $1 \leq k_1 < k_2 < \cdots < k_{R_K} \leq a_K$ so that $l_{k_q} - m'_{k_q} \geq L$ for $1 \leq q \leq R_K$. Let $\{K'\}$ be an infinite sequence of $K_L$ for all $L > 0$. If $\{R_K \mid K \in \{K'\}\}$ is bounded and if we write $B_K = \{k_1, \ldots, k_{R_K}\}$ and $A_K = \{1, 2, \ldots, a_K\} - \cup_{\{K'\}} B_{K'}$ for all $K > l + \overline{N} + 1$, then the family $(\{z_k\}_{k \in A_K}, \{l_k\}_{k \in A_K})$ satisfies the condition where $\{l_k - m'_k\}_{k \in A_K}$ is uniformly bounded with respect to $K > l + \overline{N} + 1$. Thus we have $W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \varnothing$ for some $\Lambda_b$, which is our requirement.

Therefore, when $\{R_K \mid K \in \{K'\}\}$ is unbounded we proceed our argument for the family $(\{z_k\}_{k \in B_{K'}}, \{l_k\}_{k \in B_{K'}})$. To avoid complication we may suppose that the family $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies the conditions such that each $\{l_k - m'_k\}_{k=1}^{a_K}$ is a strictly increasing sequence and its first term is sufficiently large. Then for all $K$ sufficiently large $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ admits a sequence $\{m_k\}_{k=1}^{a_K}$ of integers $l_k > m_k \geq m'_k$ such that

$$\prod_{j=m_k}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| \leq \gamma^{n-m_k+1} \quad (m_k \leq n \leq l_k) \tag{3.5}$$

for all $k$ with $0 < k \leq a_K$ (use (XI) to obtain (3.5)). Here $\gamma$ satisfies $\lambda < \gamma < 1$. By the choice of $m_k$

$$\prod_{j=1}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(m_k-j)}(z_k)) \right\| > \gamma^n \tag{3.6}$$

for all $1 \leq n \leq m_k$. Notice that $m_k \neq 0$ for all $1 \leq k \leq a_K$ and all large $K$. From (3.2), (3.3), (3.4), (3.5) and (3.6) we can conclude that if for all $K > l + \overline{N} + 1$ the finite pairs $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfy (3.2) then we have one of the following properties:

(3.7.a) there exists $\hat{q} \in V_{\Lambda_a}^- - \Lambda_a$ such that $\hat{q} \in W^s(\Lambda_b)$ for some basic set $\Lambda_b$, and such that for all large $K$ there exist finitely many points $q \in \{f^{-m_0 m_k}(z_k)\}_{k=1}^{a_K}$





so that the distance between $\hat{q}$ and $q$ is very small. Thus

$$W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \varnothing.$$

(3.7.b) for all $K$ sufficiently large there exists a sequence $\{m_k\}_{k=1}^{a_K}$ of integers such that for all large $k$ with $1 \leq k \leq a_K$

$$\begin{cases} \displaystyle\prod_{j=m_k}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| \leq \gamma^{n-m_k+1} & (m_k \leq n \leq l_k). \\ \displaystyle\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(m_k-1-j)}(z_k)) \right\| > \lambda^{n+1} & (0 \leq n \leq m_k - 1). \end{cases}$$

Thus, when the family $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies (3.2), we proceed our argument for the case (3.7.b) since (3.7.a) is our requirement.

When there exists an infinite sequence of integers $K$ such that $(\{z_k\}_{k=1}^{a_K}, \{l_k\}a_{k=1}^{K})$ satisfies

$$\prod_{j=1}^{l_k} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| \leq \lambda^{l_k}$$

for some $1 \leq k \leq a_K$, i.e. $(z_k, f^{-m_0 l_k}(z_k))$ is a $(f^{-m_0}, \lambda, F)$-string, by (XI) we can find also $0 < m_k < l_k$ satisfying (3.7.b). To develop our argument denote by $D_K$ the set of $1 \leq k \leq a_K$ satisfying the above inequality. If the cardinality of $D_K$ is bounded with respect to $K$ and if we write $A_K = \{1, 2, \ldots, a_K\} - D_K$, then the family $(\{z_k\}_{k \in A_K}, \{l_k\}_{k \in A_K})$ satisfies (3.2). For this case we may suppose that $A_K = \{1, 2, \ldots, a_K\}$ for $K$, and that there exists an infinite sequence $\{K\}$ of $K$ such that $(\{z_k\}_{j=1}^{a_K}, \{l_k\}_{j=1}^{a_K})$ satisfies (3.2) for all $1 \leq k \leq a_K$. If the cardinality of $D_K$ is unbounded with respect to $K$,then we may suppose that $D_K = \{1, 2, \ldots, a_K\}$ for large $K$. For this case we have that there exists an infinite sequence of integers $K$ such that $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies $\prod_{j=1}^{l_k} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| \leq \lambda^{l_k}$ for all $1 \leq k \leq a_K$.

Therefore, for a family $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfying (3.1) we can suppose, without loss of generality, that the family satisfies always (3.7.b).

For this family let $\{m_k\}_{k=1}^{a_K}$ be as in (3.7.b). If $\{l_k - m_k\}_{k=1}^{a_K}$ is uniformly bounded with respect to $K > l + \overline{N} + 1$, as observed above we have (3.7.a). This is our requirement.

Therefore we may suppose without loss of generality that each of $\{l_k - m_k\}_{k=1}^{a_K}$ is a strictly increasing sequence and we shall check that case. Notice that each $m_k$ is the smallest integer satisfying (3.7.b). For $1 \le k \le a_K$ and $K$ sufficiently large (XI) ensures the existence of a finite sequence

$$(3.8)\quad \begin{cases} 0 = m_{k,0} < m_k = m_{k,1} < m_{k,2} < \cdots < m_{k,j_k} \le l_k \\ \qquad \text{such that for } 1 \le i \le j_k \\ \prod_{j=m_{k,i}}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(z_k)) \right\| \le \gamma^{n-m_{k,i}+1} \quad (m_{k,i} \le n \le l_k), \\ \prod_{j=1}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(m_{k,i+1}-j)})(z_k)) \right\| > \gamma^n \\ \qquad (1 \le n \le m_{k,i+1} - m_{k,i} + 1). \end{cases}$$

Here we consider two cases:

(3.9) $A_K = \{m_{k,i+1} - m_{k,i} \mid 1 \le k \le a_K,\ 0 \le i \le j_k\}$ is uniformly bounded with respect to all large $K$.

(3.10) $A_k$ is not uniformly bounded with respect to all large $K$.

We check first the case when $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies (3.9) for all $K$ sufficiently large. For this case we have that there exists $L > 0$ such that $m_k \le L$ for all $1 \le k \le a_K$ and $K > l + \overline{N} + 1$. Since the limit point of $\{z_k\}$ is in $W^s(\Lambda_a)$ (take a subsequence if necessarily), there exists $q' \in W^s(\Lambda_a)$ such that $f^{-m_0 m_k}(z_k) \to q' (\in \overline{P}_{j_0+1}(f))$ as $k \to \infty$. Let $n_0 \ge 0$ be an integer such that $f^{m_0 n_0}(q')$ belongs firstly to the interior of $V(r_0, \Lambda_a)$.

Since $\Lambda(j_0)$ is hyperbolic, $\Lambda(j_0)$ has the continuous splitting $TM|_{\Lambda(j_0)} = E^s \oplus E^u$ such that there exist constants $c_1, c_2 > 0$ and $0 < \lambda_1 < 1$ satisfying

$$\|Df^n \mid E^s(x)\| \le c_1 \lambda_1^n$$

and

$$\|Df^n \mid E^u(x)\| \ge c_2 \lambda_1^{-n}$$

for all $x \in \Lambda(j_0)$ and $n \ge 0$. Take $\delta > 0$ and $\gamma_2$ such that $0 < \gamma_2 < e^{-\delta} < 1$. The only restriction we used about $m_0$ in (I) was to choose it arbitrarily large such that $m_0 \ge m$. Thus we can choose $m_0$ satisfying

$$\left\| Df^{m_0 j} \mid E^s(x) \right\| \le \gamma_2^j, \quad \left\| Df^{m_0 j} \mid E^u(x) \right\| \ge \gamma_2^{-j}$$

for all $x \in \Lambda(j_0)$ and $j \ge 0$.





Now, for $x \in \overline{P}_{j_0+1}(f) \cap \Lambda(j_0)$ the tangent space $T_x M$ splits into two direct sums

$$T_x M = E^s(x) \oplus E^u(x) = E(x) \oplus F(x).$$

Since $\dim E^u(x) < j_0 + 1$, we have $E^s(x) \cap F(x) \ne \{0\}$ and then

$$\left\| Df^{-m_0 j} \mid F(x) \right\| \ge \gamma_2^{-j}$$

for all $j \ge 0$.

In the definitions of $V_\Lambda^+, V_\Lambda^-$ and $V(r_0, \Lambda)$ (see section 2), the number $\varepsilon_1$ was arbitrary. Thus we can take $\varepsilon_1$ such that $d(x,y) \le \varepsilon_1$ $(x, y \in \overline{P}_{j_0+1}(f))$ implies

$$(3.11)\quad \left| \log \|Df^{-m_0} \mid F(x)\| - \log \|Df^{-m_0} \mid F(y)\| \right| < \delta.$$

When $x \in \overline{P}_{j_0+1}(f) \cap \Lambda_a$, from (3.11)

$$(3.12)\quad \|Df^{-m_0} \mid F(y)\| \ge e^{-\delta} \|Df^{-m_0} \mid F(x)\| \ge e^{-\delta}\gamma_2^{-1}.$$

Let $n_0$ be as above. Since $f^{m_0 n_0}(q') = q \in \operatorname{int} V(r_0, \Lambda_a) \cap \overline{P}_{j_0+1}(f)$ and $f^{m_0 j}(q) \in V(r_0, \Lambda_a) \cap \overline{P}_{j_0+1}(f)$ for all $j \ge 0$, from (3.12)

$$\left\| Df^{-m_0} \mid F(f^{m_0 j}(q)) \right\| \ge e^{-\delta}\gamma_2^{-1}$$

for all $j > 0$, and by the properties of dominated splitting

$$(3.13)\quad \left\| Df^{m_0} \mid E(f^{m_0 j}(q)) \right\| \le e^{\delta}\lambda\gamma_2 < 1$$

for all $j \ge 0$ (since $\lambda\gamma_2 < \gamma_2 < e^{-\delta}$).

Let $D(q, E)$ be a small closed disk satisfying (2.2), (2.3) and (2.4). Then it is checked from (3.13) that

$$D(q, E) \subset W^s(q_0)$$

where $W^s(q_0)$ $(\subset W^s(\Lambda_a))$ is the stable manifold containing $q$. Remark here that there exists a very small closed disk $D'(q', E)$ such that

$$f^{m_0 n_0}(D'(q', E)) \subset D(f^{m_0 n_0}(q'), E) = D(q, E)$$

(see Proposition 2.3, Mañé [**Ma3**]). Therefore, for $k$ sufficiently large choose a small closed disk $D'(f^{-m_0 m_k}(z_k), F)$ and take a point

$$\tilde{z} \in D'(q', E) \cap D'(f^{-m_0 m_k}(z_k), F)$$

(Figure 10). Then we have

$$f^{m_0 n_0}(\tilde{z}) \in f^{m_0 n_0}(D'(q', E)) \subset D(q, E) \subset W^s(q_0).$$

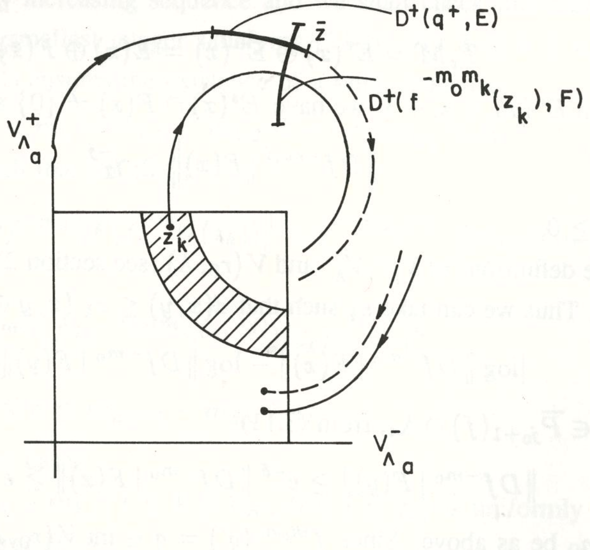


Figure 10

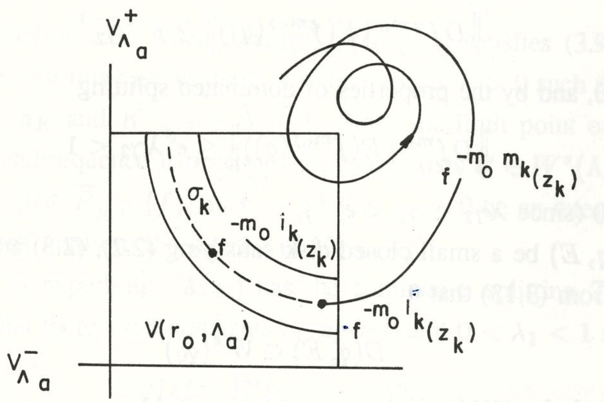


Figure 11

Therefore $\tilde{z} \in W^s(\Lambda_a)$. Since (3.7.b) holds, we can apply (XIV) for large $k$ with $1 \le k \le a_K$. Then the distance between $f^{-j}(\tilde{z})$ and $f^{-j}(f^{-m_0 m_k}(z_k))$ is sufficiently small, say

$$d(f^{-j}(\tilde{z}), f^{-j}(f^{-m_0 m_k}(z_k)) < \delta,$$

for all $0 \le j \le m_0(l_k - m_k)$.

To create a homoclinic orbit associated with $\Lambda_a$ (i.e. $\tilde{z} \in W^s_g(\Lambda_a) \cap W^u_g(\Lambda_a)$)





for a certain diffeomorphism $g$ being $C^1$ near to $f$, we prepare the following (3.14) and (3.15). Notice that each $\{l_k - m_k\}_{k=1}^{a_K}$ is a strictly increasing sequence.

(3.14) For $k > 0$ let $\sigma_k$ be the string containing $f^{-m_k l_k}(z_k)$ and $i'_k > 0$ be the largest integer such that $f^{-m_0 i'_k}(z_k) \in \sigma_k$ and $(f^{-m_0 i'_k}(z_k), f^{-m_0 m_k}(z_k))$ is a uniform $(f^{-m_0}, \gamma, F)$-string i.e.

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(m_k+j)}(z_k)) \right\| \le \gamma^{n+1}$$

for all $0 \le n \le i'_k$. Let $E_K$ be the consecutive, strictly increasing subsequence starting at the first term $i'_1 - l_1$ of $\{i'_k - l_k\}_{k=1}^{a_K}$. Then one has that either there exists an infinite sequence $\{K\}$ such that $E_K \nearrow$ as $K \to \infty$ and $\mathcal{O} \cap V^-_{\Lambda_a} = \emptyset$, where $\mathcal{O}$ denotes the set of all accumulation points of

$$\{f^{-m_0(i'_k+1)}(z_k) \mid i'_k - l_k \in E_K,\ K \in \{K\}\},$$

or $W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \emptyset$ for some basic set $\Lambda_b$. To avoid complication we suppose, without loss of generality, that if we have the first part then $E_K = \{i'_p - l_k\}_{k=1}^{a_K}$ for $K \in \{K\}$ (Figure 11).

To see (3.14) suppose that $\{i'_k - l_k\}_{k=1}^{a_K}$ is uniformly bounded (then it is clear that $\mathcal{O} \cap (V^-_{\Lambda_a} - \Lambda_a) \neq \emptyset$). But $\{l_k - m_k\}_{k=1}^{a_K}$ is not uniformly bounded with respect to $K > l + N + 1$. Thus $\{i'_k - m_k\}_{k=1}^{a_K}$ is not uniformly bounded. By maximality of $i'_k$, $(f^{-m_0(i'_k+1)}(z_k), f^{-m_0 m_k}(z_k))$ is an uniform $(f^{-m_0}, \gamma, F)$-obstruction; i.e.

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{-m_0(i'_k+1-j)}(z_k)) \right\| > \gamma^{n+1}$$

for all $0 \le n \le i'_k - m_k - 1$. Then we can find $\hat{q} \in V^-_{\Lambda_a} - \Lambda_a$ such that

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{m_0 j}(\hat{q})) \right\| \ge \gamma^{n+1}$$

for all $n \ge 0$. Thus we have $\hat{q} \in W^s(\Lambda_b)$ for some basic set $\Lambda_b$ by (XII) and so $W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \emptyset$. Therefore we obtain (3.14).

Let $r_0 > 0$ be as before (see section 2) and for $x \in M$ let $T_x(T'_x)$ be the integer satisfying

$$d(f^{-j}(x), V^-_{\Lambda_a}) \le r_0/2, \quad \text{for all} \quad 0 \le j \le T_x$$
$$(d(f^{j}(x), V^+_{\Lambda_a}) \le r_0/2, \quad \text{for all} \quad 0 \le j \le T'_x)$$

and

$$d(f^{-(T_x+1)}(x), V_{\Lambda_a}^-) > r_0/2 \quad (d(f^{(T'_x+1)}(x), V_{\Lambda_a}^+) > r_0/2).$$

Then we have that

(3.15) given a small $0 < \delta \le r_0/2$ there exist positive integers $\hat{l}$ and $L$ such that for $k \ge L$ and $x \in V(r_k, \Lambda_a)$ if one has either $d(f^{-j}(x), f^{-j}(y)) \le \delta$ for $0 \le j \le T_x$, or $d(f^j(x), f^j(y)) \le \delta$ for $0 \le j \le T'_x$, then $y \in S_{k-\hat{l}}$.

This is easily followed from (VII). In fact, since $x \in V(r_k, \Lambda_a)$, we have $d(x, V_{\Lambda_a}^-) \le r_k = r_0^{(1+\delta_0)^k}$ and by (VII) (1)

$$\gamma_1^{T_x+1} r_0/2 < \gamma_1^{T_x+1} d(f^{-(T_x+1)}(x), V_{\Lambda_a}^-) \le d(x, V_{\Lambda_a}^-) \le r_0^{(1+\delta_0)^k}.$$

Take $1 > C'_1 > \log r_0 / \log \gamma_1$. Then there exists $L_1 > 0$ such that $k \ge L_1$ implies $T_x > C'_1(1+\delta_0)^k$. Suppose $y \notin S_{k-\hat{l}}$ for all $\hat{l} > 0$. Since $T(y) \le C_2(1+\delta_0)^{k-\hat{l}}$ by (VII), there exist $L_2 \ge L_1$ and $\hat{l} > 0$ satisfying $T_x > T(y)$ when $k \ge L_2$. Thus there exists $0 \le j \le T_x$ such that $f^{-j}(y) \notin V(r_0, \Lambda_a)$ and by (VII),

$$d(f^{-j}(x), V_{\Lambda_a}^+) \le \lambda_1^j d(x, V_{\Lambda_a}^+) \le \lambda_1^j r_k < r_0/2.$$

Therefore $d(f^{-j}(x), f^{-j}(y)) > r_0/2$, which is a contradiction. The first part of (3.15) was obtained. Similarly we obtain the second part.

Now we use (3.14) for the family $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$. Then we have that either $W^u(\Lambda_a) \cap W^s(\Lambda_b) \ne \varnothing$ for some basic set $\Lambda_b$, or there exists an infinite sequence $\{K\}$ such that $\{i'_k - l_k\}_{k=1}^{a_K}$ is strictly increasing and $\mathcal{O} \cap V_{\Lambda_a}^- = \varnothing$. If we have the first part, then it is our requirement.

Therefore for the case when we have the lafer we shall derive a contradiction. Let $L$ be as in (3.15). Given a large $k \ge L$ there exists $t_k = T^l_{f^{-m_0 i'_k}(z_k)}$ such that

$$d(f^{j-m_0 i'_k}(z_k), V_{\Lambda_a}^+) \le r_0/2$$

for all $0 \le j \le t_k$ and

$$d(f^{(t_k+1)-m_0 i'_k}(z_k), V_{\Lambda_a}^+) > r_0/2.$$

As $l > 0$ was arbitrary we take it such that $l = \hat{l}$. Here $\hat{l}$ is an integer chosen as in (3.15).

Let $\tilde{z}$ be as above. Then it is concluded (by using (3.1) and (3.15)) that there exists a $(f^{-m_0}, \gamma, F)$-string $\sigma_1 \subset S_{K-\hat{l}}$ contained in the orbit of $\tilde{z}$ such that





$\sigma \cap S_{K-\hat{l}-1} = \varnothing$ for every $(f^{-m_0}, \gamma, F)$-string $\sigma \ne \sigma_1$ contained in $\mathcal{O}_{l_k}(\tilde{z})$. Use here (VIII) and (VIII)$^o$. Then there exists a small $C^1$ perturbation of $f$, by which a homoclinic orbit associated with $\Lambda_a$ is created, thus contradicting (VI).

For the case when the family $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies (3.9) we have proved that $W^u(\Lambda_a) \cap W^s(\Lambda_b) \ne \varnothing$ for some basic set $\Lambda_b$.

Next we consider the case when $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ satisfies (3.10) for all $K$ sufficiently large. As in (3.8) we have that $0 = m_{k,0} < m_k = m_{k,1} < m_{k,2} < \cdots < m_{k,j_k} \le l_k$ for all $1 \le k \le a_K$ and all large $K$. Then, by the assumption

$$\{m_{k,i} - m_{k,i-1} \mid 0 \le i \le j_k, \ 1 \le k \le a_K, \ K \ge \hat{l}\}$$

is unbounded. Thus, for $L > 0$ there exists $K_L > \hat{l} + \overline{N} + 1$ such that for all $K \ge K_L$ there exist a sequence $1 \le k_1 < \cdots < k_{R_K} \le a_K$ and $1 \le i_{k_q} \le j_{k_q}$, $1 \le q \le R_K$, so that

$$m_{k_q, i_{k_q}} - m_{k_q, i_{k_q}-1} \ge L.$$

Let $\{K'\}$ be an infinite sequence of $K_L$ for all $L > 0$. If $\{R_K \mid K \in \{K'\}\}$ is bounded and if

$$B_K = \{k_1, \ldots, k_{R_K}\} \text{ and } A_K = \{1, 2, \ldots, a_K\} - \cup_{\{K'\}} B_{K'}$$

for all $K > \hat{l} + \overline{N} + 1$, then the family $(\{z_k\}_{k \in A_K}, \{l_k\}_{k \in A_K})$ derives a contradiction by using the above argument since (3.9) is satisfied.

Therefore, when $\{R_K \mid K \in \{K'\}\}$ is unbounded we proceed our argument for the family $(\{z_k\}_{k \in B_{K'}}, \{l_k\}_{k \in B_{K'}})$. To avoid complication we may suppose that $\{K'\}$ is the sequence $\{K\}$ of $K$ satisfying $K > \hat{l} + \overline{N} + 1$ and each $B_K$ is the sequence $\{1, 2, \ldots, a_K\}$.

Since $\{m_{k,1} - m_{k,0}\}_{k=1}^{a_K} = \{m_k\}_{k=1}^{a_K}$ for all large $K$, for $L > 0$ there exists $K_0 > 0$ such that $m_k \ge L$ for all $1 \le k \le a_K$ and $K \ge K_0$. Then, for $\delta > 0$ we can find $K_\delta > 0$, $q_{K_\delta} \in \overline{P}_{j_0+1}(f)$ and finitely many points $q$ belonging to $\{f^{-m_0(m_k-1)}(z_k)\}_{k=1}^{a_{K_\delta}}$ such that the distance between $q_{K_\delta}$ and $q$ is less than $\delta$. If $\delta \to 0$, then we have $q_{K_\delta} \to \hat{q}$ since $K_\delta \to \infty$. Then by (3.7.b)

$$\prod_{j=0}^{n} \left\| Df^{-m_0} \mid F(f^{m_0 j}(\hat{q})) \right\| \ge \gamma^{n+1}$$

for all $n \ge 0$. Thus $\hat{q} \in W^s(\Lambda_b)$ for some basic set $\Lambda_b$. If $\Lambda_a = \Lambda_b$, as observed above we have a homoclinic orbit associated with $\Lambda_a$ under a certain small $C^1$ perturbation of $f$, thus contradicting (VI).

For the case when $\Lambda_a \neq \Lambda_b$ we shall show that $W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \varnothing$. The rest of this section will be devoted to obtain it.

Notice first that for $K$ sufficiently large finitely many points belonging to $\{f^{-m_0(m_k-1)}(z_k)\}_{k=1}^{a_K}$ are near to $\hat{q}$. For convenience we may suppose that all points of $\{f^{-m_0(m_k-1)}(z_k)\}_{k=1}^{a_K}$ are near to $\hat{q}$ and this holds for all $K$ sufficiently large. To simplify the notations write

$$\bar{z}_k = f^{-m_0(m_k-1)}(z_k) \text{ and } \bar{l}_k = l_k - m_k + 1$$

for all $1 \leq k \leq a_K$ and all $K$ sufficiently large (Figure 12). Obviously $f^{-m_0 \bar{l}_k}(\bar{z}_k) = f^{-m_0 l_k}(z_k)$ for all $k$ and the family $\{\mathcal{O}_{\bar{l}_k}(\bar{z}_k)\}_{k=1}^{a_K}$ satisfies one of the following properties.

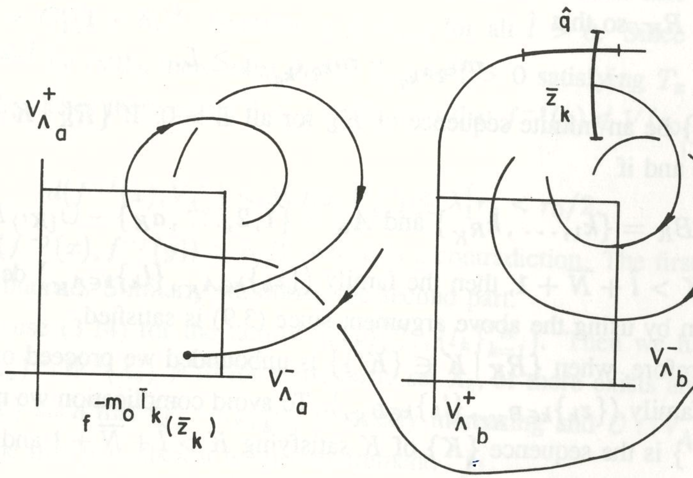


Figure 12

(3.16) There exists $K_1 > 0$ such that for all $1 \leq k \leq a_K$ and all large $K$

$$\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1}^{\Lambda_b} = \varnothing.$$

(3.17) Otherwise, i.e. for $K_1 > 0$ there exist $K > 0$ and $1 \leq k \leq a_K$ such that

$$\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1}^{\Lambda_b} \neq \varnothing \text{ and } \mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1+1}^{\Lambda_b} = \varnothing.$$

Let $\{K'\}$ be a sequence of $K$ satisfying (3.17) for $K_1 > 0$. If $\{K'\}$ is finite, then $\{K\} - \{K'\}$ satisfies (3.16). For this case (3.16) holds for all $K$ sufficiently large. If $\{K'\}$ is infinite, then there exist $K'' \in \{K'\}$ and $1 \leq k \leq a_{K''}$ such





that

$$\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1+1}^{\Lambda_b} \neq \varnothing \text{ and } \mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1+2}^{\Lambda_b} = \varnothing.$$

Let $\{K''\}$ be a sequence of $K''$ satisfying the above relation and repeat the same manner for $\{K''\}$. Then, in the repetition of $n$ time we have that either (3.16) holds for all large $K$, or

(3.18) there exists an infinite sequence $\{K^{(n)}\}$ such that for all $K^{(n)}$ there exist $0 < k_1 < k_2 < \cdots < k_n \leq a_{K^{(n)}}$ so that

$$\mathcal{O}_{\bar{l}_{k_i}}(\bar{z}_{k_i}) \cap S_{K_1+i}^{\Lambda_b} \neq \varnothing \quad \text{and} \quad \mathcal{O}_{\bar{l}_{k_1}}(\bar{z}_{k_i}) \cap S_{K_1+i+1}^{\Lambda_b} = \varnothing,$$

for all $1 \leq i \leq n$. To avoid complication for the above two relations we may suppose that

$$\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1+k}^{\Lambda_b} \neq \varnothing \text{ and } \mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1+k+1}^{\Lambda_b} = \varnothing,$$

for all $1 \leq k \leq a_{K^{(n)}}$ and $K^{(n)}$.

First we investigate the case when the family $\{\mathcal{O}_{\bar{l}_k}(\bar{z}_k)\}_{k=1}^{a_K}$ satisfies (3.18). Let $\hat{q}$ be as above. Fix $K$ sufficiently large and choose a point $\bar{z}_{k_0}$ which is near to $\hat{q}$. Notice that $\hat{q} \in W^s(\Lambda_b)$. As above we can find small closed disks $D'(\bar{z}_{k_0}, F)$ and $D'(\hat{q}, E)$ satisfying

(3.19) $$D'(\bar{z}_{k_0}, F) \cap D'(\hat{q}, E) \neq \varnothing.$$

Take $\tilde{z} \in D'(\bar{z}_{k_0}, F) \cap D'(\hat{q}, E) \subset W^s(\Lambda_b)$. By using the fact that

$$\prod_{j=0}^{n-1} \left\| Df^{-m_0} \mid F(f^{-m_0 j}(\bar{z}_{k_0})) \right\| < \gamma^n$$

for $1 \leq n \leq \bar{l}_{k_0}$, it follows that the distance between $f^{-j}(\tilde{z})$ and $f^{-j}(\bar{z}_{k_0})$ is small for $0 \leq j \leq m_0 \bar{l}_{k_0}$ (see (XIV)). Let $\hat{l}$ be as in (3.15). From the construction of $(\{z_k\}_{k=1}^{a_K}, \{l_k\}_{k=1}^{a_K})$ we have $\mathcal{O}_{l_k}(z_k) \cap (S_{k-\hat{l}}^{\Lambda_a} - S_K^{\Lambda_a}) = \varnothing$ for $1 \leq k \leq a_K$ and hence

(3.20) $$\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap (S_{K-\hat{l}}^{\Lambda_a} - S_K^{\Lambda_a}) = \varnothing \quad (1 \leq k \leq a_K).$$

Use (3.14) for the family $(\{\bar{z}_k\}_{k=1}^{a_K}, \{\bar{l}_k\}_{k=1}^{a_K})$. Then there exists a family $\{i'_k\}_{k=1}^{a_K}$ such that one of the following cases holds.

(3.21) There exists an infinite sequence $\{K\}$ such that $\{i'_k - \bar{l}_k\}_{k=1}^{a_K}$ is strictly increasing and $\mathcal{O} \cap V_{\Lambda_a}^- = \varnothing$ (here $\mathcal{O}$ is defined as in (3.14)). For this case, as observed above, there exists a $(f^{-m_0}, \gamma, F)$-string $\sigma_1$ contained in the orbit of $\tilde{z}$

such that $\sigma \cap S^{\Lambda_a}_{k-\hat{l}-1} = \emptyset$ for every $(f^{-m_0}, \gamma, F)$-string $\sigma \neq \sigma_1$ contained in the orbit of $\tilde{z}$. Then we can find certain diffeomorphism $g$ such that $W^u_g(\Lambda_a) \cap W^s_g(\Lambda_b) \neq \emptyset$ for some basic set $\Lambda_b$ (Figure 13).

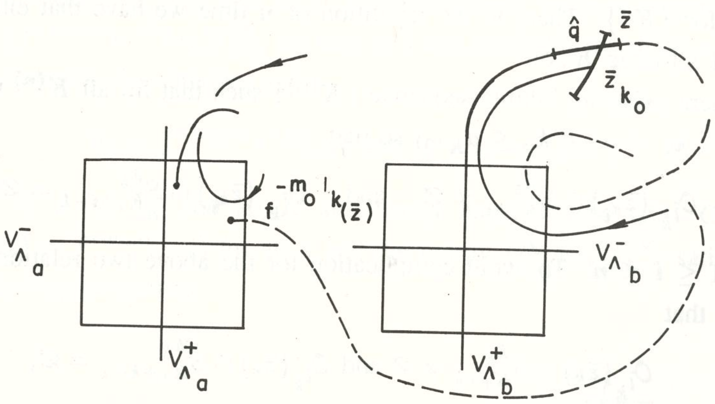


Figure 13

(3.22) When $\{i'_k - \bar{l}_k\}_{k=1}^{a_K}$ is uniformly bounded with respect to all large $K$, there exists a basic set $\Lambda_b$ such that $W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \emptyset$.

As (3.22) is our requirement, we shall derive a contradiction for the case (3.21). For large $K$ choose $\bar{z}_{k_0}$ as above and so take the smallest integer $k_1 = k_1(K) > 0$ such that

$$\mathcal{O}_{\bar{l}_{k_0}}(\bar{z}_{k_0}) \cap S^{\Lambda_b}_{k_1} \neq \emptyset \text{ and } \mathcal{O}_{\bar{l}_{k_0}}(\bar{z}_{k_0}) \cap S^{\Lambda_b}_{k_1+1} = \emptyset.$$

Take $K$ so large that $\bar{z}_{k_0}$ is very near to $\hat{q}$. Then we can choose $k_1(K)$ so large that $k_1(K) > \overline{N} + \hat{l}$. Since the family $\{\mathcal{O}_{\bar{l}_k}(\bar{z}_k)\}_{k=1}^{a_K}$ satisfies (3.18), as $K_1$ of (3.18) we take $K' = k_1(K)$. Then by choosing $K$ as in (3.18) we have that for all $0 \leq j \leq a_K - k_0$

$$\mathcal{O}_{\bar{l}_{k_0+j}}(\bar{z}_{k_0+j}) \cap S^{\Lambda_b}_{K'+j} \neq \emptyset \text{ and } \mathcal{O}_{\bar{l}_{k_0+j}}(\bar{z}_{k_0+j}) \cap S^{\Lambda_b}_{K'+j+1} = \emptyset.$$

To avoid complication we may suppose that $\bar{z}_{k_0} = \bar{z}_1$ and $\bar{l}_{k_0} = \bar{l}_1$, and then we have

$$\mathcal{O}_{\bar{l}_j}(\bar{z}_j) \cap S^{\Lambda_b}_{K'+j-1} \neq \emptyset \text{ and } \mathcal{O}_{\bar{l}_j}(\bar{z}_j) \cap S^{\Lambda_b}_{K'+j} = \emptyset,$$

for all $1 \leq j \leq a_K$ and large $K$ (notice that $K'$ depends on $K$). Notice that the family $(\{\bar{z}_k\}_{k=1}^{a_K}, \{\bar{l}_k\}_{k=1}^{a_K})$ is defined nearby the basic set $\Lambda_b$ and satisfies (3.20) for large $K$.





For $1 \leq j \leq a_K$ and large $K$ denote as $f^{-i_j}(\bar{z}_j)$ the point of $\mathcal{O}_{\bar{l}_j}(\bar{z}_j)$ that runs lastly away $S^{\Lambda_b}_{K'+j}$. Then we have

$$B_j \cap S^{\Lambda_b}(\overline{N} + j - 1) \neq \emptyset$$

where $B_j = \{\bar{z}_j, f^{-1}(\bar{z}_j), \dots, f^{-i_j}(\bar{z}_j)\}$. If this is false, then there exists a certain diffeormorphism $g$ creating a homoclinic orbit associated with $\Lambda_b$ as observed above. This is a contradiction.

Take here the point $z'_j = f^{-i}(\bar{z}_j) \in B_j$ that runs lastly away $S^{\Lambda_b}(\overline{N}+j-1)$. Let $l'_j$ be the smallest integer such that $f^{-m_0 l'_j}(z'_j)$ is the first point belonging to $S^{\Lambda_b}_{K'+j}$. Obviously $l'_j \leq \bar{l}_j$ for $1 \leq j \leq a_K$. Since $\overline{N}+j \leq K'-\hat{l} < K' < K'+j$ for all $1 \leq j \leq a_K$, we have

$$S^{\Lambda_b}_{K'+j} \subset S^{\Lambda_b}_{K'} \subset S^{\Lambda_b}_{K'-\hat{l}} \subset S^{\Lambda_b}_{\overline{N}+j} \quad (1 \leq j \leq a_K).$$

The choice of $z'_j$ and $f^{-m_0 l'_j}(z'_j)$ ensures that

$$\mathcal{O}_{l'_j}(z'_j) \cap (S^{\Lambda_b}_{\overline{N}+j} - S^{\Lambda_b}_{K'+j}) = \emptyset.$$

In fact, if this is false, by the manner as above we can find a certain diffeomorphism creating a homoclinic orbit associated with $\Lambda_b$, thus contradicting.

Therefore, for the family $(\{z'_j\}_{j=1}^{a_K}, \{l'_j\}_{j=1}^{a_K})$ we have

$$\text{(3.23)} \qquad \begin{cases} \mathcal{O}_{l'_j}(z'_j) \cap (S^{\Lambda_b}_{K'-\hat{l}} - S^{\Lambda_b}_{K'}) = \emptyset, \\ \mathcal{O}_{l'_j}(z'_j) \subset \mathcal{O}_{\bar{l}_j}(\bar{z}_j) \subset \mathcal{O}_{l_j}(z_j), \\ \mathcal{O}_{l'_j}(z'_j) \cap (S^{\Lambda_a}_{K-\hat{l}} - S^{\Lambda_a}_{K}) = \emptyset. \end{cases}$$

As observed above we have that $\{l'_k\}_{k=1}^{a_K}$ is not uniformly bounded with respect to all large $K$ and the family $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$ derives one of the following properties.

(3.24) Let $\{m'_k\}_{k=1}^{a_K}$ be the sequence satisfying (3.8) for the family $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$. Then there exists $\hat{q} \in V^-_{\Lambda_b} - \Lambda_b$ such that $\hat{q} \in W^s(\Lambda_c)$ for some basic set $\Lambda_c$, and such that for large $K$ there exist finitely many points $q \in \{f^{-m_0 m'_k}(z'_k)\}_{k=1}^{a_K}$ so that the distance between $\hat{q}$ and $q$ is very small. Thus $W^u(\Lambda_b) \cap W^s(\Lambda_c) \neq \emptyset$ as observed above.

(3.25) For all $K$ sufficiently large there exists a sequence $\{m'_k\}_{k=1}^{a_K}$ of integers satisfying (3.7.b) for $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$ and there exist finitely many points belonging to $\{f^{-m_0(m'_k-1)}(z'_k)\}_{k=1}^{a_K}$ which are near to some point $\tilde{q}'$ of some set

$W^s(\Lambda_c)$. For convenience suppose that all points of $\{f^{-m_0(m'_k-1)}(z'_k)\}_{k=1}^{a_K}$ are near to $q$ and this holds for all $K$ sufficiently large. Write

$$\overline{z}_k = f^{-m_0(m'_k-1)}(z'_k) \text{ and } \overline{l}'_k = l'_k - m'_k + 1$$

for all $k$. Then each $O_{\overline{l}'_k}(\overline{z}'_k)$ satisfies one of the following properties (similar to (3.16) and (3.18)).

(3.25.a) There exists $K_2 > 0$ such that for all large $K$ and all $1 \le k \le a_K$

$$O_{\overline{l}'_k}(\overline{z}'_k) \cap S^{\Lambda_c}_{K_2} = \varnothing.$$

(3.25.b) Otherwise, i.e. for $n > 0$ there exists an infinite sequence $\{K^{(n)}\}$ such that for all $K^{(n)}$ there exist $0 < k_1 < k_2 < \cdots < k_n \le b_{K^{(n)}}$ so that

$$O_{l'_{k_i}}(\overline{z}'_{k_i}) \cap S^{\Lambda_c}_{K_2+i} \ne \varnothing \text{ and } O_{\overline{l}'_{k_i}}(\overline{z}'_{k_i}) \cap S^{\Lambda_c}_{K_2+i+1} = \varnothing,$$

for all $1 \le i \le n$. To avoid complication for the above two relations we may suppose that

$$O_{\overline{l}'_k}(\overline{z}'_k) \cap S^{\Lambda_c}_{K_2+k} \ne \varnothing \text{ and } O_{\overline{l}'_k}(\overline{z}'_k) \cap S^{\Lambda_c}_{K_2+k+1} = \varnothing,$$

for all $1 \le k \le K^{(n)}$ and $K^{(n)}$.

First we check the case when (3.25) is satisfied. Suppose $\{O_{\overline{l}'_k}(\overline{z}'_k)\}_{k=1}^{b_K}$ satisfies (3.25.b) for all $K$ sufficiently large. For convenience put $K = K^{(n)}$ for $n$ large enough. Then, as observed above we can find a point $\overline{z}'_{k_0}$ near to a point $\hat{q}'$ of $W^s(\Lambda_c)$ such that there exist small closed disks $D'(\overline{z}'_{k_0}, F)$ and $D'(\hat{q}', E)$ such that $\varnothing \ne D'(\overline{z}'_{k_0}, F) \cap D'(\hat{q}', E) \subset W^s(\Lambda_c)$. Take $\tilde{z} \in D'(\overline{z}'_{k_0}, F) \cap D'(\hat{q}', E)$. It is checked that the distance between $f^{-j}(\tilde{z})$ and $f^{-j}(\overline{z}_{k'_0})$ is small for $0 \le j \le m_0 \overline{l}'_{k_0}$.

Since $O_{l'_k}(z'_k) \cap (S^{\Lambda_b}_{K'-\hat{l}} - S^{\Lambda_b}_{K'}) = \varnothing$ for $1 \le k \le a_K$ by (3.23), obviously

$$O_{\overline{l}'_{k_0}}(\overline{z}'_{k_0}) \cap (S^{\Lambda_b}_{K'-\hat{l}} - S^{\Lambda_b}_{K'}) = \varnothing.$$

Since $O_{\overline{l}'_{k_0}}(\overline{z}'_{k_0}) \subset O_{l'_{k_0}}(z'_{k_0})$ and $O_{l'_k}(z'_k) \cap (S^{\Lambda_a}_{K-\hat{l}} - S^{\Lambda_a}_{K}) = \varnothing$, for all $1 \le k \le a_K$ and large $K$, we have

$$O_{\overline{l}'_{k_0}}(\overline{z}'_{k_0}) \cap (S^{\Lambda_a}_{K-\hat{l}} - S^{\Lambda_a}_{K}) = \varnothing.$$

Use (3.14) for the family $(\{\overline{z}'_k\}_{k=1}^{a_K}, \{\overline{l}'_k\}_{k=1}^{a_K})$. Then there exists a family $\{i'_k\}_{k=1}^{a_K}$ such that one of the following two cases holds.





(3.26) There exists an infinite sequence $\{K\}$ such that $\{i'_k - \overline{l}'_k\}_{k=1}^{a_K}$ is strictly increasing and $O \cap V^-_{\Lambda_b} = \varnothing$ (here $O$ is defined as in (3.14)). For this case, as in (3.21), we can find a certain diffeomorphism $g$ such that $W^u_g(\Lambda_b) \cap W^s_g(\Lambda_c) \ne \varnothing$ for some basic set $\Lambda_c$.

(3.27) When $\{i'_k - \overline{l}'_k\}_{k=1}^{a_K}$ is uniformly bounded for all large $K$, there exists a basic set $\Lambda_c$ such that $W^u(\Lambda_b) \cap W^s(\Lambda_c) \ne \varnothing$.

Suppose that we have (3.26). For large $K$ choose $\overline{z}'_{k_0}$ as above and so take the smallest integer $k_2 = k_2(K, k_1(K)) > 0$ such that

$$O_{\overline{l}_{k_0}}(\overline{z}'_{k_0}) \cap S^{\Lambda_c}_{k_2} \ne \varnothing \text{ and } O_{\overline{l}'_{k_0}}(\overline{z}'_{k_0}) \cap S^{\Lambda_c}_{k_2+1} = \varnothing.$$

As observed above we can take $k_2$ so large that $k_2 > \overline{N} + \hat{l}$ by choosing a large $K$. Put $K'' = k_2$ and use (3.18). Then we have

$$O_{\overline{l}'_{k_0+1}}(\overline{z}'_{k_0+j}) \cap S^{\Lambda_c}_{K''+j} \ne \varnothing \text{ and } O_{\overline{l}'_j}(\overline{z}'_j) \cap S^{\Lambda_c}_{K''+j+1} = \varnothing,$$

for all $0 \le j \le a_K - k_0$ and large $K$. Again we may suppose that $\overline{z}'_{k_0} = \overline{z}'_1$ and $\overline{l}'_{k_0} = \overline{l}'_1$, and then we have

$$O_{\overline{l}'_j}(\overline{z}'_j) \cap S^{\Lambda_c}_{K''+j-1} \ne \varnothing \text{ and } O_{\overline{l}'_j}(\overline{z}'_j) \cap S^{\Lambda_c}_{K''+j} = \varnothing,$$

for all $1 \le j \le a_K$ and large $K$ (notice that $K''$ depends on $K$ and $K' = k_1(K)$). Then the family $(\{\overline{z}'_k\}_{k=1}^{a_K}, \{\overline{l}'_k\}_{k=1}^{a_K})$ is defined nearby the basic set $\Lambda_c$. This family satisfies (3.20) and (3.23) for large $K$.

As observed above we can find finite sequences $\{z''_k\}_{k=1}^{a_K}$ and $\{l''_k\}_{k=1}^{a_K}$ such that each $z''_k$ is a point of $O_{\overline{l}'_k}(\overline{z}'_k)$ that runs lastly away $S^{\Lambda_c}(\overline{N} + k - 1)$ and each $l''_k$ is the smallest integer such that $f^{-m_0 l''_k}(z''_k)$ is the first point belonging to $S^{\Lambda_c}_{K''}$. Then it is easily checked that the family $(\{z''_k\}_{k=1}^{a_K}, \{l''_k\}_{k=1}^{a_K})$ satisfies

$$(3.28)\quad \begin{cases} O_{l''_k}(z''_k) \cap (S^{\Lambda_c}_{K''-\hat{l}} - S^{\Lambda_c}_{K''}) = \varnothing, \\ O_{l''_k}(z''_k) \subset O_{l'_k}(z'_k), \\ O_{l''_k}(z''_k) \cap ([S^{\Lambda_a}_{K-\hat{l}} - S^{\Lambda_a}_{K}] \cup [S^{\Lambda_b}_{K'-\hat{l}} - S^{\Lambda_b}_{K'}]) = \varnothing. \end{cases}$$

Therefore the family $(\{z''_k\}_{k=1}^{a_K}, \{l''_k\}_{k=1}^{a_K})$ satisfies again one of (3.24) and (3.25).

Suppose that we have (3.27). Here let $m'_k$ be an integer satisfying all the properties of $m_k$ of (3.14) for $1 \le k \le K$ and large $K$. Then each $\{\overline{l}'_k - m_k\}_{k=1}^{a_K}$ is a strictly increasing sequence. Thus each $\{i'_k - m'_k\}_{k=1}^{a_K}$ is a strictly increasing sequence except for a certain bounded set. Let $\delta > 0$ be small. Then there exist $K_\delta \in \{K\}$, $q_{K_\delta} \in \overline{P}_{j_0+1}(f)$ and finitely many points $q \in A = \{f^{-m_0(i'_k+1)}(\overline{z}'_k)\}_{k=1}^{a_K}$ such that the distance between $q_{K_\delta}$ and $q$ is very small. Since $q_{K_\delta} \to \hat{q}$ as $\delta \to 0$, as in the proof of (3.14) we have that $\hat{q} \in V_{\Lambda_b}^- - \Lambda_b$ and that $\hat{q} \in W^u(\Lambda_b) \cap W^s(\Lambda_c)$ for some $\Lambda_c$. Thus $f^{m_0 N}(\hat{q}) \in V_{\Lambda_c}^+$ for some $N > 0$ and so the distance between $f^{m_0 N}(q_{K_\delta})$ and $V_{\Lambda_c}^+$ is small for large $K_\delta$. Thus the distance between $f^{m_0 N}(q)$ and $V_{\Lambda_c}^+$ is small. To avoid complication, for all $K_\delta$ and all $q \in A$ we may suppose that the distance between $f^{m_0 N}(q)$ and $V_{\Lambda_c}^+$ is small.

By construction we have that for all $1 \le k \le a_{K_\delta}$ and all $K_\delta$

$$\mathcal{O}_{\overline{l}'_k}(\overline{z}'_k) \subset \mathcal{O}_{l'_k}(z'_k) \subset \mathcal{O}_{\overline{l}_k}(\overline{z}_k) \subset \mathcal{O}_{l_k}(z_k).$$

The family $\{\mathcal{O}_{\overline{l}_k}(\overline{z}_k)\}_{k=1}^{a_{K_\delta}}$ satisfies (3.17) and hence it satisfies (3.18). To simplify the notations we may suppose that (3.18) holds for all large $K_\delta$ and we write $K = K_\delta$.

To obtain our requirement for the family $\{\mathcal{O}_{\overline{l}_k}(\overline{z}_k)\}_{k=1}^{a_K}$ we adapt the method used to construct the family $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$. In fact take $\overline{z}_{k_0}$ as in (3.19) by replacing $\Lambda_b$ with $\Lambda_c$ and as above let $k_1(K) = k_1$ be the smallest integer such that $\mathcal{O}_{\overline{l}_{k_0}}(\overline{z}_{k_0}) \cap S_{k_1}^{\Lambda_c} \neq \emptyset$ and $\mathcal{O}_{\overline{l}_{k_0}}(\overline{z}_{k_0}) \cap S_{k_1+1}^{\Lambda_c} = \emptyset$. Since $\{\mathcal{O}_{\overline{l}_k}(\overline{z}_k)\}_{k=1}^{a_K}$ satisfies (3.18), as $K_1$ of (3.18) we put $K'' = k_1(K)$ for large $K$. Then we have

$$\mathcal{O}_{\overline{l}_{k_0+j}}(\overline{z}_{k_0+j}) \cap S_{K''+j}^{\Lambda_c} \neq \emptyset \text{ and } \mathcal{O}_{\overline{l}_{k_0+j}}(\overline{z}_{k_0+j}) \cap S_{K''+j+1}^{\Lambda_c} = \emptyset,$$

for all $0 \le j \le a_K - k_0$. To simplify the notations as above we may suppose that $\overline{z}_{k_0} = \overline{z}_1$ and $\overline{l}_{k_0} = \overline{l}_1$, and then we have

$$\mathcal{O}_{\overline{l}_j}(\overline{z}_j) \cap S_{K''+j-1}^{\Lambda_c} \neq \emptyset \text{ and } \mathcal{O}_{\overline{l}_j}(\overline{z}_j) \cap S_{K''+j}^{\Lambda_c} = \emptyset,$$

for all $1 \le j \le a_K$. Notice that the family $(\{\overline{z}_k\}_{k=1}^{a_K}, \{\overline{l}_k\}_{k=1}^{a_K})$ is defined nearby the basic set $\Lambda_c$ and satisfies (3.20).

Denote as $f^{-j_i}(\overline{z}_j)$ the point of $\mathcal{O}_{\overline{l}_j}(\overline{z}_j)$ that runs lastly away $S_{K''+j}^{\Lambda_c}$. Then we have $B_j \cap S^{\Lambda_c}(\overline{N}+j-1) \neq \emptyset$ where $B_j = \{\overline{z}_j, f^{-1}(\overline{z}_j), \ldots, f^{-j_i}(\overline{z}_j)\}$. Take the point $z''_j \in B_j$ that runs lastly away $S^{\Lambda_c}(\overline{N}+j-1)$. Let $l''_j$ be the smallest integer such that $f^{-m_0 l''_j}(z''_j)$ is the first point belonging to $S_{K''+j}^{\Lambda_c}$. Then





for the family $(\{z''_j\}_{j=1}^{a_K}, \{l''_j\}_{j=1}^{a_K})$ we have

$$(3.29) \qquad \begin{cases} \mathcal{O}_{l''_j}(z''_j) \cap (S_{K'-\hat{l}}^{\Lambda_c} - S_{K''}^{\Lambda_c}) = \emptyset, \\ \mathcal{O}_{l''_j}(z''_j) \subset \mathcal{O}_{\overline{l}_j}(\overline{z}_j) \subset \mathcal{O}_{l_j}(z_j), \\ \mathcal{O}_{l''_j}(z''_j) \cap (S_{K-\hat{l}}^{\Lambda_a} - S_K^{\Lambda_a}) = \emptyset. \end{cases}$$

The family $(\{z''_k\}_{k=1}^{a_K}, \{l''_k\}_{k=1}^{a_K})$ satisfies one of (3.24) and (3.25).

For the two cases (3.26) and (3.27) we constructed the families satisfying (3.28) and (3.29) respectively. In any case suppose (3.25.b) is satisfied and repeat the above argument for them. Let us do it for the family $(\{z''\}_{k=1}^{a_K}, \{l''\}_{k=1}^{a_K})$ satisfying (3.28). Then we shall reach a contradiction. In the similar way we have the same conclusion for the family satisfying (3.29) and so we omit the proof of this case.

First we consider the case when (3.25.b) is always satisfied in the repetition. Since the number of basic sets is finite, in the repetition of finite time we can find, by using (3.21) and (3.26), a diffeomorphism $g$ such that $g = f$ in the complement of the union of certain neighborhoods being nearby basic sets $\Lambda_a, \Lambda_b, \ldots$ and $g$ creates a cycle among the basic sets. However this is a contradiction.

Therefore, in the above repetition, that (3.25.b) is always satisfied is false. Thus we reach the case that satisfies (3.16) (or (3.25.a)) in the repetition of finite time. Therefore it will be enough to check the case when the family $\{\mathcal{O}_{\overline{l}_k}(\overline{z}_k)\}_{k=1}^{a_K}$ satisfies (3.16) (or (3.25.a)).

Since $\{\overline{l}_k\}_{k=1}^{a_K}$ is not uniformly bounded with respect to $K$, as we did for the family $\{l_k\}_{k=1}^{a_K}$ we may suppose that each $\{\overline{l}_k\}_{k=1}^{a_K}$ is a strictly increasing sequence. Thus the family $(\{\overline{z}_k\}_{k=1}^{a_K}, \{\overline{l}_k\}_{k=1}^{a_K})$ admits the sequence $0 = m_{k,0} < m_{k,1} < \cdots < m_{k,j_k} \le \overline{l}_k$, $1 \le k \le a_K$, satisfying (3.8), for which we have one of (3.9) and (3.10).

If (3.10) is satisfied, as observed above we can suppose that for $L > 0$ there exists $K_0 > 0$ so that $m_k \ge L$ for all $1 \le k \le a_K$ and $K \ge K_0$. Thus there exists $\hat{q} \in \overline{P}_{j_0+1}(f)$ such that there exists finitely many points $q$ belonging to $\{f^{-m_0(m_k-1)}(z_k)\}_{k=1}^{a_K}$ such that the distance between $q$ and $\hat{q}$ is small, and such that, by (3.8) and (VII), we have $\hat{q} \in W^s(\Lambda_c)$ for some $\Lambda_c$. If $\Lambda_a$ and $\Lambda_c$ agree, then we have a contradiction since a homoclinic orbit associated with $\Lambda_a$ is created under a certain small $C^1$ perturbation of $f$. Therefore $\Lambda_a \neq \Lambda_c$.

We claim that $\Lambda_b \neq \Lambda_c$. Indeed, suppose $\Lambda_b = \Lambda_c$. Since finitely many points belonging to $\{f^{-m_0(m_k-1)}(\bar{z}_k)\}_{k=1}^{a_K}$ are near to $\hat{q}$, we have $\hat{q} \in W^s(\Lambda_c) = W^s(\Lambda_b)$. Thus for every $L > 0$ there exist $K > 0$ and $1 \le k \le a_K$ such that $\mathcal{O}_{\bar{l}_k}(\bar{z}) \cap S_L^{\Lambda_b} \neq \varnothing$. But this is contrary to (3.16) (or (3.25.a)).

Henceforth let $K$ denote sufficiently large integers. Since finitely many points belonging to $A = \{f^{-m_0(m_k-1)}(\bar{z}_k)\}_{k=1}^{a_K}$ are near to $\hat{q}$, we may suppose that each point of $A$ is near to $\hat{q}$ and write

$$\bar{\bar{z}}_K = f^{-m_0(m_k-1)}(\bar{z}_k) \text{ and } \bar{\bar{l}}_k = \bar{l} - m_k + 1,$$

$1 \le k \le a_K$. Then the family $\{\mathcal{O}_{\bar{\bar{l}}_k}(\bar{\bar{z}}_k)\}_{k=1}^{a_K}$ satisfies one of (3.16) and (3.18) (or (3.25.a) and (3.25.b)). Without loss of generality we can suppose that (3.16) (or (3.25.a)) is satisfied. Let $0 = m_{k,0} < m_{k,1} < \cdots < m_{k,j_k} \le \bar{\bar{l}}_k$, $1 \le k \le a_K$, be the sequence satisfying (3.8) for the family $\{\mathcal{O}_{\bar{\bar{l}}_k}(\bar{\bar{z}}_k)\}_{k=1}^{a_K}$.

If the sequence satisfies (3.10), as above there exists a basic set $\Lambda_d$ such that $\Lambda_a, \Lambda_b, \Lambda_c$ and $\Lambda_d$ are mutually distinct. Since the number of the basic sets is finite, the repetition of this manner derives a contradiction. Thus we arrive at the case that satisfies (3.9) in the repetition of finite time.

Now it is enough to check the case when the family $\{\mathcal{O}_{\bar{l}_k}(\bar{z}_k)\}_{k=1}^{a_K}$ satisfying (3.16) (or (3.25.a)) obeys (3.9). For any basic set $\Lambda_j$ there exists $R_i$ such that $\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{R_i}^{\Lambda_i} = \varnothing$, for all $1 \le k \le a_K$ and all large $K$. Denote as $K_1$ the maximum of the integers $R_i$. Then $\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_{K_1}^{\Lambda_b} = \varnothing$, for all $1 \le k \le a_K$ and all large $K$. By the manner mentioned in the first part of this section construct nearby $\Lambda_b$ pairs $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$ for all $K > 0$, and repeat the same argument for the pairs. Then a family $(\{\bar{z}'_k\}_{k=1}^{a_K}, \{\bar{l}'_k\}_{k=1}^{a_K})$ which satisfies one of (3.24) and (3.25) is constructed nearby a certain basic set $\Lambda_c$ by using the family $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$. From the above argument we can suppose that the family $\{\mathcal{O}_{\bar{l}'_k}(\bar{z}'_k)\}_{k=1}^{a_K}$ satisfies (3.25) and moreover (3.25.a). Then it satisfies either (3.9) or (3.10). We suppose, without loss of generality, that (3.9) is satisfied. Then there exists $K_2 > 0$ such that for any basic set $\Lambda_i$,

$$\mathcal{O}_{\bar{l}'_k}(\bar{z}'_k) \cap S_{K_2}^{\Lambda_i} = \varnothing \text{ and hence } \mathcal{O}_{\bar{l}'_k}(\bar{z}'_k) \cap S_{K_2}^{\Lambda_c} = \varnothing,$$

for all $k$ and all large $K$.

Here construct nearby $\Lambda_c$ pairs $(\{z''_k\}_{k=1}^{a_K}, \{l''_k\}_{k=1}^{a_K})$ by the manner as in the first part of this section and continue the above argument for the pairs. As above a family $(\{\bar{z}''_k\}_{k=1}^{a_K}, \{\bar{l}''_k\}_{k=1}^{a_K})$ is constructed nearby a certain basic set $\Lambda_d$ by using the family $(\{\bar{z}''_k\}_{k=1}^{a_K}, \{\bar{l}''_k\}_{k=1}^{a_K})$. Then we can find $K_3 > 0$ so that $\mathcal{O}_{\bar{l}'_k}(\bar{z}'_k) \cap$





$S_{K_3}^{\Lambda_i} = \varnothing$ for any basic set $\Lambda_i$, and so $\mathcal{O}_{\bar{l}'_k}(\bar{z}'_k) \cap S_{K_3}^{\Lambda_d} = \varnothing$, for all $k$ and all large $K$. In this repetition we have a diffeomorphism $g$ such that $g$ coincides with $f$ in the complement of the union of certain neighborhoods of basic sets $\Lambda_j$ and $g$ creates a cycle among basic sets $\Lambda_j$.

For instance, if $\Lambda_c$ and $\Lambda_a$ agree, then there exists a positive integer $L$ such that for all $1 \le k \le a_K$ and all large $K$

$$\mathcal{O}_{\bar{l}_k}(\bar{z}_k) \cap S_L^{\Lambda_b} = \varnothing, \quad \mathcal{O}_{\bar{l}'_k}(\bar{z}'_k) \cap S_L^{\Lambda_a} = \varnothing.$$

Notice that $f^{-m_0 l_k}(z_k) = f^{-m_0 \bar{l}_k}(\bar{z}_k) \in S_L^{\Lambda_a}$ and $f^{-m_0 l'_k}(z'_k) = f^{-m_0 \bar{l}'_k}(\bar{z}'_k) \in S_L^{\Lambda_b}$ for all $k$ and all large $K$. Then we have, as observed before, a diffeormorphism $g$ such that $g = f$ in the complement of the union of certain neighborhoods of $\Lambda_a$ and $\Lambda_b$ and $g$ creates a cycle between $\Lambda_a$ and $\Lambda_b$. This is contrary to (VI) (figure 14).

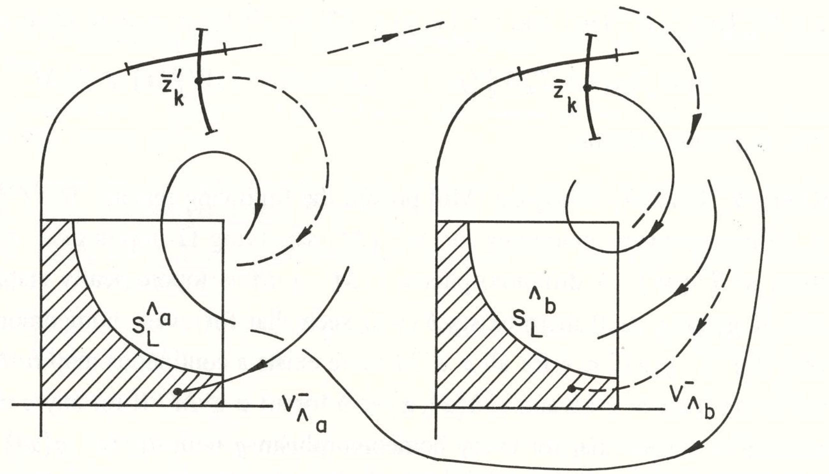


Figure 14

It only remains to check the case when the family $(\{z'_k\}_{k=1}^{a_K}, \{l'_k\}_{k=1}^{a_K})$ satisfies (3.24). For this case apply the method used to construct a family $(\{z''_k\}_{k=1}^{a_K}, \{l''_k\}_{k=1}^{a_K})$ nearby $\Lambda_c$ for the case (3.27). Then we shall reach a contradiction by running on the argument mentioned above.

Consequently the hypothesis $\Lambda_a \cap \overline{P}_{j_0+1}(f) \neq \varnothing$ leads up the phenomenon of being $W^u(\Lambda_a) \cap W^s(\Lambda_b) \neq \varnothing$ for $f$ and some $\Lambda_b$. However this is again contrary to (VI). Therefore $\Lambda(j_0) \cap \overline{P}_{j_0+1}(f) = \emptyset$ and $f$ satisfies Axiom A.

## 4. Proof of Theorem 2

Since $KS(M) \subset \mathcal{F}^1(M)$, it is enough to check that $f \in KS(M)$ satisfies strong transversality. Suppose that there exist $f \in KS(M)$ and $x \in M - \Omega(f)$ such that $T_xW_f^s(x)+T_xW_f^u(x) \neq T_xM$. Since $x \notin \Omega(f)$, for $\delta > 0$ small enough we have $B_\delta(x) \cap f^n(B_\delta(x)) = \emptyset$ for all $n \neq 0$. Let $\varepsilon > 0$ and take $p_1, p_2 \in P(f)$ and $y_1, y_2 \in B_\delta(x)$ satisfying

$$\begin{cases} y_1 \in W_f^s(p_1) & \text{and} \quad y_2 \in W_f^u(p_2), \\ C_\delta^s(y_1, f) & \text{is} \quad \varepsilon\text{-}C^1 \quad \text{near to} \quad C_\delta^s(x,f), \\ C_\delta^u(y_2, f) & \text{is} \quad \varepsilon\text{-}C^1 \quad \text{near to} \quad C_\delta^u(x,f), \end{cases}$$

where $C_\delta^\sigma(z,f)$ is the connected component of $z$ in $W_f^\delta(z) \cap B_\delta(x)$ $(\sigma = s,u)$. Therefore, for $0 < \delta_1 < \delta$ small enough there exist a diffeomorphism $g \in KS(M)$ and $p_1, p_2 \in P(g)$ such that $g = f$ on $M - \{B_\delta(x) \cup f^{-1}(B_\delta(x))\}$, $x \in W_g^s(p_1) \cap W_g^u(p_2)$ and $C_{\delta_1}^\sigma(x,g) = C_{\delta_1}^\sigma(x,f)$ for $\sigma = s,u$, from which

$$T_xW_g^s(p_1) + T_xW_g^u(p_2) = T_xW_f^s(x) + T_xW_f^u(x) \neq T_xM.$$

This is a contradiction.

**Remark.** Recently Moriyasu [**Mo**] proved the following results. The $C^1$ interior of diffeomorphisms belonging to $\mathrm{Diff}^1(M)$ satisfying $\Omega$- topological stability is equal to $\mathcal{F}^1(M)$. A diffeomorphism $f: M \to M$ is topologically stable if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for every homeomorphism $g$ with $d(f(x), g(x)) < \delta$ for all $x \in M$ there exists a continuous map $h: M \to M$ satisfying $h \circ g = f \circ h$ and $d(h(x), x) < \delta$ for all $x \in M$. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every homeomorphism $g$ with $d(f(x), g(x)) < \delta$ for $x \in M$ there is a continuous map $h: \Omega(g) \to \Omega(f)$ $(h(\Omega(g)) \subset \Omega(f))$ satisfying $h \circ g = f \circ h$ on $\Omega(g)$ and $d(h(x), x) < \varepsilon$ for all $x \in \Omega(g)$, then $f$ is $\Omega$-topologically stable.

Restricting to 3-closed manifolds, Sakai [**Sak**] proved that the $C^1$ interior of all diffeomorphisms having the pseudo orbit tracing property is characterized as the set of Axiom A diffeomorphisms with strong transversality.

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