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Rigidity of Isometric Immersions of Higher Codimension

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Abstract. Given an isometric immersion $f: M^n \to \mathbb{R}^N$ into Euclidean space, we provide sufficient conditions on f so that any 1-regular isometric immersion of M^n into $\mathbb{R}^{N+1}$ is necessarily obtained as a composition of f with a local isometric immersion $\mathbb{R}^N \supset U \to \mathbb{R}^{N+1}$. This result has several applications.

1. Introduction

Let $f: M^n \to \mathbb{R}^N$ be an isometric immersion into Euclidean space of an n-dimensional connected Riemannian manifold. Even assuming f to be rigid, a large set of non-congruent isometric immersions of M^n into $\mathbb{R}^{N+1}$ can be produced by composing f with elements of the infinite-dimensional family of (local) isometric immersions of $\mathbb{R}^N$ into $\mathbb{R}^{N+1}$. The main purpose of this paper is to provide sufficient conditions on f to ensure that any isometric immersion of M^n into $\mathbb{R}^{N+1}$ is a composition of f with an isometric immersion as above.

A classical rigidity theorem due to Allendoerfer [**All**] states that f is rigid in $\mathbb{R}^N$ if the type number τ of its vector valued second fundamental form $\alpha: TM \times TM \to TM^\perp$ satisfies $\tau(x) \geq 3$ for all $x \in M^n$. Recall that the *type number* of a symmetric bilinear form $\gamma: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is the largest integer τ for which there are τ vectors $X_1, \ldots, X_\tau$ in $\mathbb{R}^n$ such that the τp vectors $B_{\xi_j}(X_i)$, $1 \leq i \leq \tau$, $1 \leq j \leq p$, are linearly independent, where $\xi_1, \ldots, \xi_p$ is a basis of $\mathbb{R}^p$ and $B_{\xi_j}: \mathbb{R}^n \to \mathbb{R}^n$ is given by $\left\langle B_{\xi_j} X, Y \right\rangle = \langle \gamma(X,Y), \xi_j \rangle$. We define the *rank* ρ of γ as

$$\rho = \min\{\text{rank } B_\xi : \xi \in \mathbb{R}^p, \xi \neq 0\}.$$

Then $\rho \geq \tau$. Observe also that $\rho > 0$ implies $S(\gamma) = \text{span}\{\gamma(X,Y) : X, Y \in \mathbb{R}^n\} = \mathbb{R}^p$.

Recall that an isometric immersion is called 1-*regular* if the first normal

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spaces $N_1(x) = \text{span}\{\alpha(X,Y) : X, Y \in T_xM\}$ have equal dimension. Our main result is the following extension of Allendoerfer's theorem.

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric embedding whose second fundamental form has type number $\tau \geq 3$ and rank $\rho \geq 4$, everywhere. Then every 1-regular isometric immersion $g\colon M^n \to \mathbb{R}^{N+1}$ is a composition $g = h \circ f$, where $U \subset \mathbb{R}^N$ is a neighborhood of $f(M^n)$ and $h\colon U \subset \mathbb{R}^N \to \mathbb{R}^{N+1}$ is an isometric immersion.*

Theorem 1 was obtained by Erbacher ([**Er**]) for $M^n = S^n_c$ and $N = n+1$. Counter-examples exist if we drop either the 1-regularity or the rank hypothesis. Henke ([**He**]) showed that local isometric immersions $S^n_c \supset V \to \mathbb{R}^{n+2}$ may not be a composition of immersions near umbilical points. As for the rank condition, there exist local isometric immersions of $S^3_c \supset V \to \mathbb{R}^5$ which are not compositions of immersions as in the above theorem (see [**D-T**]).

Remark. Theorem 1 is not true if we do not ask f to be an embedding. It is not too difficult to verify that an n-dimensional tube with self intersections along a curve in $\mathbb{R}^{n+1}$ may admit a 1-regular isometric embedding in $\mathbb{R}^{n+2}$. However, if f is just an immersion, the proof of the theorem shows that $g(M^n)$ is contained in a flat hypersurface of $\mathbb{R}^{N+1}$.

On the other hand, if we restrict ourselves to the class of minimal immersions, then the assumptions of Theorem 1 may be weakened.

Theorem 2. *Let $f\colon M^n \to \mathbb{R}^N$ be a minimal isometric immersion with type number $\tau(x_0) \geq 3$ at a point $x_0 \in M^n$. Then any minimal isometric immersion $g\colon M^n \to \mathbb{R}^{N+1}$ is congruent to f in $\mathbb{R}^{N+1}$.*

Finally, we apply Theorem 1 to the study of Riemannian manifolds which can be isometrically immersed in both $\mathbb{R}^N$ and S^N_c. The case of codimension one was already considered by do Carmo-Dajczer ([**C-D**]), where it is shown that the manifold must be conformally flat. Conversely, a simply-connected conformally flat hypersurface L^N of S^{N+1}_c, $N \geq 4$, without umbilical points, can be isometrically immersed in $\mathbb{R}^{N+1}$ (see [**C-Y**]) as a 1-parameter envelope of spheres (see [**A-D**]). In particular, any submanifold M^n of L^N admits isometric immersions into S^{N+1}_c and $\mathbb{R}^{N+1}$.

We say that an isometric immersion $f\colon M \to \tilde{M}$ has an *umbilical direction* $\xi \in T_xM^\perp$ at $x \in M$ if $A_\xi = \lambda I$, $0 \neq \lambda \in \mathbb{R}$. We conclude this paper with the following result.

Theorem 3. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric embedding free of umbilical directions, whose second fundamental form has type number $\tau \geq 3$ and rank $\rho \geq 4$, everywhere. If M^n admits an isometric embedding $g\colon M^n \to S^{n+p}_c$, then there exist a conformally flat manifold N^{n+p-1}_{cf} and isometric immersions $k\colon M^n \to N^{n+p-1}_{cf}$, $h_1\colon N^{n+p-1}_{cf} \to \mathbb{R}^{n+p}$ and $h_2\colon N^{n+p-1}_{cf} \to S^{n+p}_c$ so that $f = h_1 \circ k$ and $g = h_2 \circ k$.*

Although all our results refer to submanifolds of Euclidean space, they remain valid for submanifolds of the Euclidean sphere or the hyperbolic space by similar arguments.

1. Some linear algebra results

We will make use of the theory of flat bilinear forms to prove some lemmas which describe the pointwise structure of the second fundamental forms of the immersions involved in our theorems.

Lemma 1. *Let $\alpha\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear form with type number $\tau \geq 3$ and rank $\rho \geq 4$. Assume that $\gamma\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{p+1}$ is a symmetric bilinear form which verifies for all $X, Y, Z, W \in \mathbb{R}^n$, that*

$$\langle \alpha(X,Y), \alpha(Z,W)\rangle - \langle \alpha(X,W), \alpha(Z,Y)\rangle =$$
$$= \langle \gamma(X,Y), \gamma(Z,W)\rangle - \langle \gamma(X,W), \gamma(Z,Y)\rangle\,.$$

Then there exists an orthogonal sum decomposition $\mathbb{R}^{p+1} = \mathbb{R}^p \oplus \mathbb{R}$, such that

(i) $\pi_{\mathbb{R}^p} \circ \gamma = \alpha$,

(ii) *rank* $\pi_{\mathbb{R}} \circ \gamma \leq 1$,

where $\pi_{\mathbb{R}^p}$ (respectively $\pi_{\mathbb{R}}$) denotes the orthogonal projection onto $\mathbb{R}^p$ (respectively $\mathbb{R}$).

Proof. Consider the symmetric bilinear form

$$\beta = \alpha \oplus \gamma\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \oplus \mathbb{R}^{p+1} \approx \mathbb{R}^{2p+1},$$

where $\mathbb{R}^{2p+1}$ is endowed with the nondegenerate inner product $\langle\langle\ ,\ \rangle\rangle$ defined by

$$\ll (\xi,\xi'), (\eta,\eta') \gg = \langle \xi, \eta\rangle - \langle \xi', \eta'\rangle\,.$$

Then β is flat in the sense that it verifies

$$\ll \beta(X,Y), \beta(Z,W) \gg - \ll \beta(X,W), \beta(Z,Y) \gg = 0$$

for all $X, Y, Z, W \in \mathbb{R}^n$.

Given $X \in \mathbb{R}^n$, we define a linear transformation $B(X): \mathbb{R}^n \to \mathbb{R}^{2p+1}$ by $B(X)(Z) = \beta(X,Z)$. We say that $X \in \mathbb{R}^n$ belongs to the set $RE(\beta)$ of *regular elements* if, and only if,

$$\dim B(X)(\mathbb{R}^n) = q = \max\{\dim B(Y)(\mathbb{R}^n) : Y \in \mathbb{R}^n\}.$$

The basic property of an element $X \in RE(\beta)$ (see [**Mo**], p. 246) is that for any vector $n \in \operatorname{Ker} B(X)$,

$$\beta(\mathbb{R}^n, n) \subset \mathcal{U}(X) = B(X)(\mathbb{R}^n) \cap B(X)(\mathbb{R}^n)^{\perp}. \tag{1}$$

Let $k_0 = \min\{\dim \mathcal{U}(X) : X \in \mathbb{R}^n, X \neq 0\}$, and define

$$RE^*(\beta) = \{X \in RE(\beta) : \dim \mathcal{U}(X) = k_0\}.$$

Since $RE^*(\beta)$ is open and dense in $\mathbb{R}^n$ (see [**D-R**], p. 214) and α has type number $\tau \geq 3$, there exist $X_1, X_2, X_3 \in RE^*(\beta)$ such that the vectors

$$\{B_{\xi_j} X_i : 1 \leq i \leq 3, 1 \leq j \leq p\}$$

are linearly independent for a basis $\xi_1, \ldots, \xi_p$ of $\mathbb{R}^p$. Now, the subspace

$$S = \{Z \in \mathbb{R}^n : \alpha(X_i, Z) = 0, 1 \leq i \leq 3\}$$

satisfies

$$S = [\operatorname{span}\{B_{\xi_j} X_i : 1 \leq i \leq 3, 1 \leq j \leq p\}]^{\perp},$$

and therefore,

$$\dim S = n - 3p. \tag{2}$$

Since $B(X_2)(\operatorname{Ker} B(X_1)) \subset \mathcal{U}(X_1)$ by (1), thus, the linear transformation

$$D(X_2) = B(X_2)|_{\operatorname{Ker} B(X_1)} : \operatorname{Ker} B(X_1) \to \mathcal{U}(X_1)$$

satisfies

$$\operatorname{Ker} D(X_2) = \operatorname{Ker} B(X_1) \cap \operatorname{Ker} B(X_2),$$

and

$$\dim \operatorname{Ker} D(X_2) \geq \dim \operatorname{Ker} B(X_1) - k_0.$$

Similarly,

$$D(X_3) = B(X_3)|_{\operatorname{Ker} D(X_2)} : \operatorname{Ker} D(X_2) \to \mathcal{U}(X_1) \cap \mathcal{U}(X_2)$$

satisfies

$$\operatorname{Ker} D(X_3) = \bigcap_{j=1,2,3} \operatorname{Ker} B(X_j),$$

and

$$\begin{aligned} \dim \operatorname{Ker} D(X_3) &\geq \dim \operatorname{Ker} D(X_2) - \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2) \\ &\geq n - q - k_0 - \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2). \end{aligned} \tag{3}$$

Since $\bigcap_{j=1}^{3} \operatorname{Ker} B(X_j) \subset S$, we obtain from (2) and (3) that

$$n - 3p \geq n - q - k_0 - \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2).$$

Using

$$q \leq 2p + 1 - k_0, \quad \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2) \leq k_0, \tag{4}$$

we get that $k_0 \geq p - 1$. Hence, either $k_0 = p - 1$ or $k_0 = p$.

Case $k_0 = p - 1$**.** Under this assumption, inequalities (4) must be equalities. We conclude that (i) $q = p + 2$, and (ii) $\mathcal{U}(X_i) = \mathcal{U}$ for $i = 1, 2, 3$.

We claim that $\mathcal{U}^{\perp} = S(\beta)$. From $\mathcal{U} \subset \operatorname{Im} B(X_j)^{\perp}$, we have $\mathcal{U}^{\perp} \supset \operatorname{Im} B(X_j)^{\perp\perp} = \operatorname{Im} B(X_j)$. Since $\dim \mathcal{U}^{\perp} = 2p + 1 - \dim \mathcal{U} = p + 2$, we get

$$\operatorname{Im} B(X_i) = \mathcal{U}^{\perp}, \quad 1 \leq i \leq 3.$$

Clearly, given $Y \in \mathbb{R}^n$, there exists $\varepsilon > 0$ such that the set of vectors $\overline{X}_1 = X_1 + \varepsilon Y$, X_2, X_3, have the same properties than the set X_1, X_2, X_3. Hence, $\operatorname{Im} B(\overline{X}_1) = \mathcal{U}^{\perp}$, and therefore, $\operatorname{Im} B(Y) \subset \mathcal{U}^{\perp}$. This proves the claim.

Consider orthonormal bases $\xi_1, \ldots, \xi_p$ of $\mathbb{R}^p$ and $\eta_1, \ldots, \eta_{p+1}$ of $\mathbb{R}^{p+1}$, such that

$$\mathcal{U} = \operatorname{span}\{\delta_j = \xi_j + \eta_j, 1 \leq j \leq p - 1\}.$$

It follows from the claim that

$$0 = \langle\langle \beta(Y,Z), \delta_j \rangle\rangle = \langle \alpha(Y,Z), \xi_j \rangle - \langle \gamma(Y,Z), \eta_j \rangle$$

for all $Y, Z \in \mathbb{R}^n$ and $1 \leq j \leq p - 1$. Set

$$\phi = \sum_{j=1}^{p-1} \langle \alpha, \xi_j \rangle \, \xi_j, \quad \psi = \langle \alpha, \xi_p \rangle \, \xi_p, \quad \theta = \sum_{h=p}^{p+1} \langle \gamma, \eta_h \rangle \, \eta_h.$$

From the claim and flatness of β, it follows that α and γ split orthogonally as

$$\alpha = \phi \oplus \psi, \quad \gamma = \phi \oplus \theta,$$

where the symmetric bilinear form

$$\psi \oplus \theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbf{L} = \operatorname{span}\{\xi_p, \eta_p, \eta_{p+1}\}$$

is flat with respect to the induced Lorentzian metric on $\mathbf{L}$ as a subspace of $\mathbb{R}^{2p+1}$. Since rank $B_{\xi_p} \geq 4$, we have from Corollary 2 of **[Mo]** that $S(\psi \oplus \theta) \neq \mathbf{L}$. Hence Corollary 3 of **[Mo]** applies, and therefore η_p, η_{p+1} can be chosen so that

$$\langle \alpha, \xi_p \rangle = \langle \gamma, \eta_p \rangle, \quad \text{rank } \langle \gamma, \eta_{p+1} \rangle \leq 1.$$

This concludes the proof of this case.

Case $k_0 = p$. The argument for this case is similar and left to the reader. □

Lemma 2. *The conclusions in Lemma 1 with* $\pi_{\mathbb{R}} \circ \gamma = 0$ *remain valid if instead of* $\rho \geq 4$*, we assume trace* $\alpha =$ *trace* $\gamma = 0$.

Proof. First notice that rank $\pi_{\mathbb{R}} \circ \gamma \leq 1$ and trace $\pi_{\mathbb{R}} \circ \gamma = 0$ imply that $\pi_{\mathbb{R}} \circ \gamma = 0$. To prove Lemma 1, condition $\rho \geq 4$ was used only once, namely to deal with the bilinear form $\psi \oplus \theta$. In this case, instead of **[Mo]**, we can use the following result whose proof is part of the arguments in (**[B-D-J]**, pp. 435–436). □

Lemma 3. *Let* $B: \mathbb{R}^n \to \mathbb{R}^n$ *be a symmetric linear transformation with rank* $B \geq 3$*, and let* $\beta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ *be a symmetric bilinear form. Assume trace* $B =$ *trace* $\beta = 0$*, and that for all* $X, Y, Z, W \in \mathbb{R}^n$*, we have*

$$\langle BX, Y \rangle \langle BZ, W \rangle - \langle BX, W \rangle \langle BZ, Y \rangle = \\ = \langle \beta(X,Y), \beta(Z,W) \rangle - \langle \beta(X,W), \beta(Z,Y) \rangle .$$

Then $\dim S(\beta) = 1$ *and* $\langle \beta(X,Y), \delta \rangle = \langle BX, Y \rangle$*, where* $S(\beta) =$ *span*$\{\delta\}$, $\|\delta\| = 1$.

2. The proofs of the theorems

For the proof of Theorem 1 we will need the following result.

Lemma 4. *Let* $f: M^n \to \mathbb{R}^{n+p}$ *be an isometric immersion whose second fundamental form* α *satisfies that* $S(\alpha) = TM^{\perp}$ *everywhere and splits continuously and orthogonally as*

$$\alpha = \gamma \oplus \langle A_\eta, \ \rangle \eta$$

for some unit vector field $\eta \in TM^{\perp}$*. Assume rank* $A_\eta = 1$ *everywhere, and that* $\|\gamma(X,Y)\|^2$ *is a smooth function for any* $X, Y \in TM$*. Then*

(i) *If* γ *has rank* $\rho \geq 3$*, and* η *is chosen so that* A_η *has a positive eigenvalue, then* η *is smooth.*

(ii) *Suppose that* γ *has type number* $\tau \geq 3$ *and rank* $\rho \geq 4$*. Then* η *is constant along Ker* A_η *in* $\mathbb{R}^{n+p}$.

Proof. (i) First we show that for any hyperplane $W \subset T_xM$, we have at $x \in M^n$,

$$S(\gamma) = \text{span}\{\gamma(Z,Y) : Z, Y \in W\}.$$

Otherwise, there exists $\delta \in T_xM^{\perp}$, so that $\langle \gamma(Z,Y), \delta \rangle = 0$ for all $Z, Y \in W$. This implies that rank $A_\delta \leq 2$, which is a contradiction.

Fixed a point $x_0 \in M^n$, let $X_1, \ldots, X_n$ be an orthonormal basis of $T_{x_0}M$ such that $A_\eta X_1 = \mu X_1, \mu > 0$. Extend locally $X_1, \ldots, X_n$ to linearly independent vector fields and set $Y_j = X_1 + X_j$, $2 \leq j \leq n$. By the above, we may assume that at any point in a neighborhood of $x_0 \in M^n$, we have that

$$S(\gamma) = \text{span}\{\gamma(Y_{i_k}, Y_{j_k}) : 1 \leq k \leq p-1\}.$$

Clearly, the set of vector fields

$$\alpha(Y_{i_k}, Y_{j_k}) = \gamma(Y_{i_k}, Y_{j_k}) + \left\langle A_\eta Y_{i_k}, Y_{j_k} \right\rangle \eta, \quad 1 \leq k \leq p-1$$

are linearly independent, and the functions $\psi_k = \left\langle A_\eta Y_{i_k}, Y_{j_k} \right\rangle$ satisfy:

(1) $\psi_k(x_0) = \mu > 0$,

(2) $\psi_k^2 = \left\|\alpha(Y_{i_k}, Y_{j_k})\right\|^2 - \left\|\gamma(Y_{i_k}, Y_{j_k})\right\|^2 \in C^\infty(M^n)$.

Consequently, all functions ψ_k are positive and smooth in a neighborhood U of x_0.

At each point $x \in U$, η is a solution of the system of linear equations

$$\frac{1}{\psi_k} \left\langle \alpha(Y_{i_k}, Y_{j_k}), \eta \right\rangle = 1, \quad 1 \leq k \leq p-1.$$

Hence, η belongs to the intersection of $p-1$ affine hyperplanes $H_k \perp \alpha(Y_{i_k}, Y_{j_k})$ and the unit sphere $S_1^{p-1} \subset \mathbb{R}^p$. The line $\mathbb{R} = \bigcap_{k=1}^{p-1} H_k$ is orthogonal to all $\alpha(Y_{i_k}, Y_{j_k})$ and meets S_1^{p-1} in at least one point, since solution exists. To conclude the proof it is sufficient to show that $\mathbb{R} \cap S_1^{p-1}$ contains two different points. But if $\mathbb{R}$ is tangent to S_1^{p-1}, then η would be orthogonal to $\mathbb{R}$, and therefore $\eta \in \text{span}\{\alpha(Y_{i_k}, Y_{j_k}) : 1 \leq k \leq p-1\}$, which is a contradiction.

(ii) Choose η to be smooth and a local vector field $X_1 \neq 0$ so that $AX_1 = \mu X_1$, $\mu > 0$. By Codazzi's equation

$$\mu \langle [Y,Z], X_1 \rangle X_1 = A_{\nabla_Z^{\perp}\eta} Y - A_{\nabla_Y^{\perp}\eta} Z \tag{5}$$

for all $Y, Z \in \operatorname{Ker} A_\eta$. At $x \in M^n$, consider the linear map $\phi\colon \operatorname{Ker} A_\eta \to L$ defined by

$$\phi(X) = \nabla^\perp_X \eta,$$

where L is the orthogonal complement to η in $T_x M^\perp$. Suppose that $r = \dim \operatorname{Im} \phi$ is positive. For a basis $\delta_1, \ldots, \delta_r$ of $\operatorname{Im} \phi$, we have from (5)

$$A_{\delta_j} Y \in \operatorname{span}\{X_1\}, \quad 1 \le j \le r$$

for all $Y \in \operatorname{Ker} \phi$. Hence $\dim \operatorname{Ker} A_{\delta_j} \ge \dim \operatorname{Ker} \phi - 1$. It follows that

$$\dim \bigcap_{j=1}^{r} \operatorname{Ker} A_{\delta_j} \ge \dim \operatorname{Ker} \phi - 1 - (r-1) \ge n - 2r - 1.$$

For $r = 1$, this is a contradiction to $\rho \ge 4$. The assumption $\tau \ge 3$ implies that $\dim \bigcap_{j=1}^{r} \operatorname{Ker} A_{\delta_j} \le n - 3r$. This provides a contradiction for $r \ge 2$. Hence $r = 0$, and this concludes the proof. □

Proof of Theorem 1. At each point, the second fundamental forms α_f and α_g satisfy the conditions in Lemma 1 by the Gauss equations and our assumptions on the type number and rank of α_f. Therefore, α_g splits orthogonally at each point as

$$\alpha_g = \gamma \oplus \left\langle A^g_\eta, \ \right\rangle \eta$$

where η is a unit normal vector, $\gamma = \alpha_f$ and either $A^g_\eta = 0$ or A^g_η has rank one with a positive eigenvalue. In particular, A^g_η has constant rank since we are assuming that g is 1-regular.

If $A^g_\eta = 0$, it follows from Lemma 28 in [Sp] that g reduces codimension by one dimension and the proof follows from Allendoerfer's result. If rank $A^g_\eta = 1$, we easily conclude from the rank assumption on $\gamma = \alpha_f$ that η is continuous. Thus, Lemma 4(i) applies, and the orthogonal splitting has to be smooth. Let $\tau\colon T_f M^\perp \to L = \{\eta\}^\perp$ be the smooth vector bundle isometry along $g \circ f^{-1}\colon f(M) \to g(M)$ so that $\gamma = \tau \circ \alpha_f$. We claim that τ is parallel where L is considered with the connection ∇' induced by $\nabla^{f\perp}$. A straightforward computation using Lemma 4(ii) shows that γ satisfies the "Codazzi equation"

$$(\nabla'_Y \gamma)(Z, W) = (\nabla'_Z \gamma)(Y, W)$$

for all $Y, Z, W \in TM$. The claim now follows from Theorem 1 of [No].

Let $X \in TM$ be the proper unit vector field so that $A^g_\eta X = \mu X$, $\mu > 0$. We define a vector bundle isometry $T\colon T_f M^\perp \oplus \operatorname{span}\{X\} \to L \oplus \operatorname{span}\{X\}$ by

$T(\delta + cX) = \tau(\delta) + cX$. Let $\Lambda \subset L \oplus \operatorname{span}\{X\}$ denote the vector subbundle whose $(N - n)$ dimensional fibers are the orthogonal complements to $\tilde\nabla_X \eta = -\mu X + \nabla^\perp_X \eta$. Since $\mu \neq 0$, each fibre is transversal to TM. Therefore, the same holds for the subbundle $\Omega \subset T_f M^\perp \oplus \operatorname{span}\{X\}$ defined by $T\Omega = \Lambda$. Thus, the map $F\colon \Omega \to \mathbb{R}^N$ defined by

$$F(x, \xi) = f(x) + \xi$$

restricted to a neighborhood U of the 0-section is an embedding and parameterizes a tubular neighborhood of $f(M)$.

Now consider the map $G\colon \Omega \to \mathbb{R}^{N+1}$ defined by

$$G(x, \xi) = g(x) + T(\xi).$$

We claim that $G|_U$ is an isometric immersion with respect to the flat metric induced by F. To see this observe that for $\xi = \delta + cX \in \Omega$ and any $Z \in TM$, we have

$$\begin{aligned} G_*(x,\xi)Z &= \tilde\nabla_Z (g + T(\xi)) \\ &= g_*(x)(Z - A^g_{\tau(\delta)} Z + \nabla_Z cX) + \nabla^{g\perp}_Z \tau(\delta) + \alpha_g(Z, cX). \end{aligned}$$

But, $\left\langle T(\xi), \tilde\nabla_X \eta \right\rangle = 0$, thus

$$0 = \left\langle \tilde\nabla_X (\tau(\delta) + cX), \eta \right\rangle = \left\langle \nabla^{g\perp}_Z \tau(\delta) + \alpha_g(Z, cX), \eta \right\rangle.$$

We conclude that

$$\begin{aligned} G_*(x,\xi)Z &= g_*(x)(Z - A^g_{\tau(\delta)} Z + \nabla_Z cX) + \nabla'_Z \tau(\delta) + \gamma(Z, cX) \\ &= (g \circ f^{-1})_*(f_*(x)(Z - A^f_\delta Z + \nabla_Z cX)) + \\ &\quad \tau(\nabla^{f\perp}_Z \delta + \alpha_f(Z, cX)) \end{aligned}$$

and now the claim follows easily. By the above, the map

$$h = (G|_U) \circ (F|_U)^{-1}\colon F(U) \to G(U)$$

is an isometric immersion and $g = h \circ f$. This concludes the proof. □

Proof of Theorem 2. Since minimal immersions are real analytic, it is sufficient to argue for an open subset $V \subset M$ where g is 1-regular and f has type number $\tau \ge 3$. Moreover, we may assume that $p = N - n \ge 2$, since the case $p = 1$ was already considered in [B-D-J]. We conclude from Lemma 2 that the first normal spaces $N_1(x)$ of g have all dimension p. By Lemma 28 of [Sp], g reduces codimension to p and, therefore, g is congruent to f. □

Proof of Theorem 3. Let $\tilde{g}: M^n \to \mathbb{R}^{n+p+1}$ be the isometric immersion obtained by composing g with the inclusion of S_c^{n+p} in $\mathbb{R}^{n+p+1}$. Because f is free of umbilical directions, we argue that $\tilde{g}$ is 1-regular. In fact, if at some point the dimension of the first normal space of $\tilde{g}$ is less than $p+1$, then the second fundamental forms of f and $\tilde{g}$ must be congruent. This implies that f has an umbilical direction which is a contradiction. Since Theorem 1 applies, there exists an isometric immersion $h: U \subset \mathbb{R}^{n+p} \to \mathbb{R}^{n+p+1}$ such that $\tilde{g} = h \circ f$. Using that g is an embedding, we can easily see from the construction of h in the proof of Theorem 1, that U can be taken so that h is an embedding transversal to S_c^{n+p}. Set $N_{cf}^{n+p-1} = h(U) \cap S_c^{n+p}$. □

Remark. The assumption that f is free of umbilical directions is not essential in the sense that, if there exists an umbilical vector field globally defined, then the proof of the theorem still works.

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