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On the Geometry of Non-Classical Curves

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Abstract. In this paper we relate the numerical invariants attached to a projective curve, called the order sequence of the curve, to the geometry of the varieties of tangent linear spaces to the curve and to the Gauss maps of the curve.

1. Introduction

Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an irreducible projective curve defined over an algebraically closed field K. We will assume that X is non-degenerate, that is, it is not contained in any hyperplane. For a natural number m, let $\mathcal{B}^m$ and $\mathcal{K}^m$ be respectively the image and the kernel of the map of locally free sheaves over X^{sm},

$$V_{X^{\mathrm{sm}}} \xrightarrow{v^m} P^m(F),$$

where $F = i^*\mathcal{O}_{\mathbb{P}^N}(1)$, with i the inclusion map of X into $\mathbb{P}^N$, and where $P^m(F)$ is the sheaf of m-principal parts of F. (see [**K**], [**L**] or [**P**].) It is shown in [**L,Prop. 1**] that there is an increasing sequence $(\varepsilon_n)_{n=0,\dots,N}$ of non-negative integers characterized by the following conditions:

$$\mathrm{rk}(\mathcal{B}^j) = n+1, \quad \forall j \quad \text{such that} \quad \varepsilon_n \leq j < \varepsilon_{n+1}.$$

This sequence is called the *order sequence* of the curve X. We always have that $\varepsilon_0 = 0$, $\varepsilon_1 = 1$ and $\varepsilon_N \leq \deg(X)$. When $\varepsilon_n = n$ for all n, we say that X has a classical order sequence or shortly that X is classical. It is known that if $\mathrm{char}(K) = 0$ or if $\mathrm{char}(K) > \deg(X)$, then X is classical.(See for example [**L, Thm. 15**].) So non-classical curves may only occur in positive characteristic. Since the curves we are going to study are non-classical, we will assume from now on that $\mathrm{char}(K) = p > 0$.

The relationship among this approach and the classical theory of Wronskians as found in [**FKS**] or in [**S-V**] is the following:

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Let $K(X)$ be the field of rational functions on X, and let $X_0, \ldots, X_N$, be homogeneous coordinates of $\mathbb{P}^N$. We denote by f_j, for $j = 0, 1, \ldots, N$, the rational function X_j/X_0 on X. Let t be a separating variable of $K(X)/K$. We denote by D_t^m the Hasse differential operator of order m with respect to t. (For the properties of these operators needed here, we refer to **[H-1]**.) Then the order sequence of X is the minimal sequence, with respect to the lexicographic order, such that the determinant of the wronskian matrix,

$$\mathcal{W} = \left(D_t^{\varepsilon_i} f_j\right)_{i,j=0,\ldots,N}, \tag{1}$$

is not zero as a rational function on X. Geometrically, the order sequence of X represents all possible intersection multiplicities of X with the hyperplanes of $\mathbb{P}^N$ at a general point of X, that is, points in an open dense set of X which we denote by U in the sequel.

Let $\mathbb{P}^{N*}$ be the dual projective space of $\mathbb{P}^N$. Define for $n = 1, \ldots, N-1$, the set $C^{(n)}X$ as the closure in $\mathbb{P}^N \times \mathbb{P}^{N*}$ of the projective bundle $\mathbb{P}((\mathcal{K}^{\varepsilon_n})^\vee)$. So we have a chain of varieties,

$$C^{(N-1)}X \subset C^{(N-2)}X \subset \cdots \subset C^{(1)}X \subset \mathbb{P}^N \times \mathbb{P}^{N*},$$

which we call the *higher order conormal varieties* of X, and whose projections on the second factor are:

$$X^{(N-1)} \subset X^{(N-2)} \subset \cdots \subset X^{(1)} \subset \mathbb{P}^{N*},$$

which we call the *higher order dual varieties* of X. In particular, we have that $C^{(1)}X$ is the conormal variety CX of X, and its projection $X^{(1)}$ onto the second factor is the dual hypersurface X' of X. Finally, $X^{(N-1)}$ is the strict dual of X.

The variety $C^{(n)}X$ is irreducible of dimension $N-n$. The variety $X^{(n)}$ is then irreducible, and since X is non-degenerate, it has dimension $N-n$ too. Set theoretically we have that $C^{(n)}X$ is the closure in $\mathbb{P}^N \times \mathbb{P}^{N*}$ of any one of the following sets:

$$\{(P,H) \in U \times \mathbb{P}^{N*} / I(P, X.H) > \varepsilon_n\},$$

or

$$\{(P,H) \in U \times \mathbb{P}^{N*} / H \supset T_P^n X\},$$

where $I(P, X.H)$ denotes the intersection multiplicity of X and H at P and $T_P^n X$ is the osculating linear space of dimension n to X at P. For P general,

the linear space $T_P^n X$ is generated by the points

$$\left(D_t^{\varepsilon_j} f_0(P); \ldots, D_t^{\varepsilon_j} f_N(P)\right), \quad j = 0, \ldots n.$$

We denote by

$$\pi_n : C^{(n)}X \longrightarrow X,$$

and by,

$$\pi'_n : C^{(n)}X \longrightarrow X^{(n)},$$

the natural projections.

For n such that $1 \le n \le N-1$, we define the n-th Gauss map,

$$\gamma_n : X \longrightarrow \mathbb{P}^M,$$

where $M = \binom{N+1}{n+1} - 1$, as the rational map from X to the grassmannian of n-planes in $\mathbb{P}^N$, associated to the surjection $V_{X^{sm}} \longrightarrow \mathcal{B}^{\varepsilon_n}$, followed by the Plücker embedding of the grassmannian into projective space. This map associates to each point P of U the Plücker coordinates of the linear space $T_P^n X$. We will denote by $\tilde{X}^{(n)}$ the closure in $\mathbb{P}^M$ of the set $\gamma_n(U)$. Remark, for future use, that π_{N-1} is birational,

$$\tilde{X}^{(N-1)} = X^{(N-1)},$$

and

$$\gamma_{N-1} = \pi'_{N-1} \circ \pi_{N-1}^{-1}.$$

The main result of this paper is that for all $n = 1, \ldots N-1$, the inseparable degrees of the maps π'_n and γ_n are equal to the highest power of p that divides ε_{n+1}. This is a generalization of the *Generic Order of Contact Theorem* of Hefez and Kleiman (**[H-K,Theorem 3.5]**). The result for curves in $\mathbb{P}^3$ was previously obtained by the first author and announced in **[H-2]**.

2. Inseparability of rational maps

Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be projective curves and let $G: X \longrightarrow Y$ be a rational dominating map. The inseparable degree of G, denoted by $\deg_i G$, is the inseparable degree of the field extension $K(X)/K(Y)$. By standard ramification theory, $\deg_i G$ is the ramification index of G at any point of its general fiber, so it is the ramification index of G at a general point of X.

Let $K((t))$ be the field of fractions of the ring of formal power series in t with coefficients in K. A parameterization of X at a point $P \in X$ is a point

$$P(t) = (P_0(t); \ldots ; P_n(t)) \in \mathbb{P}^n_{K((t))},$$

which is not rational over K and such that $P(0) = P$ and $P(t) \in X(K((t)))$. The parameterization will be called primitive if it is not rational over $K((t^r))$ for all r such that $r \geq 2$.

Lemma 1. *Let t be a separating variable of $K(X)/K$. If P is a general point of X and $P(t)$ is a primitive parameterization of X at P, then $\deg_i G \geq p^r$ if and only if $G(P(t))$ is rational over $K((t^{p^r}))$.*

Proof. Let $G = (G_0; \ldots ; G_m)$. One of the coordinates of $G(P)$ is not zero; without loss of generality we may assume that $G_0(P) \neq 0$, so $G_0(P(t))$ is invertible in $K[[t]]$. Put $g_i = G_i/G_0$. So the ramification index of G at P is equal to

$$\min\{\text{ord}(g_i(P(t)) - g_i(P(0))); i = 1, \ldots , m\}.$$

It is clear now that if $G(P(t))$ is rational over $K((t^{p^r}))$, then $\deg_i G \geq p^r$. Conversely, suppose that $\deg_i G \geq p^r$, then we have that $[D^s_t g_i(P(t))]_{t=0} = 0$ for all s such that $1 \leq s < p^r$, so as rational functions on X we have that $D^s_t g_i = 0$ for all s as before. Now from basic properties of the Hasse differential operators and the fact that the coefficients of $g_i(P(t))$ are the Hasse derivatives of the rational function g_i evaluated at P, it follows that $g_i(P(t)) \in K[[t^{p^r}]]$, from which the result follows.

The following result gives a criterion for rationality of parametrizations over $K((t^{p^r}))$

Lemma 2. *A point $Q(t)$ of $\mathbb{P}^m_{K((t))}$ is rational over $K((t^{p^r}))$ if and only if $Q(t)$ and $D^{p^i}_t Q(t)$ are linearly dependent over $K((t))$ for all $i = 0, \ldots , r-1$.*

Proof. Suppose that $Q(t)$ is rational over $K((t^{p^r}))$, then we have,

$$Q(t) = h(t)(H_0(t^{p^r}); \ldots ; H_m(t^{p^r})),$$

with $h(t) \in K((t))^*$. Differentiating both members of the above equality we get, for all i with $0 \leq i \leq r-1$, that

$$D^{p^i}_t Q(t) = (D^{p^i}_t h(t))(H_0(t^{p^r}); \ldots ; H_m(t^{p^r})).$$

This shows that $Q(t)$ and $D^{p^i}_t Q(t)$ are dependent for all $i = 0, \ldots , r-1$.

The converse will be proved by induction over r. Without loss of generality we will assume $Q_m(t) \neq 0$. Suppose that $Q(t)$ and $Q'(t)$ are linearly dependent over $K((t))$, then

$$Q_i(t)Q'_j(t) = Q_j(t)Q'_i(t), \quad \forall\, i,j = 0, \ldots , m.$$

Hence for all $i = 0, \ldots , m$, we have

$$\left(\frac{Q_i(t)}{Q_m(t)}\right)' = \frac{Q_m(t)Q'_i(t) - Q'_m(t)Q_i(t)}{(Q_m(t))^2} = 0.$$

This implies that

$$\frac{Q_i(t)}{Q_m(t)} \in K((t^p)), \quad \forall\, i = 0, \ldots , m,$$

so $Q(t)$ is rational over $K((t^p))$, proving the assertion for $r = 1$.

Assume now that the result holds for r and suppose that $Q(t)$ and $D^{p^i}_t Q(t)$ are linearly dependent over $K((t))$, for all $i = 0, \ldots , r$. By the inductive hypothesis we have that $Q(t)$ is rational over $K((t^{p^r}))$, that is, there exist $h_i(t^{p^r}) \in K((t^{p^r}))$, $i = 0, \ldots , m$ such that

$$\frac{Q_i(t)}{Q_m(t)} = h_i(t^{p^r}), \quad i = 0, \ldots , m.$$

Now,

$$\begin{aligned} D^{p^r}_t Q_i(t) = D^{p^r}_t (Q_m(t)h_i(t^{p^r})) &= \sum_{j=0}^{p^r} D^{p^r-j}_t Q_m(t) D^j_t h_i(t^{p^r}) \\ &= h_i(t^{p^r}) D^{p^r}_t Q_m(t) + Q_m(t) D^{p^r}_t h_i(t^{p^r}), \end{aligned}$$

from which we get for $i = 0, \ldots , m$, that

$$D^{p^r}_t \left(\frac{Q_i(t)}{Q_m(t)}\right) = \frac{Q_m(t)D^{p^r}_t Q_i(t) - Q_i(t)D^{p^r}_t Q_m(t)}{(Q_m(t))^2} = 0.$$

It then follows that

$$\frac{Q_i(t)}{Q_m(t)} \in K((t^{p^{r+1}})),$$

hence $Q(t)$ is rational over $K((t^{p^{r+1}}))$.

Lemmas 1 and 2 yield the following result:

Proposition 1. *Let t be a separating variable of $K(X)/K$. If P is a general point of X and $P(t)$ is a primitive parameterization of X at P, then $\deg_i G \geq$*

p^r if and only if, for all $i = 0, \dots, r-1$, the vectors $G(P(t))$ and $D_t^{p^i} G(P(t))$ are linearly dependent over $K((t))$.

3. The inseparable degrees of the Gauss maps

The theorem in this section will relate the inseparable degrees of the Gauss maps with the order sequence of a given curve. This together with section and projection techniques, which we will develop in the next section, will allow us to prove the results for the maps π'_n.

In the proof of the theorem we will need the following two lemmas.

Lemma 3. *Let ε, c, j and α be positive integers such that p does not divide c and*

$$0 < cp^\alpha - \varepsilon \le j < p^\alpha,$$

then

$$\binom{\varepsilon + j}{\varepsilon} \equiv 0 \pmod{p}.$$

Proof. The hypotheses imply that

$$(c-1)p^\alpha < \varepsilon < cp^\alpha.$$

The above inequalities imply that there exists an integer t with $0 < t < p^\alpha$ such that

$$\varepsilon = (c-1)p^\alpha + t.$$

So,

$$\varepsilon + j = cp^\alpha + (t + j - p^\alpha).$$

From the above equality and the fact that $\varepsilon + j \ge cp^\alpha$, it follows that

$$u = t + j - p^\alpha \ge 0.$$

The inequality $j < p^\alpha$ and the above one, give

$$0 \le u < t.$$

Now, from a well known congruence among binomial coefficients (see for example [**H-1, 3.5**]), and the fact that $0 \le u < t < p^\alpha$, it follows that

$$\binom{\varepsilon + j}{\varepsilon} = \binom{cp^\alpha + u}{(c-1)p^\alpha + t} \equiv \binom{c}{c-1}\binom{u}{t} = 0 \pmod{p}.$$

Notation. For a square matrix $A = (a_{r,s})$ we will denote by $\operatorname{cof}(a_{i,j})$ the cofactor of the element $a_{i,j}$.

Lemma 4. *Let $A = (a_{\lambda,\mu})$ be an $n \times n$ matrix with coefficients in $K((t))$, and m a positive integer. Then we have,*

$$D_t^m \det(A) = \sum_{j_1 + \cdots + j_n = m} \det\left(D_t^{j_\lambda} a_{\lambda,\mu}\right).$$

Proof. The proof will be by induction on n. For $n = 1$, the equality is trivially satisfied. Suppose now the result is true for $n-1$. Then,

$$\begin{aligned}
\sum_{j_1+\cdots+j_n=m} \det\left(D_t^{j_\lambda} a_{\lambda,\mu}\right) &= \sum_{j_1=0}^{m} \sum_{j_2+\cdots+j_n=m-j_1} \det\left(D_t^{j_\lambda} a_{\lambda,\mu}\right) \\
&= \sum_{j_1=0}^{m} \sum_{j_2+\cdots+j_n=m-j_1} \sum_{\mu=1}^{n} (D_t^{j_1} a_{1,\mu}) \operatorname{cof}\left(D_t^{j_1} a_{1,\mu}\right) \\
&= \sum_{j_1=0}^{m} \sum_{\mu=1}^{n} (D_t^{j_1} a_{1,\mu}) \sum_{j_2+\cdots+j_n=m-j_1} \operatorname{cof}\left(D_t^{j_1} a_{1,\mu}\right) \\
&= \sum_{\mu=1}^{n} \sum_{j_1=0}^{m} (D_t^{j_1} a_{1,\mu}) D_t^{m-j_1} \operatorname{cof}(a_{1,\mu}) = \sum_{\mu=1}^{n} D_t^m (a_{1,\mu} \operatorname{cof}(a_{1,\mu})) \\
&= D_t^m \det(a_{\lambda,\mu}).
\end{aligned}$$

In the theorem below we will use the following notation. For $n = 1, \dots, N$, let $I = (i_0, \dots, i_n)$ and $J = (j_0, \dots, j_n)$ be $(n+1)$-tuples of integers such that

$$0 \le i_0 < i_1 < \cdots < i_n, \quad 0 \le j_0 < j_1 < \cdots < j_n \le N.$$

Define

$$\mathcal{W}(I, J) = \left(D_t^{i_\lambda} f_{j_\mu}\right)_{\lambda,\mu=0,\dots,n}.$$

where t and the f's are as in (1).

Theorem 1. *For all $n = 1, \dots, N-1$, the inseparable degree of the map γ_n is the highest power of p that divides ε_{n+1}.*

Proof. The map γ_n has coordinates

$$\gamma_{n,J} = \det \mathcal{W}(\varepsilon_0, \dots, \varepsilon_n, J),$$

with $J = (j_0, \dots j_n)$ such that $0 \le j_0 < j_1 < \cdots < j_n \le N$.

Let $P(t)$ be a primitive parameterization of X at a general point P. We will use the following notation

$$\mathcal{W}_t(I,J) = \left(D_t^{i_\lambda} P_{j_\mu}(t)\right)_{\lambda,\mu=0,\dots,n}.$$

By Lemma 4 we have

$$D_t^m \gamma_{n,J}(P(t)) = \sum_{\nu_0+\cdots+\nu_n=m} \binom{\nu_0+\varepsilon_0}{\varepsilon_0}\cdots\binom{\nu_n+\varepsilon_n}{\varepsilon_n} \times \det \mathcal{W}_t(\nu_0+\varepsilon_0,\dots,\nu_n+\varepsilon_n,J). \quad (2)$$

Write $\varepsilon_{n+1} = cp^\alpha$ with c not divisible by p. For $m < p^\alpha$, we have from Lemma 3, where we put $\varepsilon = \varepsilon_\lambda$, that the non-vanishing terms in (2) are multiples of determinants of matrices with rows of the form,

$$\left(D_t^{\varepsilon_\lambda+j} P_{j_0}(t),\dots,D_t^{\varepsilon_\lambda+j} P_{j_n}(t)\right), \quad (3)$$

with $j \le m < p^\alpha$ and $0 \le \lambda \le n$ such that $\varepsilon_\lambda + j < \varepsilon_{n+1}$.

Now, by the minimality of the order sequence, we have that the vectors,

$$\left(D_t^{\varepsilon_\lambda+j} P_0(t),\dots,D_t^{\varepsilon_\lambda+j} P_N(t)\right), \quad \lambda = 0,\dots n,$$

with $\varepsilon_\lambda + j < \varepsilon_{n+1}$ are linear combinations, with coefficients in $K((t))$ of the vectors

$$\left(D_t^{\varepsilon_r} P_0(t),\dots,D_t^{\varepsilon_r} P_N(t)\right), \quad r = 0,\dots,n. \quad (4)$$

Hence for all m with $m < p^\alpha$, and all J, it follows from (2) that there exists an element $\varphi_m \in K((t))$, not depending on J, such that

$$D_t^m \gamma_{n,J}(P(t)) = \varphi_m \gamma_{n,J}(P(t)).$$

This implies by Proposition 1 that $\deg_i \gamma_n \ge p^\alpha$.

To show that $\deg_i \gamma_n = p^\alpha$, we have, according to Proposition 1, to show that $D_t^{p^\alpha}\gamma_n(P(t))$ and $\gamma_n(P(t))$ are linearly independent over $K((t))$.

From (2) and Lemma 3, we have that the non-vanishing summands of the expression of $D_t^{p^\alpha}\gamma_{n,J}(P(t))$ may be expressed as determinants of matrices with rows as in (3) with either $\varepsilon_\lambda + j < \varepsilon_{n+1}$ or $j = p^\alpha$ and $\varepsilon_\lambda + j \ge \varepsilon_{n+1}$.

It then follows from this, and from the first part of the proof that $D_t^{p^\alpha}\gamma_{n,J}(P(t))$ is a multiple of $\gamma_{n,J}(P(t))$ (with factor independent of J), plus terms of the form

$$\binom{\varepsilon_\lambda + p^\alpha}{\varepsilon_\lambda} \det \mathcal{W}_t(\varepsilon_0,\dots,\varepsilon_{\lambda-1},\varepsilon_\lambda+p^\alpha,\varepsilon_{\lambda+1},\dots,\varepsilon_n,J) \quad (5)$$

with λ such that $\varepsilon_\lambda + p^\alpha \ge \varepsilon_{n+1}$.

Since for any λ, the vector

$$\left(D_t^{\varepsilon_\lambda+p^\alpha} P_0(t),\dots,D_t^{\varepsilon_\lambda+p^\alpha} P_N(t)\right),$$

is a linear combination of vectors as in (4), but with $r = 0,\dots,N$, we have that (5) is equal to

$$b_\lambda \det \mathcal{W}_t(\varepsilon_0,\dots,\varepsilon_n,J) + a_{n+1}^\lambda \det \mathcal{W}_t(\varepsilon_0,\dots,\hat{\varepsilon_\lambda},\dots,\varepsilon_n,\varepsilon_{n+1},J) + \cdots + a_N^\lambda \det \mathcal{W}_t(\varepsilon_0,\dots,\hat{\varepsilon_\lambda},\dots,\varepsilon_n,\varepsilon_N,J),$$

with b_λ and a_μ^λ in $K((t))$ and independent of J.

Since $\binom{cp^\alpha}{(c-1)p^\alpha} \not\equiv 0 \bmod p$, we have that $(c-1)p^\alpha$ is an order (see for example **[S-V, Cor.1.9]**), say

$$\varepsilon_s = (c-1)p^\alpha.$$

It then follows that $\varepsilon_s + p^\alpha = \varepsilon_{n+1}$ and therefore $b_s = 0$. Also for $\mu = n+2,\dots,N$ we have $a_\mu^s = 0$ and

$$a_{n+1}^s = (-1)^{n-\lambda}\binom{cp^\alpha}{(c-1)p^\alpha} \not\equiv 0 \pmod p.$$

So we have, for some d in K

$$D_t^{p^\alpha}\gamma_{n,J}(P(t)) = d\gamma_{n,J}(P(t)) + a_{n+1}^s \det \mathcal{W}_t(\varepsilon_0,\dots,\hat{\varepsilon}_s,\dots,\varepsilon_n,\varepsilon_{n+1},J) + \sum_{\mu\ge n+1}\sum_{\lambda>s} a_\mu^\lambda \det \mathcal{W}_t(\varepsilon_0,\dots,\hat{\varepsilon}_\lambda,\dots,\varepsilon_n,\varepsilon_\mu,J) \quad (6)$$

We will now verify that $\gamma_n(P(t))$ and $D_t^{p^\alpha}\gamma_n(P(t))$ are linearly independent over $K((t))$. Suppose the opposite true, then from (6) we have that $\det \mathcal{W}_t(\varepsilon_0,\dots,\hat{\varepsilon}_s,\dots,\varepsilon_n,\varepsilon_{n+1},J)$ is a linear combination, with coefficients independent of J of terms of the form

$$\det \mathcal{W}_t(\varepsilon_{i_0},\dots,\varepsilon_{i_n},J), \quad (\varepsilon_{i_0},\dots,\varepsilon_{i_n}) \ne (\varepsilon_0,\dots,\hat{\varepsilon}_s,\dots,\varepsilon_n,\varepsilon_{n+1}).$$

This implies that the row

$$(\det \mathcal{W}_t(\varepsilon_0,\dots,\hat{\varepsilon}_s,\dots,\varepsilon_n,\varepsilon_{n+1},J))_J$$

of the matrix $\bigwedge^{n+1}\mathcal{W}_t(\varepsilon_0,\dots,\varepsilon_N,0,\dots,N)$ is a linear combination of other rows. This is a contradiction since the above matrix is invertible because the matrix $\mathcal{W}_t(\varepsilon_0, \dots, \varepsilon_N, 0, \dots, N)$ is invertible (see for example [**F, Theorem** 1]).

4. Projections

Using Theorem 1 and projection arguments, we will prove in this section our result concerning the maps π'_n

Let W be a codimension r vector subspace of the $(N+1)$-dimensional K-vector space V and consider the induced linear projection

$$proj_W : \mathbb{P}(V) - \mathbb{P}(V/W) \longrightarrow \mathbb{P}(W).$$

Suppose that W is such that X doesn't meet the center of projection $\mathbb{P}(V/W)$, so the composition map,

$$X \overset{i}{\hookrightarrow} \mathbb{P}(V) - \mathbb{P}(V/W) \overset{proj_W}{\longrightarrow} \mathbb{P}(W),$$

is well defined. If we put $Y = \mathbb{P}(V)$ and $Z = \mathbb{P}(W)$, we have

$$F = i^*\mathcal{O}_Y(1) = (proj_W \circ i)^*\mathcal{O}_Z(1).$$

Let U be the open subset of X^{sm} such that $proj_W \circ i\,|_U$ is an embedding and for all m, the map $V_{X,x} \longrightarrow \mathcal{B}^m_x$ is onto for all x in U. Such U is not empty if W is general and $N \geq 3$.

Let $\mathcal{B}^m_1$ be the image of W_U via the morphism $v^m : V_U \longrightarrow P^m(F)$ and let $\mathcal{K}^m_1$ be the kernel of the restriction $v^m\,|_{W_U}$. There is clearly a commutative diagram of sheaves on U:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}^m & \longrightarrow & V_U & \longrightarrow & \mathcal{B}^m & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{K}^m_1 & \longrightarrow & W_U & \longrightarrow & \mathcal{B}^m_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad (7)$$

Denote by $\varepsilon'_0, \dots, \varepsilon'_{N-r}$ the order sequence of $X_W = proj_W(X)$.

Proposition 2. *If $N \geq 3$ and W is general of codimension r in V, then $\varepsilon'_n = \varepsilon_n$, for all $n = 0, \dots, N-r$.*

Proof. It is obviously sufficient to prove the result when $r = 1$, because we may iterate the process.

Choose coordinates $X_0, \dots, X_N$ for V and $Y_0, \dots, Y_{N-1}$ for W in order

that $X_0 = Y_0$ and $proj_W$ is given by

$$Y_i = \sum_{j=0}^{N} a_{ij} X_j, \quad i = 0, \dots, N-1, \qquad (8)$$

where $A = (a_{ij})_{i,j}$ is an $N \times (N+1)$-matrix with entries in K and of rank N.

Let t be a separating variable of $K(X)$. Since W is general, we have that $K(X_W) = K(X)$, so t is also a separating variable of $K(X_W)$. Let $f_j = X_j/X_0$ for $j = 0, \dots, N$ and $f'_j = Y_j/Y_0$ for $j = 0, \dots, N-1$, considered as functions on X and X_W respectively. From (8) we get

$$\mathcal{W}'(i_0, \dots, i_{N-1}) = \mathcal{W}(i_0, \dots, i_{N-1}).A^T,$$

where A^T is the transposed matrix of A,

$$\mathcal{W}(i_0, \dots, i_{N-1}) = \left(D_t^{i_\lambda} f_j\right)_{\lambda=0,\dots,N-1; j=0,\dots,N}$$

and

$$\mathcal{W}'(i_0, \dots, i_{N-1}) = \left(D_t^{i_\lambda} f'_j\right)_{\lambda=0,\dots,N-1; j=0,\dots,N-1}.$$

Since W is general, the matrix A is general, so it may be chosen in order that for all $i_0, \dots, i_{N-1}$ we have $Ker\,(\mathcal{W}(i_0, \dots, i_{N-1})) \not\subset Im\left(A^T\right)$, where A^T and $\mathcal{W}(i_0, \dots, i_{N-1})$ are viewed respectively as linear transformations from $K(X)^N$ to $K(X)^{N+1}$ and from $K(X)^{N+1}$ to $K(X)^N$. From this it follows that for all $i_0, \dots, i_{N-1}$, the matrices $\mathcal{W}'(i_0, \dots, i_{N-1})$ and $\mathcal{W}(i_0, \dots, i_{N-1})$ have the same rank, and therefore $\varepsilon_n = \varepsilon'_n$, for all $n = 0, \dots, N-1$.

Proposition 3. *The inseparable and separable degrees of π'_n, for $1 \leq n \leq N-2$, are invariant under general central projections.*

Proof. Let W be general and of codimension one in V . Put $U_1 = proj_W(U)$. For $1 \leq n \leq N-2$, diagram (7), for $m = \varepsilon_n$, yields the following exact diagram of sheaves on U,

$$\begin{array}{ccccc} V_U^\vee & \longrightarrow & (\mathcal{K}^{\varepsilon_n})^\vee & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ W_U^\vee & \longrightarrow & (\mathcal{K}_1^{\varepsilon_n})^\vee & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

This gives the following cartesian diagram:

$$\begin{array}{ccccc} C^nU & \hookrightarrow & U \times \mathbb{P}(V)^* & \longrightarrow & \mathbb{P}(V)^* \\ \uparrow & \square & \uparrow & \square & \uparrow \\ C^nU_1 & \hookrightarrow & U \times \mathbb{P}(W)^* & \longrightarrow & \mathbb{P}(W)^* \end{array}$$

It then follows that for $1 \leq n \leq N-2$, and for all $Q \in U_1^{(n)} \subset U^{(n)}$, the fibers of $C^{(n)}U \to U^{(n)}$ and $C^{(n)}U_1 \to U_1^{(n)}$ over Q are isomorphic as schemes. Now the result follows by observing that for general W, a general point of $X_W^{(n)}$ is a general point of $X^{(n)}$. The result for any codimension follows by induction.

Theorem 2. *For all n, with $1 \leq n \leq N-1$, the inseparable degree of π'_n is the highest power of p that divides ε_{n+1}.*

Proof. For $n = N-1$, the result follows from Theorem 1 and the fact that $\gamma_{N-1} = \pi'_{N-1} \circ \pi_{N-1}^{-1}$,with π_{N-1} birational.

Suppose now that $n \leq N-2$. By a sequence of general central projections, project X onto $X_W \subset \mathbb{P}^{n+1}$. Now applying Theorem 1 at the level n to X_W, the result follows from the above discussed case, Proposition 2 and Proposition 3.

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