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Linear systems on curves with no Weierstrass points

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Abstract. We study order-sequences of linear systems on smooth curves and establish the formula: $b_j + b_{N-j} \leq b_N$ for all j, where $\{b_0 < b_1 < \ldots < b_N\}$ is the order-sequence of a linear system on a curve. As an application of the formula, we describe all linear systems on curves which have no Weierstrass points.

0. Introduction

In the characteristic-free approach to Weierstrass points of a linear system on a curve, we meet the concept of *Weierstrass order-sequences* of a linear system (see, Schmidt [**14, 15**], Matzat [**13**], Laksov [**11, 12**] and Stöhr-Voloch [**16**].)

Let $\mathcal{D}$ be a linear system of projective dimension N on a smooth curve C over an algebraically closed field and let $P \in C$. A nonnegative integer m is a Hermite invariant of $\mathcal{D}$ at P if there is a divisor $D \in \mathcal{D}$ such that the multiplicity of D at P is m. It is obvious that there are $N+1$ Hermite invariants $\{\mu_0(P) < \ldots < \mu_N(P)\}$ of $\mathcal{D}$ at P. A basic result is that there are $N+1$ integers $\{b_0 < \ldots < b_N\}$ such that the Hermite invariants of $\mathcal{D}$ at P coincide with $\{b_0 < \ldots < b_N\}$ for all but finitely many points $P \in C$. This sequence is called the *Weierstrass order-sequence* of $\mathcal{D}$. A point $P \in C$ is a Weierstrass point if $\{\mu_0(P) < \ldots < \mu_N(P)\} \neq \{b_0 < \ldots < b_N\}$. If the characteristic of the ground field is zero, then every Weierstrass order-sequence is classical, that is $b_j = j$ for every j. In positive characteristic p, however, this is not always true. In this case, as Schmidt has shown [**14, Satz 6**], each Weierstrass order-sequence $\{b_0 < \ldots < b_N\}$ has the following property: *if b is a nonnegative integer such that*

$$\binom{b_k}{b} \not\equiv 0 \mod p$$

for some k, then $b = b_j$ for some j. Conversely, as Stöhr and Voloch have shown

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[16], a sequence of nonnegative integers with this property is the Weierstrass order-sequence of a certain linear system of a curve.

Recently, relations between projective geometry of curves and their Weierstrass order-sequences have been studied by several authors e.g., Ballico-Russo [1], Garcia-Voloch [2], Hefez-Kakuta [3], Hefez-Voloch [4], Homma [6, 7, 8], Homma-Kaji [9], Kaji [10]. In this paper, we prove the following formula on Weierstrass order-sequences.

Theorem I. *If $\{b_0 < \ldots < b_N\}$ is a Weierstrass order-sequence of a linear system on a curve, then we have*

$$b_j + b_{N-j} \leq b_N$$

for all $j = 0, \ldots, N$.

As an application of the formula, we prove the following theorem. A Weierstrass order-sequence $\{b_0 < \ldots < b_N\}$ will be called *symmetric* if the equalities $b_j + b_{N-j} = b_N$ $(j = 0, \ldots, N)$ hold.

Theorem II. *Let C be a smooth curve over an algebraically closed field k and L a line bundle on C. Let V be a nonzero k-subspace of $H^0(C, L)$ and $\mathcal{D} = \mathbb{P}V$ the linear system corresponding to V. Then C has no $\mathcal{D}$-Weierstrass points if and only if $C = \mathbb{P}^1$, the Weierstrass order-sequence $\{b_0 < \ldots < b_N\}$ of $\mathcal{D}$ is symmetric and b_N coincides with the degree of $\mathcal{D}$. In this case, taking suitable coordinates S and T of $\mathbb{P}^1$, V is spanned by $\{S^{b_j}T^{b_{N-j}} \mid j = 0, \ldots, N\}$ in $H^0(\mathbb{P}^1, \mathcal{O}(b_N))$.*

1. Abstract Order-Sequences

Throughout this section, we fix a prime number p.

Let m, n be two nonnegative integers with p-adic expansions:

$$m = \alpha_e p^e + \alpha_{e-1}p^{e-1} + \ldots + \alpha_0 \quad (0 \leq \alpha_i < p)$$
$$n = \beta_e p^e + \beta_{e-1}p^{e-1} + \ldots + \beta_0 \quad (0 \leq \beta_i < p).$$

Then we denote by $m \succ_p n$ (or $n \prec_p m$) if $\alpha_i \geq \beta_i$ for all i. It is easy to show that

$$\binom{m}{n} \not\equiv 0 \mod p$$

if and only if $m \succ_p n$. In this case, we say that m dominates n or m is a dominator for n.

Definition 1.1. An AO (= abstract order)-sequence of dimension N with respect to p is an sequence of $N+1$ nonnegative integers $\{b_0 < b_1 < \cdots < b_N\}$ with the following property: if b is a nonnegative integer such that $b \prec_p b_k$ for some k, then $b = b_j$ for some j. A member of an AO-sequence will be called an *order*.

Note that $b_0 = 0$ by the definition. In particular, the only example of an AO-sequence of dimension 0 is $\{0\}$. Thus, from now on, we assume that $N \geq 1$.

Definition 1.2. An AO-sequence $\{b_0 < b_1 < \ldots < b_N\}$ is said to be of separable type if $b_1 = 1$.

Remark 1.3. If an AO-sequence $\{b_0 < b_1 < \ldots < b_N\}$ is not of separable type, then b_1 is a positive power of p and divides every b_i. In this case,

$$\{b_0(=0) < b_1/b_1(=1) < b_2/b_1 < \ldots < b_N/b_1\}$$

is an AO-sequence of separable type.

Proof. A proof of this fact is an easy exercise. □

Remark 1.4. Let $\mathcal{D}$ be a linear system of projective dimension $N > 0$ on a curve. It is easy to show that if B is the set of base points of $\mathcal{D}$, then $\mathcal{D}$'s Weierstrass order-sequence coincides with $\mathcal{D}(-B)$'s. So every Weierstrass order-sequence is an AO-sequence (cf. **[16, Cor. 1.8]**). When the linear system $\mathcal{D}$ has no base points, the corresponding morphism $\Phi_{\mathcal{D}}: C \to \mathbb{P}^N$ is separable if and only if the Weierstrass order-sequence of $\mathcal{D}$ is of separable type.

Remark 1.5. A sequence of nonnegative integers $\{a_0 < \ldots < a_N\}$ is the Weierstrass order-sequence of a linear system on a curve if and only if it is an AO-sequence.

Proof. See **[16, Remark after Prop. 1.6]**. □

In the rest of this section, we take up several properties of AO-sequences, which will be used in the next section.

Lemma 1.6. *Let $B = \{b_0 < \ldots < b_N\}$ be an AO-sequence. If b_k is a maximal element with respect to the order $\succ_p$ i.e., $b_i \succ_p b_k$ implies $b_i = b_k$, then $B \backslash \{b_k\}$ is also an AO-sequence.*

Proof. Let b' be a nonnegative integer such that $b' \prec_p b_l$ for some $b_l \in B \backslash \{b_k\}$. So we have $b' \in B$. Suppose $b' = b_k$, then $b_l = b_k$ by maximality of b_k, which is a contradiction. So we have $b' \in B \backslash \{b_k\}$. □

Corollary 1.7. *Let $B = \{b_0 < \ldots < b_N\}$ be an AO-sequence and M be any integer such that $0 \leq M \leq N$. Then $\{b_0 < \ldots < b_M\}$ is also an AO-sequence.*

Proof. This is a consequence of Lemma 1.6. □

Corollary 1.8. *Let $B = \{b_0 < \ldots < b_N\}$ be an AO-sequence. For a fixed element b_k, let $D = \{b_i \in B \mid b_i \succ_p b_k\}$. Then $B \backslash D$ is also an AO-sequence.*

Proof. This is also an easy consequence of Lemma 1.6. □

Let m be a nonnegative integer. We denote by $\operatorname{coeff}_{p^i} m$, the coefficient of p^i of the p-adic expansion of m.

Definition 1.9. The *height* of a positive integer m is the maximum in the integers i with $\operatorname{coeff}_{p^i} m \neq 0$. For an AO-sequence $B = \{b_0 < \ldots < b_N\}$, we define the height of B, denoted height B, to be the height of b_N.

Example 1.10. Obviously, if an order b of an AO-sequence is of height 0, then every nonnegative integer less than b is also an order. Therefore, if B is an AO-sequence of dimension N and of height 0, then B coincides with $\{0, 1, \ldots, N\}$.

2. A Basic Formula

The purpose of this section is to prove the following theorem.

Theorem 2.1. *Let $\{b_0 < \ldots < b_N\}$ be a Weierstrass order-sequence of a linear system on a curve. Then we have $b_j + b_{N-j} \leq b_N$ for every $j = 0, \ldots, N$.*

When the characteristic of the ground field is zero, since $b_j = j (j = 0, \ldots, N)$, the assertion is trivial. So we may assume that the characteristic of the ground field is $p > 0$. In this case, as explained before, each Weierstrass order-sequence is an AO-sequence with respect to p. So our theorem is a consequence of the following theorem.

Theorem 2.2. *Let $\{b_0 < \ldots < b_N\}$ be an AO-sequence with respect to a prime number p. Then we have $b_j + b_{N-j} \leq b_N$ for every $j = 0, \ldots, N$.*

We start with an example.

Example 2.3. An AO-sequence $\{b_0 < \ldots < b_N\}$ is said to be *classical* if $b_j = j$ for every $j = 0, \ldots, N$. Obviously, every classical AO-sequence is *symmetric*, that is $b_j + b_{N-j} = b_N$ for every j. In particular, Theorem 2.2 is true for all AO-sequences of height 0 (cf. Example 1.10).

Now we prove Theorem 2.2. Let $\mathcal{A}$ be the family of AO-sequences with respect to p. If $B = \{b_0 < \ldots < b_N\} \in \mathcal{A}$, we define the *index* of B to be $(N, b_N, b_{N-1}, \ldots, b_0)$. We define a total order on $\mathcal{A}$ by the lexicographic order of indices. Obviously, every nonempty subset of $\mathcal{A}$ has the minimum element with respect to the total order.

Suppose that there is an AO-sequence for which the conclusion of Theorem 2.2 does not hold. Let $B = \{b_0 < \ldots < b_N\}$ be the minimum element of the such AO-sequences. By Example 2.3, we have height B (say e) ≥ 1. Let j be the minimum integer such that $b_j + b_{N-j} > b_N$. Note that $j > 0$ because $b_0 + b_N = b_N$ and that $b_j \leq b_{N-j}$ because of the minimality of j. Setting

$$\operatorname{coeff}_{p^e} b_j = \alpha$$
$$\operatorname{coeff}_{p^e} b_{N-j} = \beta$$
$$\operatorname{coeff}_{p^e} b_N = \gamma,$$

we must have $0 \leq \alpha \leq \beta \leq \gamma < p$ and $\gamma \neq 0$.

Claim 1. *If b_k is a maximal element of B with respect to the order $\succ_p$, then we have $k \geq N - j$.*

Proof. By Lemma 1.6, $B \backslash \{b_k\}$ is also an AO-sequence. Suppose $k < N - j$ and put

$$B \backslash \{b_k\} = \{b'_0 < \ldots < b'_{N-1}\}.$$

Then we have $b'_i = b_i$ if $i < k$ and $b'_i = b_{i+1}$ if $i \geq k$, in particular, $b'_j \geq b_j$, $b'_{(N-1)-j} = b_{N-j}$ and $b'_{N-1} = b_N$. Hence we have

$$b'_j + b'_{(N-1)-j} \geq b_j + b_{N-j} > b_N = b'_{N-1}.$$

Hence $B \backslash \{b_k\}$ gives a counter-example to the assertion of Theorem 2.2, which is a contradiction, because $B \backslash \{b_k\}$ is smaller than B with respect to their indices. □

Claim 2. *Let ϵ be an integer such that $0 \leq \epsilon \leq \gamma$. Let*

$$\tilde{B}(\epsilon) := \{b \in B \mid \epsilon p^e \leq b < (\epsilon + 1) p^e\}$$
$$B(\epsilon) := \{b - \epsilon p^e \mid b \in \tilde{B}(\epsilon)\}.$$

Then we have

$$B(0) \supset B(1) \supset \ldots \supset B(\gamma)$$

and each $B(\epsilon)$ forms an AO-sequence.

Proof. First we prove $B(\epsilon) \supset B(\epsilon+1)$ for each ϵ with $0 \le \epsilon \le \gamma - 1$. Let $c \in B(\epsilon+1)$. Since $c < p^e$, we have $(\epsilon+1)p^e + c \succ_p \epsilon p^e + c$. Hence $\epsilon p^e + c \in B$, because $(\epsilon+1)p^e + c \in B$. This means $c \in B(\epsilon)$.

Next we prove that $B(\epsilon)$ forms an AO-sequence. Let c' be a nonnegative integer such that $c' \ {}_{p}\!\!\prec c$ for some $c \in B(\epsilon)$. Since $\epsilon p^e + c \succ_p \epsilon p^e + c'$ and $\epsilon p^e + c \in B$, we have $\epsilon p^e + c' \in B$. So we have $c' \in B(\epsilon)$. □

Claim 3. $B(0) = \ldots = B(\beta)$.

Proof. We may assume that $\beta > 0$. Let $c \in B(0)$. From Claim 1, there is an integer $b \in B$ such that $b \succ_p c$ and $b \ge b_{N-j}$. Letting $\mathrm{coeff}_{p^e} b = \delta$, we have $\delta \ge \beta$ because $b_{N-j} \in \tilde{B}(\beta)$. Writing as $b = \delta p^e + c'$, we have $c' \succ_p c$, because $c < p^e$ and $b \succ_p c$. Hence $\delta p^e + c' \succ_p \beta p^e + c$ and hence $\beta p^e + c \in B$. So $c \in B(\beta)$. Therefore, by Claim 2, we have $B(0) = \ldots = B(\beta)$. □

Claim 4. *Let $B(0) = \{c_0 < c_1 < \ldots < c_n\}$. Then we have*

$$B(0) = \ldots = B(\beta) = \ldots = B(\gamma - 1) \supset B(\gamma)$$

and

$$B(\gamma) = \{c_0 < c_1 < \ldots < c_h\}$$

for some $h \le n$. So we have $\gamma(n+1) + h = N$.

Proof. The last assertion is a consequence of the preceding assertion because

$$N = {}^{\#}B - 1 = \sum_{\epsilon=0}^{\gamma} {}^{\#}B(\epsilon) - 1.$$

To prove the first assertion, we may assume that $B(0) \supsetneqq B(\gamma)$. Let δ be the minimum integer such that $B(0) \supsetneqq B(\delta)$. Note that $\delta > \beta$ by Claim 3. For each integer ϵ such that $\delta \le \epsilon \le \gamma$, put

$$\tilde{B}_+(\epsilon) := \{\epsilon p^e + c \mid c \in B(0)\}.$$

Obviously,

$$B_+ := B(0) \cup \tilde{B}(1) \cup \ldots \cup \tilde{B}(\delta - 1) \cup \tilde{B}_+(\delta) \cup \ldots \cup \tilde{B}_+(\gamma)$$

is also an AO-sequence. Note that $B_+ \supset B$ and the first successive $\delta(n+1)$ elements of B_+ are contained in B. Let $B' = \{b'_0 < b'_1 < \ldots < b'_N\}$ be the first successive $N+1$ integers of B_+. From Corollary 1.7, B' is an AO-sequence. Since $\delta > \beta$, we have $b'_j = b_j$, $b'_{N-j} = b_{N-j}$ and $b'_N \le b_N$. Hence we have

$$b'_j + b'_{N-j} = b_j + b_{N-j} > b_N \ge b'_N,$$

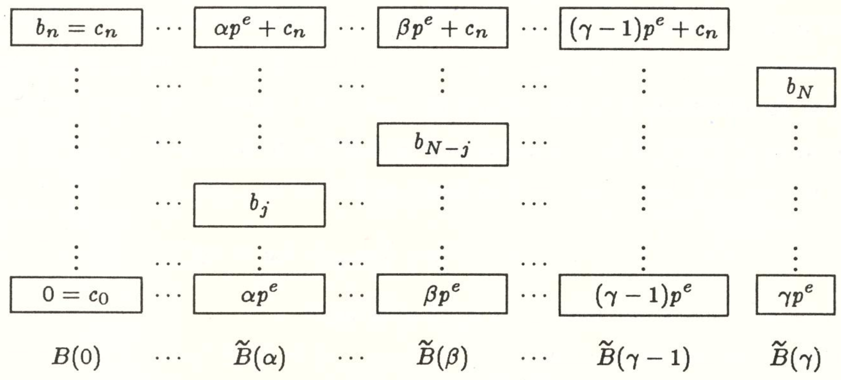
which means B' is a counter-example to the conclusion of Theorem 2.2. Since B is the minimum member of among all counter-examples and B' is smaller than or equal to B with respect to the total order of $\mathcal{A}$, B must coincides with B'. □

By Claim 4,

$$b_N = \gamma p^e + c_h$$

and there are integers f, g with $0 \le f, g \le n$ such that

$$b_{N-j} = \beta p^e + c_g$$
$$b_j = \alpha p^e + c_f.$$



Claim 5. $\alpha = \beta = 0$ *and* $\gamma = 1$.

Proof. Since

$$j = {}^{\#}\{b \in B \mid b > b_{N-j}\} = h + 1 + (\gamma - 1 - \beta)(n+1) + (n - g)$$

and

$$j = {}^{\#}\{b \in B \mid b < b_j\} = \alpha(n+1) + f,$$

we have

$$h + 1 + (n-g) + (\gamma - \beta - 1)(n+1) = f + \alpha(n+1). \tag{1}$$

Since $n - g \ge 0$ and $f < n+1$, (1) implies

$$(\gamma - \beta - 1)(n+1) \le f + \alpha(n+1) < (\alpha+1)(n+1).$$

Hence we have

$$\gamma \le \alpha + \beta + 1. \tag{2}$$

Similarly, since $n - g < n + 1$ and $h \le n$, (1) implies

$$\alpha + \beta \le \gamma. \tag{3}$$

From (2) and (3), $\gamma = \alpha + \beta$ or $\alpha + \beta + 1$.

First we consider the case $\gamma = \alpha + \beta$. Substituting $\alpha + \beta$ for γ in the equation (1), we have

$$h + 1 + (n - g) + (\alpha - 1)(n + 1) = f + \alpha(n + 1).$$

So we have $h = f + g$. On the other hand, since

$$(\alpha + \beta)p^e + c_f + c_g = b_j + b_{N-j} > b_N = \gamma p^e + c_h$$

and $\gamma = \alpha + \beta$, we have $c_f + c_g > c_h$. Those mean $\{c_0 < c_1 < \ldots < c_h\}$, which is an AO-sequence because of Claim 2, is a counter-example to Theorem 2.1. Since the height of the AO-sequence is less than e, that contradicts the choice of B as the minimum member among all counter-examples. Therefore γ must be equal to $\alpha + \beta + 1$. Substituting $\alpha + \beta + 1$ for γ in the equation (1), we have $(n + 1) + h = f + g$. On the other hand, since

$$(\alpha + \beta)p^e + c_f + c_g = b_j + b_{N-j} > b_N = \gamma p^e + c_h$$

and $\gamma = \alpha + \beta + 1$, we have $c_f + c_g > p^e + c_h$.

Now, consider the sequence

$$B' = \{c_0 < c_1 < \ldots < c_n < p^e + c_0 < \ldots < p^e + c_h\}.$$

Obviously, B' is an AO-sequence of height e and of dimension $n + h + 1$. Writing as $B' = \{b'_0 < b'_1 < \ldots < b'_{n+h+1}\}$, we have $b'_i = c_i$ if $i \leq n$ and $b'_{n+i} = p^e + c_{i-1}$. Since $n + h + 1 - f = g$ and $c_f + c_g > p^e + c_h$, we have

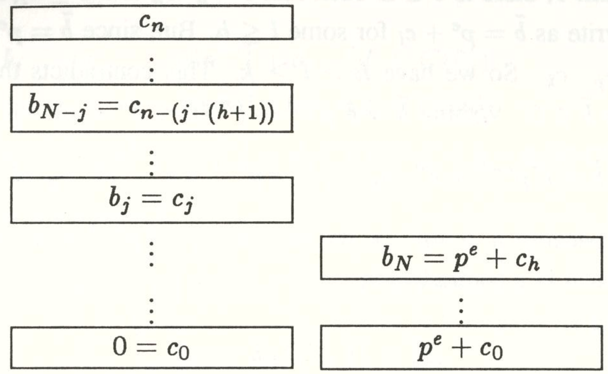
$$b'_f + b'_{(n+h+1)-f} = c_f + c_g > p^e + c_h = b'_{n+h+1}.$$

This means B' is also a counter-example to the conclusion of Theorem 2.2. Since $B' \subset B$, we have $B' = B$ by the minimality of B. In particular, we have $\alpha = \beta = 0$ and $\gamma = 1$. □

By the previous Claims, our situation was reduced to the following: There is an AO-sequence $C = \{c_0 < c_1 < \ldots < c_n\}$ of height $< e$ such that

$$\begin{aligned} B &= \{b_0 < \ldots < b_N\} \\ &= \{c_0 < c_1 < \ldots < c_n < p^e + c_0 < \ldots < p^e + c_h\} \end{aligned}$$

for some $h \leq n$ and $b_j, b_{N-j} \in B(0) = C$. Note that $N = n + h + 1$ and $h + 1 \leq j$ (because $b_{N-j} \in C$).



Claim 6. $e \geq 2$.

Proof. Suppose that $e = 1$. Then since C is of height 0, $C = \{0, 1, \ldots, n\}$ and $n < p$. Hence $b_j = c_j = j$ and $b_{N-j} = c_{n-(j-(h+1))} = n + h + 1 - j$. Since $n < p$, we get $b_j + b_{N-j} = n + h + 1 \leq p + h = b_N$, which is absurd. □

Setting

$$\mathrm{coeff}_{p^{e-1}}\, b_j = \alpha'$$
$$\mathrm{coeff}_{p^{e-1}}\, b_{N-j} = \beta'$$

we must have $0 \leq \alpha' \leq \beta' < p$.

Let ϵ' be an integer such that $0 \leq \epsilon' \leq p - 1$ and let

$$\tilde{C}(\epsilon') := \{c \in C \mid \epsilon' p^{e-1} \leq c < (\epsilon' + 1)p^{e-1}\}$$
$$C(\epsilon') := \{c - \epsilon' p^{e-1} \mid c \in \tilde{C}(\epsilon')\}.$$

Then, by arguments similar to those in the proof of Claim 2, we can show that

$$C(0) \supset C(1) \supset \ldots \supset C(p - 1)$$

and each $C(\epsilon')$ is an AO-sequence.

Claim 7. $C(0) = \ldots = C(\beta')$.

Proof. Suppose that there is an order $d \in C(0)$ such that $d \notin C(\beta')$. Let $D_d := \{b \in B \mid b \succ_p d\}$.

First we show that if $b \in D_d$, then $b \leq c_h$ or $p^e \leq b$. Suppose the contrary: let $c_k \in C$ such that $c_k \succ_p d$ and $c_h < c_k$. Writing as $c_k = \epsilon' p^{e-1} + \epsilon''$, we get $\epsilon'' \succ_p d$ because $d < p^{e-1}$. If $\epsilon' \geq \beta'$, then $c_k \succ_p \beta' p^{e-1} + d$ and then $\beta' p^{e-1} + d \in C$. Hence $d \in C(\beta')$, which is a contradiction. So $\epsilon' < \beta'$. In particular, $c_k < b_{N-j}$ because

$$c_k = \epsilon' p^{e-1} + \epsilon'' < \beta' p^{e-1} \leq b_{N-j}.$$

Hence, by Claim 1, there is $\tilde{b} \in B$ such that $\tilde{b} \succ_p c_k$ and $\tilde{b} \geq b_{N-j}$. If $\tilde{b} \geq p^e$, then we can write as $\tilde{b} = p^e + c_l$ for some $l \leq h$. But, since $\tilde{b} = p^e + c_l \succ_p c_k$, we have $c_l \succ_p c_k$. So we have $h \geq l \geq k$. This contradicts the assumption $c_h < c_k$. Thus $\tilde{b} \in C$. Writing $\tilde{b} = \delta' p^{e-1} + \delta''$ $(0 \leq \delta'' < p^{e-1})$, since

$$\tilde{b} = \delta' p^{e-1} + \delta'' \succ_p c_k = \epsilon' p^{e-1} + \epsilon'', \quad \epsilon'' \succ_p d$$

and $\delta' \geq \beta'$ (because $\tilde{b} \geq b_{N-j}$), we have $\tilde{b} \succ_p \beta' p^{e-1} + d$. Hence we get $d \in C(\beta')$, which is a contradiction.

Now letting

$$D_d^0 = \{b \in D_d | b \leq c_h\}$$
$$D_d^+ = \{b \in D_d | p^e \leq b\},$$

by the preceding remark D_d is the disjoint union of D_d^0 and D_d^+. Moreover, for each $c_k (0 \leq k \leq h)$, $c_k \in D_d^0$ if and only if $p^e + c_k \in D_d^+$. So we have $^\# D_d^0 = {}^\# D_d^+ = \frac{1}{2} {}^\# D_d$. We consider the sequence $B \backslash D_d$, which is an AO-sequence by Corollary 1.8. Put $B \backslash D_d = \{b'_0 < \ldots < b'_{N'}\}$. Then we have $b_j = b'_{j - \# D_d^0}$ and $b_{N-j} = b'_{N' - (j - \# D_d^0)}$ because $b_{N-j} = b'_{N-j-\# D_d^0}$ and $N' = N - {}^\# D_d = N - 2^\# D_d^0$. Since

$$b'_{j-\# D_d^0} + b'_{N'-(j-\# D_d^0)} = b_j + b_{N-j} > b_N \geq b'_{N'} \ ,$$

the AO-sequence $B \backslash D_d$ is also a counter-example to the conclusion of Theorem 2.2. Since B is the minimum member among all counter-examples, we can conclude that $D_d = \emptyset$. □

Let us write as $C(0) = \{d_0 < d_1 < \ldots < d_r\}$.

Claim 8. $C(0) = \ldots = C(\beta') = \ldots = C(p-1)$.

Proof. For each integer ϵ' such that $0 \leq \epsilon' \leq p-1$, let

$$\tilde{C}_+(\epsilon') = \{\epsilon' p^{e-1} + d_\mu | \mu = 0, 1, \ldots, r\}.$$

Then $\tilde{C}_+(0) \cup \ldots \cup \tilde{C}_+(p-1) \cup \{p^e + c_0, \ p^e + c_1, \ldots, p^e + c_h\}$ is an AO-sequence. Let $B' = \{b'_0 < \ldots < b'_N\}$ be the AO-sequence which consists of the first successive $N+1$ integers in the above sequence. Since $\tilde{C}_+(\epsilon') = \tilde{C}(\epsilon')$ for all ϵ' with $0 \leq \epsilon' \leq \beta'$ by Claim 7, we get $b'_j = b_j$ and $b'_{N-j} = b_{N-j}$. Hence B' is also a counter-example to Theorem 2.2. So B must coincides with B'. □

End of Proof. From Claim 8, we can write as

$$\begin{aligned} b_j &= \alpha' p^{e-1} + d_s \quad (d_s \in C(0)) \\ b_{N-j} &= \beta' p^{e-1} + d_t \quad (d_t \in C(0)) \\ b_N &= p^e + \gamma' p^{e-1} + d_u \quad (0 \leq \gamma' < p, \ d_u \in C(0)). \end{aligned}$$

Note that, by Claim 8,

$$B = \left(\bigcup_{\epsilon'=0}^{p-1} \{\epsilon' p^{e-1} + d_\mu | d_\mu \in C(0)\} \right) \cup \left(\bigcup_{\epsilon'=0}^{\gamma'-1} \{p^e + \epsilon' p^{e-1} + d_\mu | d_\mu \in C(0)\} \right) \cup \{p^e + \gamma' p^{e-1} + d_\mu | \ 0 \leq \mu \leq u\}.$$

Since

$$\begin{aligned} j &= {}^\#\{b \in B | b > b_{N-j} (= \beta' p^{e-1} + d_t)\} \\ &= (u+1) + \gamma'(r+1) + (p-1-\beta')(r+1) + r - t \end{aligned}$$

and

$$\begin{aligned} j &= {}^\#\{b \in B | b < b_j (= \alpha' p^{e-1} + d_s)\} \\ &= \alpha'(r+1) + s, \end{aligned}$$

we have

$$(p - 1 - \beta' + \gamma')(r+1) + u + 1 + r - t = \alpha'(r+1) + s. \tag{4}$$

Since $u + 1 + r - t > 0$ and $s < r + 1$, (4) implies

$$(p - 1 - \beta' + \gamma')(r+1) < (\alpha' + 1)(r+1).$$

Hence we have

$$p + \gamma' \leq \alpha' + \beta' + 1. \tag{5}$$

Similarly, since $u + 1 \leq r + 1$ and $r - t \leq r$, (4) implies

$$\alpha' + \beta' \leq p + \gamma'. \tag{6}$$

From (5) and (6), we get $\gamma' = \alpha' + \beta' - p$ or $\alpha' + \beta' + 1 - p$.

Case 1. First we consider the case $\gamma' = \alpha' + \beta' - p$. From (4), we get $u = s + t$. On the other hand, since

$$b_j + b_{N-j} = (\alpha' + \beta') p^{e-1} + d_s + d_t = p^e + \gamma' p^{e-1} + d_s + d_t,$$

$b_N = p^e + \gamma' p^{e-1} + d_u$ and $b_j + b_{N-j} > b_N$, we obtain $d_s + d_t > d_u$. This means that the AO-sequence $\{d_0 < \ldots < d_u\}$ is also a counter-example to Theorem 2.2. But the AO-sequence is of height $< e$, which contradicts the minimality of B.

Case 2. Next we consider the case $\gamma' = \alpha' + \beta' + 1 - p$. By the same argument used in the first case, we have $u + (r+1) = s + t$ and $d_s + d_t > p^{e-1} + d_u$, which mean the AO-sequence

$$\{d_0 < \ldots < d_r < p^{e-1} + d_0 < \ldots < p^{e-1} + d_u\}$$

is also a counter-example. This is a contradiction because the height of the sequence is $e - 1$ ($< e$). So we can conclude that Theorem 2.2 is true for every AO-sequence.

3. Proof of Theorem II

Now we prove Theorem II stated in Introduction. Our proof is divided into two parts.

Theorem 3.1. *Let $\mathcal{D}$ be a linear system on a smooth curve C of degree $d > 0$ and of (projective) dimension $N > 0$ and let $\{b_0 < \ldots < b_N\}$ be the Weierstrass order-sequence of $\mathcal{D}$. Then C has no $\mathcal{D}$-Weierstrass points if and only if $C = \mathbb{P}^1$, $b_N = d$ and $b_j + b_{N-j} = b_N$ for all $j = 0, 1, \ldots, N$.*

Proof. Let W be the ramification divisor of $\mathcal{D}$ (see [12] or [16]; in the terminology of [12], W is the highest Wronskian of $\mathcal{D}$). The divisor W has the following properties:

(i) W is effective of degree

$$(b_0 + \ldots + b_N)(2g-2) + (N+1)d,$$

where g is the genus of C;

(ii) A point $P \in C$ is a $\mathcal{D}$-Weierstrass point if and only if $P \in \operatorname{Supp} W$.

Therefore, C has no $\mathcal{D}$-Weierstrass points if and only if

$$(b_0 + \ldots + b_N)(2-2g) = (N+1)d. \tag{7}$$

If the equation (7) holds, then $g = 0$ because $g > 0$ implies $(N+1)d \leq 0$, which is absurd. Hence our condition is equivalent to

$$\begin{cases} g = 0 \\ 2(b_0 + \ldots + b_N) = (N+1)d. \end{cases} \tag{8}$$

On the other hand, using the fact: $b_N \leq d$ and the inequality proved in Theorem 2.1, we have

$$2(b_0 + \ldots + b_N) \leq (N+1)b_N \leq (N+1)d \tag{9}$$

and all the equalities in (9) hold if and only if $b_N = d$ and $b_j + b_{N-j} = b_N$ for any $j = 0, \ldots, N$. This completes the proof. □

The second step is to prove the following lemma.

Lemma 3.2. *Let V be an $(N+1)$-dimensional subspace of $H^0(\mathbb{P}^1, \mathcal{O}(d))$ and $\{\mu_0 < \ldots < \mu_N\}$ a sequence of nonnegative integers (not necessary an AO-sequence) such that $\mu_N = d$ and $\mu_j + \mu_{N-j} = \mu_N$ ($j = 0, \ldots, N$). Suppose that there are two points $P_1, P_2 \in \mathbb{P}^1$ such that the Hermite invariants of $\mathcal{D} = \mathbb{P}(V)$ at $P_i (i = 1, 2)$ coincides with $\{\mu_0 < \ldots < \mu_N\}$. Then, taking suitable coordinates S, T of $\mathbb{P}^1$, the $N+1$ elements*

$$S^{\mu_0}T^{\mu_N}, S^{\mu_1}T^{\mu_{N-1}}, \ldots, S^{\mu_N}T^{\mu_0}$$

forms a basis of V.

Proof. Choose coordinates S and T of $\mathbb{P}^1$ such that $P_1 = 0 = (0{:}1)$, $P_2 = \infty = (1{:}0)$ and put $s = \dfrac{S}{T}$. Now we consider the isomorphism

$$H^0(\mathbb{P}^1,\ \mathcal{O}(d)) \xrightarrow{\sim} \mathcal{L}(d\infty)$$
$$S^k T^{d-k} \mapsto s^k,$$

where

$$\mathcal{L}(d\infty) = \{f \in k(\mathbb{P}^1) \mid \operatorname{div} f + d\infty \succ 0\} \cup \{0\}.$$

We denote by $\mathcal{L}(\mathcal{D}, d\infty)$, the image of V under the isomorphism. To prove our assertion, it suffices to show that $\mathcal{L}(\mathcal{D}, d\infty)$ is generated by $\{s^{\mu_0}, s^{\mu_1}, \ldots, s^{\mu_N}\}$. Since the Hermite invariants of $\mathcal{D}$ at P_1 are $\{\mu_0 < \ldots < \mu_N\}$, we can choose a basis $\varphi_0, \ldots, \varphi_N$ of $\mathcal{L}(\mathcal{D}, d\infty)$ such that

$$\begin{cases} \varphi_0 = s^{\mu_0} + \sum_{i=\mu_0+1}^{\mu_N} a_{0,i}s^i \\ \quad\vdots \\ \varphi_\alpha = s^{\mu_\alpha} + \sum_{i=\mu_\alpha+1}^{\mu_N} a_{\alpha,i}s^i \\ \quad\vdots \\ \varphi_N = s^{\mu_N}. \end{cases} \tag{10}$$

Furthermore, replacing $\varphi_0, \dots, \varphi_N$ by

$$\begin{aligned}
\varphi'_N &= \varphi_N \\
\varphi'_{N-1} &= \varphi_{N-1} - a_{N-1,\mu_N}\varphi'_N \\
&\vdots \\
\varphi'_\alpha &= \varphi_\alpha - (a_{\alpha,\mu_N}\varphi'_N + \dots + a_{\alpha,\mu_{\alpha+1}}\varphi'_{\alpha+1}) \\
&\vdots \\
\varphi'_0 &= \varphi_0 - (a_{0,\mu_N}\varphi'_N + \dots\dots + a_{0,\mu_1}\varphi'_1),
\end{aligned}$$

we may assume that

$$a_{\alpha,\mu_k} = 0 \quad (k = \alpha+1, \dots, N) \quad \text{for any} \quad \alpha \quad \text{in (10).} \qquad (11)$$

We show that $\varphi_\alpha = s^{\mu_\alpha}$ $(\alpha = 0, \dots, N)$ under the assumption (11). Suppose the contrary. For a number α such that $\varphi_\alpha \neq s^{\mu_\alpha}$, let β be the maximum number such that $a_{\alpha,\beta} \neq 0$. Put $t = \dfrac{T}{S}$ and consider the isomorphism

$$\begin{gathered}
H^0(\mathbb{P}^1, \mathcal{O}(d)) \xrightarrow{\sim} \mathcal{L}(d \cdot 0) \\
S^k T^{d-k} \mapsto t^{d-k}.
\end{gathered}$$

Denoting by $\mathcal{L}(\mathcal{D}, d\cdot 0)$ the image of V under the isomorphism, we get the isomorphism from $\mathcal{L}(\mathcal{D}, d\infty)$ to $\mathcal{L}(\mathcal{D}, d\cdot 0)$ via V. The isomorphism send φ_α to

$$t^{\mu_N} \cdot \varphi_\alpha\left(\frac{1}{t}\right) = t^{\mu_N - \mu_\alpha} + \dots + a_{\alpha,\beta} t^{\mu_N - \beta}.$$

Since t is a local parameter at ∞ and $a_{\alpha,\beta} \neq 0$, $\mu_N - \beta$ must be a Hermite invariant of $\mathcal{D}$ at ∞. Hence we have $\mu_N - \beta = \mu_\gamma$ for some γ. Since $\mu_j + \mu_{N-j} = \mu_N$ for all j, we have $\beta = \mu_{N-\gamma}$, which contradicts (11). This completes the proof.

Addendum. (This note was added in December, 1991). Recently, two nice papers concerning Theorem I have appeared. The first one is

A. Garcia, *Some arithmetic properties of order-sequences of algebraic curves* (Preprint, Oct. 1991) in which he has given a short proof of this theorem. The second one is

E. Esteves, *A geometric proof of an inequality of order sequences* (preprint, Oct. 1991), in which he has given a geometric proof of a generalization of

Theorem I.

Added in proof

Recently, the author found a very short proof of Theorem 1 and another proof of Esteve's generalization which was mentioned in Addendum. The proofs are, however, rather tricky. We will discuss them in another paper.

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