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Survival of Multidimensional contact Process in Random Environments

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Abstract. We consider contact processes in dimension $d \geq 2$, with death rates identically one and random infection rates i.i.d distributed on the space. We show that the process may survive although the distribution λ of the infection rate is such that the expectation of $[\log(1+\lambda)]^{d-\epsilon}$ is as close to zero as one wishes.

1. Introduction and Statement of Results

Consider the d dimensional integer lattice $\mathbb{Z}^d$ and suppose that for each (unordered) pair of elements x and y at Euclidean distance one there is an edge joining then. Denote the edge joining x to y by xy and the set of all edges by $\mathbb{E}_d$. Let $Y = \{\text{all subsets of } \mathbb{Z}^d\}$ and suppose that λ_{xy} is a collection of i.i.d. random variables indexed by $\mathbb{E}_d$, defined in a probability space (Ω, P) and taking values in $\mathbb{R}_+$. Call elements of Ω environments and for each environment ω define a Markov process on Y in the following way: Give in another probability space $(\Omega_\omega, P_\omega)$ two collections of Poisson processes $N_{xy}(\cdot)$ and $M_x(\cdot)$ such that

a) all these Poisson processes are independent,

b) the collections N_{xy} and M_x are indexed by $\mathbb{E}_d$ and $\mathbb{Z}^d$ respectively,

c) $\forall x \in \mathbb{Z}^d$, M_x has intensity one and $\forall xy \in \mathbb{E}_d$, N_{xy} has intensity $\lambda_{xy}(\omega)$.

Then consider $\mathbb{Z}^d \times [0,\infty)$ as a subspace of $\mathbb{R}^d \times [0,\infty)$, and add to it the following (randomly placed) objects:

i) Horizontal edges joining (x,s) to (y,s) for each s such that $N_{xy}(s) - N_{xy}(s-) = 1$.

ii) $\delta' s$ at points (x,s) such that $M_x(s) - M_x(s-) = 1$.

For $s \leq t$ and $x, y \in \mathbb{Z}^d$, say that there exists a path from (x,s) to (y,t) if there is a chain of upward vertical and horizontal edges which starts at (x,s),

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ends at (y,t) and does not cross any δ. Denote this event by

$$\{(x,s) \longrightarrow (y,t)\}$$

For any $A \subset \mathbb{Z}^d$ say that the event

$$\{(x,s) \xrightarrow{A} (y,t)\}$$

occurs if there exists a chain as above entirely contained in $A \times [s,t]$. Note that for a fixed environment the events.

$$(x,s_1) \xrightarrow{A_1} (y,t_1) \quad \text{and} \quad (u,s_2) \xrightarrow{A_2} (v,t_2)$$

are independent if either

$$(s_1,t_1) \cap (s_2,t_2) = \emptyset$$

or

$$A_1 \cap A_2 = \emptyset.$$

Note also that

$$\{(x,s) \xrightarrow{A} (y,t)\} \subset \{(x,s) \xrightarrow{B} (y,t)\}$$

if $A \subset B$.

Now, fix an $x \in \mathbb{Z}^d$, let

$$A_0^x = \{x\}$$

and for $t > 0$

$$A_t^x = \{y \in \mathbb{Z}^d : (x,0) \to (y,t)\}.$$

The process A_t^x is called the contact process in the environment ω with initial condition $\{x\}$. For a more general initial state $B \in Y$, we define A_t^B as $\bigcup_{x\in B} A_t^x$. We say that the contact process in the environment ω survives if for some $x \in \mathbb{Z}^d$, $P_\omega(A_t^x \neq \emptyset,\ \forall t) > 0$.

Note that the set of environments in which the process survives is translation invariant. Hence, by the Ergodic Theorem, it has probability zero or one. If this probability is one, we say that the contact process in the random environment (Ω, P) survives and if it is zero that it dies out. The expectation operators corresponding to P and P_ω are denoted by E and E_ω respectively. Finally, λ will be a random variable arbitrarily chosen among the λ_{xy}'s.

Liggett (1992) showed that in dimension 1, the contact process in a random environment (CPIRE) dies out if $E\log\lambda < 0$. In higher dimensions there exists no $c \in \mathbb{R}$ for which $E\log\lambda < c$ implies extinction. This was observed by Holley

who gave the following argument: for any c there exists a distribution of λ satisfying both $E\log\lambda < c$ and $P(\lambda \geq 2)$ greater than the critical value for bond percolation on $\mathbb{Z}^d$. For this distribution, there exists for almost all environment an infinite path of bonds for which $\lambda \geq 2$. Since the one dimensional contact process survives when $\lambda \equiv 2$ (Holley and Liggett (1978)), this yields an imbedded one dimensional process that survives. In this paper we build up on Holley's argument to show that in dimension $d \geq 2$, for any $\epsilon, \delta > 0$ there exists a CPIRE that survives for which $E[\log(1+\lambda)]^{d-\epsilon} < \delta$. To do so we will only need distributions of λ taking two values λ_0 and λ_1. For this reason, in the sequel we assume that

$$P(\lambda = \lambda_1) = p \quad \text{and} \quad P(\lambda = \lambda_0) = 1 - p$$

where $0 \leq p \leq 1$. We now state our results:

Theorem 1.1. *Given $\lambda_0 > 0$, there exists a constant $C(\lambda_0)$ such that $\forall 0 < p \leq 1$ the CPIRE survives if $\lambda_1 \geq \exp[C(\lambda_0)p^{-\frac{1}{d}}]$*

Corollary 1.2. *Given $\epsilon, \delta > 0$, there exists a CPIRE that survives for which $E[\log(1+\lambda)]^{d-\epsilon} < \delta$.*

To derive the corollary from the theorem observe that if

$$\lambda_1 = \exp[C(\lambda_0)p^{-\frac{1}{d}}],$$

then

$$E[\log(1+\lambda)]^{d-\epsilon} \leq [\log(1+\lambda_0)]^{d-\epsilon} + p[\log(1+\exp(C(\lambda_0)p^{-\frac{1}{d}}))]^{d-\epsilon}.$$

Take now $\lambda_0 > 0$ and such that the first term of the right hand side is less than $\frac{\delta}{2}$. Then note that the second term goes to zero as $p \downarrow 0$.

Recent work of A. Klein (1991) shows that the CPIRE dies out if $E[\log(1+\lambda)]^{f(d)}$ is small enough, where $f(d)$ is a function growing as $2d^2$. It would be interesting to know whether this situation or the one described in Corollary (1.2) holds when considering $E[\log(1+\lambda)]^\alpha$ for values of α in the interval $[d, f(d)]$.

To prove Theorem (1.1) we need to introduce two percolation models on $X = \mathbb{Z}_+ \times \mathbb{Z}_+$.

Oriented site percolation

Call elements of X sites and consider a family $\{Z_x\}_{x\in X}$ of i.i.d. 0 or 1 valued random variables, such that $P(Z_x = 1) = p$. Then say that a site y is open (closed) if $Z_y = 1 (Z_y = 0)$.

Standard techniques show that there exists a critical value $p_c \in (0,1)$ such that $\forall p > p_c$ there exists with probability one an infinite sequence $z_0, z_1, \ldots$, of open sites such that:

$$\forall i \in \mathbb{N} \quad z_i - z_{i-1} = (1,0) \quad \text{or} \quad z_i - z_{i-1} = (0,1).$$

Oriented bond percolation with finite range dependence.

To each element $x \in X$ we associate two 0 or 1 valued random variables U_x and V_x. Assume that x is joined by a bond to $x + (0,1)$ and by another bond to $x + (1,1)$. Say that the first (second) of these bonds is open if $U_x = 1 (V_x = 1)$ and that it is closed if $U_x = 0$ $(V_x = 0)$. For $x, y \in X$ say that y can be reached from x if there exists a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that for all $i \in \{1, \ldots, n\}$ either $x_i - x_{i-1} = (0,1)$ and $U_{x_{i-1}} = 1$, or $x_i - x_{i-1} = (1,1)$ and $V_{x_{i-1}} = 1$. Then define C_0 as $\{y \in X: y \text{ can be reached from } (0,0)\}$ and denote by $|C_0|$ the cardinal of this random set. Suppose now that the random vector (U_x, V_x) satisfies the following finite range dependence condition:

$\forall x \in X$, (U_x, V_x) is independent of

$$\{(U_y, V_y): y \in X \backslash \{x, x+(1,0), x+(2,0), x-(1,0), x-(2,0)\}\} \quad (1.3)$$

Then, contour methods similar to the ones used in Toom (1968) show that there exists a value $p_0 \in (0,1)$ such that for any model as above and for which

$$\inf_{x \in X} \{\min(P(U_x = 1), P(V_x = 1))\} > p_0$$

we have $P(|C_0| = \infty) > 0$.

In the next section we prove some preliminary lemmas involving these percolation models and the proof of Theorem (1.1) is given in Section 3.

2. Preliminary Results

Our first lemma is a consequence of the result concerning oriented site percolation mentioned in Section 1. Before stating it, we introduce some notation that will remain valid throghout this paper. Fix a real number $A \geq 1$ and such that $1 - e^{-A} > p_c$, denote by [ ] the integer part of its argument, by $(e_1, \ldots, e_d)$ the canonical basis in $\mathbb{R}^d$ and by | | the Euclidean norm. Finally let

$$\ell(p) = \left[\left(\frac{A}{p}\right)^{\frac{1}{d}}\right]$$

and note that $\ell(p) \geq 1$.

Lemma 2.1. *For all p suficiently small there exists for P almost all environment* ω, *a sequence* $(k_i)_{i \in \mathbb{Z}_+}$ *in* $\mathbb{Z}^d$ *and a subsequence* $(k_{i_j})_{j \in \mathbb{Z}_+}$ *satisfying the following properties:*

a) $k_i \neq k_j \quad \forall i \neq j$,

b) $|k_{i+1} - k_i| = 1 \quad \forall i \in \mathbb{Z}_+$,

c) $i_0 = 0$,

d) $i_{j+1} - i_j < 2d\ell(p)$,

e) $\forall j \in \mathbb{Z}_+ \ \lambda_{x_j y_j}(\omega) = \lambda_1$, *where* $x_j = k_{i_j}$ *and* $y_j = k_{i_j+1}$.

Proof. For $z \in \mathbb{Z}^d$ let A_{pz} be the set of edges in Z^d whose vertices are contained in:

$$\left\{\ell(p) \cdot z + \sum_{i=1}^{d} a_i e_i : 1 \leq a_i \leq \ell(p) - 1 \quad i = 1, \ldots d\right\},$$

where $\ell(p) \cdot z$ is the element of $\mathbb{Z}^d$ obtained by multiplying each coordinate of z by $\ell(p)$. The number of such edges is $d(\ell(p) - 1)^{d-1}(\ell(p) - 2)$, which for p small enough is bigger than $\frac{A}{p}$. Hence, for those values of p, the probability that for at least one edge in A_{pz} the corresponding λ is equal to λ_1 is bigger than

$$1 - (1-p)^{\frac{A}{p}} > 1 - e^{A} > p_c.$$

Since $A_{pz} \cap A_{pz'} = \emptyset$ if $z \neq z'$, the events

$$B_z = \{\lambda = \lambda_1, \quad \text{for at least one edge in} \quad A_{pz}\} \quad z \in \mathbb{Z}^d,$$

are independent and we can apply the result mentioned in the previous section on oriented site percolation, to conclude that for almost all environment there exists a sequence $(z_i)_{i \in \mathbb{Z}_+}$ in $\mathbb{Z}^d$ such that $\forall i \in \mathbb{Z}_+$, $z_{i+1} - z_i \in \{e_1, \ldots, e_d\}$ and A_{pz_i} has at least one edge for which $\lambda = \lambda_1$. The sequence k_i and the subsequence k_{i_j} are easily derived from (z_i).

Note. Since p_c is the critical value for two dimensional oriented site percolation, the sequence z_i can be taken among the elements of $\mathbb{Z}^d$ whose last $d-2$ coordinates are 0. This property will not be needed later.

In the sequel k_i, k_{i_j}, x_j and y_j will be as in the above lemma. We now define $\forall j \in \mathbb{Z}_+$:

$$E_j = \{x_j, y_j\}, \quad F_j = \{k_{i_j}, k_{i_j+1}, \ldots, k_{i_{j+1}}, k_{i_{j+1}+1}\},$$

and

$$\delta = P(T < \min\{S, 1\}),$$

where T and S are independent exponentially distributed ramdom variables of parameters λ_0 and 1 respectively.

Lemma 2.2. *Let $0 \leq s < t$ and suppose that ω is an environment for which the conclusion of Lemma (2.1) holds.*

Then, $\forall j \in \mathbb{Z}_+$ we have:

a) $P_\omega((x_j, s) \xleftrightarrow{E_j} (x_j, t)) \geq 1 - 2\frac{t-s+1}{\lambda_1+1}$,

b) $P_\omega((y_j, s) \xrightarrow{E_j} (y_j, t)) \geq 1 - 2\frac{t-s+1}{\lambda_1+1}$,

and

c) $P_\omega((x_j, s) \xrightarrow{E_j} (y_j, t)) \geq 1 - 2\frac{t-s+1}{\lambda_1+1} - e^{-\lambda_1(t-s)}$.

Moreover if $\frac{\lambda_1 - 1}{2} \geq t - s \geq 2d\ell(p)k$ for some $k \in \mathbb{N}$, then $\forall j \in \mathbb{Z}_+$ we also have

d) $P_\omega((y_j, s) \xrightarrow{F_j} (x_{j+1}, t)) \geq \left[1 - 2\left(\frac{t-s+1}{\lambda_1+1}\right)\right]^2 (1 - (1 - \delta^{2d\ell(p)})^k)$.

Proof. It clearly suffices to consider the case in which $s = 0$ and $j = 0$. This has the advantage of simplifying the notation in this proof. To prove part a) note that the complement of

$$\{(x_0, 0) \xrightarrow{E_0} (x_0, t)\}$$

is contained in $\bigcup_{i=1}^4 A_i$ where

$A_1 = \{$For some $u \geq 0$ $M_{x_0}(u) = 1$ and $N_{x_0y_0}(u) = 0\}$,

$A_2 = \{$For some $0 \leq u < t$ $M_{x_0}(t) = M_{x_0}(u) + 1$ and $N_{x_0y_0}(t) = N_{x_0y_0}(u)\}$,

$A_3 = \{$For some $0 \leq u < t$ $M_{x_0}(u) = M_{x_0}(u-) + 1$ and for some $v \geq u$ $M_{y_0}(v) - M_{y_0}(u) = 1$ and $N_{x_0y_0}(v) = N_{x_0y_0}(u)\}$

$A_4 = \{$For some $0 \leq u < t$ $M_{y_0}(u) = M_{y_0}(u-) + 1$ and for some $v \geq u$ $M_{x_0}(v) - M_{x_0}(u) = 1$ and $N_{x_0y_0}(v) = N_{x_0y_0}(u)\}$.

Hence

$$P_\omega((x_0, 0) \xrightarrow{E_0} (x_0, t)) \geq 1 - \sum_{i=1}^{4} P_\omega(A_i).$$

We now give upper bounds for $P_\omega(A_i), 1 \leq i \leq 4$. Clearly

$$P_\omega(A_1) = \frac{1}{\lambda_1+1}, \quad P_\omega(A_2) \leq \frac{1}{\lambda_1+1},$$

$$P_\omega(A_3 | M_{x_0}(t) = n) \leq \frac{n}{\lambda_1+1} \quad \forall n \quad \text{and} \quad P_\omega(A_4 | M_{y_0}(t) = n) \leq \frac{n}{\lambda_1+1} \quad \forall n.$$

Therefore:

$$\sum_{i=1}^{4} P_\omega(A_i) \leq \frac{1}{\lambda_1+1} + \frac{1}{\lambda_1+1} + \sum_{n=0}^{\infty} \frac{n}{\lambda_1+1}[P_\omega(M_{x_0}(t) = n) + P_\omega(M_{y_0}(t) = n)]$$

$$= \frac{2}{\lambda_1+1} + 2\frac{t}{\lambda_1+1} = 2\frac{t+1}{\lambda_1+1},$$

which proves a). To prove part b) define $B_i (1 \leq i \leq 4)$ as A_i interchanging the roles of x_0 and y_0. Then procede as in the proof of part a). To prove part c) define

$$C = \{N_{x_0y_0}(t) = 0\}$$

and note that the complement of

$$\{(x_0, 0) \xrightarrow{E_0} (y_0, t)\}$$

is contained in $A_1 \cup B_2 \cup A_3 \cup A_4 \cup C$.

Part c) now follows from the estimates already obtained and the fact that $P_\omega(C) = e^{-\lambda_1 t}$. To prove part d) write:

$$P_\omega((y_0, 0) \xrightarrow{F_0} (x_1, t)) \geq$$

$$P_\omega(\text{ For some } 0 \leq u \leq t, (y_0, 0) \xrightarrow{F_0} (x_1, u) \text{ and } (x_1, u) \xrightarrow{E_1} (x_1, t)) \geq$$

$$P_\omega(\text{ For some } 0 \leq u \leq t, (y_0, 0) \xrightarrow{F_0} (x_1, u)) \cdot \inf_{u \in [0,t]} P_\omega((x_1, u) \xrightarrow{E_1} (x_1, t)),$$

where the second inequality follows from the comments made immediately after introducing the notation

$$\{(x, 0) \xrightarrow{A} (y, t)\}.$$

Hence, by part a)

$$P_\omega((y_0, 0) \xrightarrow{F_0} (x_1, t)) \geq$$

$$\left[1 - 2\tfrac{t+1}{\lambda_1+1}\right] \cdot P_\omega \text{ (for some } 0 \leq u \leq t \ (y_0, 0) \xrightarrow{F_0} (x_1, u)). \tag{2.3}$$

To give a lower bound to the second term in the right hand side, define, for $0 \le j \le k$, $s_j = j\,(i_1 - 1)$, where i_1 is as in Lemma (2.1), and for

$$0 \le j \le k-1 \quad D_j = \{(y_0, s_j) \xrightarrow{F_0} (x_1, u) \text{ for some } s_j < u \le s_{j+1}\}.$$

Then,

$$P_\omega(\text{ for some } 0 < u \le t \ (y_0, 0) \xrightarrow{F_0} (x_1, u)) \ge$$

$$P_\omega\left(\bigcup_{j=0}^{k-1}\left[D_j \cap \left\{(y_0,0) \xrightarrow{E_0} (y_0, s_j)\right\}\right]\right) \ge$$

$$P_\omega\left(\bigcup_{j=0}^{k-1} D_j\right) \inf_{0 \le j \le k-1} P_\omega\left((y_0,0) \xrightarrow{E_0} (y_0, s_j)\right).$$

The last inequality being again a consequence of the comment following the definition of

$$\{(x,s) \xrightarrow{A} (y,t)\}.$$

By part a) the right hand side above is bounded below by

$$P_\omega\left(\bigcup_{j=0}^{k-1} D_j\right) \cdot \left[1 - 2\frac{(k)(i_1-1)+1}{\lambda_1+1}\right].$$

By Lemma (2.1) $i_1 < 2d\ell(p)$, hence

$$P_\omega(\text{ for some } 0 < u \le t, \ (y_0,0) \xrightarrow{F_0} (x_1,u)) \ge \left[1 - 2\tfrac{t+1}{\lambda_1+1}\right] P_\omega\left(\bigcup_{j=0}^{k-1} D_j\right).$$

In view of (2.3), it now suffices to show that

$$P_\omega\left(\bigcup_{j=0}^{k-1} D_j\right) \ge [1 - (1 - \delta^{2d\ell(p)})^k] \tag{2.4}$$

To prove this, note that the events D_j are independent and have the same probability. Therefore

$$P_\omega\left(\bigcup_{j=0}^{k-1} D_j\right) = 1 - (1 - P(D_0))^k \tag{2.5}$$

Since $s_1 - s_0 = i_1 - 1$ and $F_0 \supset \{k_1 = y_0, \ldots, k_{i_1} = x_1\}$, it is clear from the definition of δ that $P_\omega(D_0) \ge \delta^{i_1 - 1}$, and since $i_1 - 1 < 2d\ell(p)$, this implies that $P_\omega(D_0) \ge \delta^{2d\ell(p)}$, which together with (2.5) yields (2.4).

3. Proof of Theorem (1.1)

In this proof we adopt the following convention: for $a \in \mathbb{R}$ we let $\lceil a \rceil = \inf\{z \in$

$\mathbb{Z}: z \ge a\}$. We will show the following statement, which is easily seen to imply the theorem: there exist constants c_1 and c_2 ((depending on δ and hence on λ_0, but not on p) such that for p small enough and

$$\lambda_1 \ge c_1 p^{-\frac{1}{d}} \lceil \exp(c_2 p^{-\frac{1}{d}}) \rceil$$

the CPIRE survives. Since we have to deal only with small values of p, it suffices to show that the contact process survives on environments satisfying the conclusion of Lemma (2.1). To prove this, choose positive integers L and K, such that

$$\left(1 - \frac{2}{L}\right)^2 \left(1 - e^{-K}\right) > p_0$$

and

$$1 - \frac{3}{L} > p_0,$$

where p_0 is the value appearing in Section 1 and related to oriented bond percolation models satisfying the finite range dependence condition (1.3). Then, let

$$n = K\lceil \delta^{-2d\ell(p)} \rceil \quad \text{and} \quad t_0 = 2nd\ell(p)$$

where δ is the constant defined just before Lemma (2.2). Suppose that

$$\lambda_1 \ge 2t_0 L = 4LKd\ell(p)\lceil \delta^{-2d\ell(p)} \rceil.$$

We will now show that $P_\omega^{x_0}(A_t \ne \varnothing, \ \forall t) > 0$; where $(x_i)_{i \in \mathbb{Z}_+}$ and $(y_i)_{i \in \mathbb{Z}_+}$ are the sequences related to ω as in the conclusion of Lemma (2.1). For this purpose, it suffices to show that

$$P_\omega(A_{nt_0}^{x_0} \cap \{x_i, y_i : i \in \mathbb{Z}_+\} \ne \varnothing \ \ \forall n) > 0.$$

To prove this inequality we compare the contact process in the environment ω to an oriented bond percolation model satisfying (1.3) and defined in the probability space $(\Omega_\omega, P_\omega)$ in the following way: For $i, n \in \mathbb{Z}_+$ identify (x_i, nt_0) to $(2i, n)$ and (y_i, nt_0) to $(2i+1, n)$. Then let

$$U_{2i,n} = 1_{\{(x_i, nt_0) \xrightarrow{E_i} (x_i, (n+1)t_0)\}},$$

$$V_{2i,n} = 1_{\{(x_i, nt_0) \xrightarrow{E_1} (y_i, (n+1)t_0)\}},$$

$$U_{2i+1,n} = 1_{\{(y_i, nt_0) \xrightarrow{E_i} (y_i, (n+1)t_0)\}}$$

and

$$V_{2i+1,n} = 1_{\{(y_i, nt_0) \xrightarrow{F_i} (x_{i+1},(n+1)t_0)\}},$$

where E_i and F_i are the subsets of $\mathbb{Z}^d$ defined after the proof of Lemma (2.1). The random variables $U_{j,n}$ and $V_{j,n}$ define an oriented bond percolation model as in Section 1. This percolation model satisfies condition (1.3) and is related to the contact process in the environment ω by the following inclusion

$$\{|C_0| = \infty\} \subset \{A^{x_0}_{nt_0} \cap \{x_i, y_i : i \in \mathbb{Z}_+\} \neq \emptyset \; \forall n\}.$$

Hence, to complete the proof it suffices to show that $P_\omega(|C_0| = \infty) > 0$. To do so we give the following lower bounds:

a) $P_\omega(U_{2i,n} = 1) \geq 1 - 2\dfrac{t_0+1}{\lambda_1+1} \geq 1 - 2\dfrac{t_0+1}{2t_0 L} \geq 1 - \frac{2}{L}$,

where the first inequality follows for part a) of Lemma (2.2), the second from our choice of λ_1 and the third from the fact that $t_0 \geq 1$. In the same manner we can also prove that:

b) $P_\omega(U_{2i+1,n} = 1) \geq 1 - \frac{2}{L}$

Then using part c) of Lemma (2.2) we obtain

c)

$$\begin{aligned} P_\omega(V_{2i,n} = 1) &\geq 1 - 2\frac{t_0+1}{\lambda_1+1} - e^{-\lambda_1 t_0} \\ &\geq 1 - \frac{2}{L} - e^{-\lambda_1} \geq 1 - \frac{2}{L} - \frac{1}{\lambda_1} \geq 1 - \frac{3}{L}, \end{aligned}$$

where the last inequality follows from the choice of λ_1 and the fact that $t_0 \geq 1$. Finally, using part d) of Lemma (2.2) we get:

d)

$$\begin{aligned} P_\omega(V_{2i+1,n} = 1) &\geq \left(1 - 2\frac{t_0+1}{\lambda_1+1}\right)^2 \left(1 - \left(1 - \delta^{2d\ell(p)}\right)^n\right) \\ &\geq \left(1 - \frac{2}{L}\right)^2 \left(1 - \left(1 - \delta^{2d\ell(p)}\right)^{\delta^{-2d\ell(p)}.K}\right) \\ &\geq \left(1 - \frac{2}{L}\right)^2 \left(1 - e^{-K}\right), \end{aligned}$$

where the second inequality follows from our choice of n and the third from the fact that for $y \geq 1$

$$\left(1 - \frac{1}{y}\right)^y \leq e^{-1}.$$

By our choice of L and K, all the lower bounds are greater than p_0 and this implies that $P_\omega(|C| = \infty) > 0$.

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