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Entropy of non-uniformly hyperbolic plane billiards

N. Chernov and R. Markarian

Abstract. We prove exact formulas for measure theoretic entropy of plane billiards systems with absolutely-focusing boundaries with non-vanishing Lyapunov exponents. In particular, our formulas hold for the billiards introduced by Wojtkowski, Markarian, Donnay and Bunimovich. As an illustration, we calculate the entropy of a "perturbation" of the boundary of a polygon by absolutely focusing "ripples". .

1. Introduction

In 1970 Ya. G. Sinai proved that plane billiard systems with strictly concave boundary are hyperbolic and ergodic. L. A. Bunimovich (1974, 1979) extended these results to billiards with a strictly convex boundary with regular form (consisting of circular arcs). Recently M. Wojtkowsky (1986) and R. Markarian (1988) introduced far larger classes of plane billiards with focusing boundaries which are also (non-uniformly) hyperbolic. Finally, R. Markarian (1990), V. Donnay (1991) and L.A. Bunimovich (1991) introduced more general classes of convex curves which may serve as parts of the boundary of chaotic billiards. Donnay and Bunimovich have called them (absolutely) focusing arcs. Note that billiards with smooth (C^6) convex boundary have caustics that occupy a set of positive measure (L'azutkin, 1973) and are therefore not chaotic.

Sinai (1970) obtained exact formulas for the measure-theoretic entropy of plane billiards with concave boundaries. Chernov provided another proof of the Sinai formulas and extended them to Bunimovich billiards with circular arcs. Here we will prove the same formulas for all non-uniformly hyperbolic billiards with absolutely focusing boundaries, including those studied by Wojtkowsky and

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Markarian.

2. Notations

A complete and fairly detailed introduction to the theory of billiards can be found in (Cornfeld et al., 1982). As this paper is a continuation of (Chernov, 1991), the same notation has been used. The following paragraphs provide a brief summary of the basic notations used in this paper. Differences with other papers by R. Markarian should be observed.

A plane billiard is the dynamical system describing the free motion of a point mass inside an open, bounded, connected region Q of the plane, with elastic reflections at the boundary. The boundary consists of a finite set of closed curves ∂Q_1 that can be either C^{r+1}, $r \geq 3$, with non-zero curvature, or real analytic. In both cases, the curvature $|K|$ is bounded. The regular components of the boundary

$$\partial \tilde{Q}_i = \partial Q_i \setminus \bigcup_{j \neq i} \partial Q_j$$

can have positive (dispersing components), strictly negative (focusing) or zero curvature (neutral). Angles between adjacent regular components are not equal to zero. The union of regular components with positive curvature is represented by ∂Q^+.

Let $x = (q, v)$ be a moving point $q \in \overline{Q}$, and $\|v\| = 1$. The billiard flow $\{S^t\}$ can be defined on $\mathcal{M} = \overline{Q} \times S^1$ (the phase space). It preserves the Liouville measure $d\mu = c_\mu dq dv$. Here $c_\mu = (2\pi|Q|)^{-1}$ is the normalizing factor and $|Q|$ stands for the area of Q.

Consider the cross-section

$$M = \{(q, v), q \in \partial Q, < v, n(q) > \geq 0\}$$

of the phase space $\mathcal{M}$, where $n(q)$ denotes the inward unit normal vector to ∂Q. The coordinate system in M is constituted by the arc length parameter r along ∂Q and the angle φ between $n(q)$ and v. Clearly $|\varphi| \leq \pi/2$.

Consider the measure $d\nu = c_\nu \cos\varphi dr d\varphi$. Here $c_\nu = (2|\partial Q|)^{-1}$ is the normalizing factor and $|\partial Q|$ stands for the length of ∂Q. The free path of a point $x \in \mathcal{M}$ until the first reflection is denoted as $t(x)$, and the natural projection from $\mathcal{M}$ to $\overline{Q}$ and from M to ∂Q as π. If $n(\pi(S^{t(x)}x)$ is well-defined and $S^{t(x)+}x \in M$ for $x \in M$, then $Tx = S^{t(x)+}x$ defines almost everywhere the first return map $T\colon M \to M$. T is called the billiard map.

If $N \subset M$ is the set of points where T^k is not defined or not continuous for some $k \in \mathbb{Z}$, then T is a C^r, $r \geq 3$, diffeomorphism from $H = M \setminus N$ onto H. $\nu(H) = 1$ and T preserves the measure ν.

Strelcyn, in (Katok, Strelcyn, 1986), part V.7, described classes of arcs - including those indicated above - that may constitute parts of the boundary of billiards whose billiard map is a smooth map with singularities (or discontinuous dynamical system). Then Pesin's theory may be applied in order to construct invariant manifolds and to obtain explicit formulas for the entropy.

We will say that the system is non-uniformly hyperbolic or chaotic if the Pesin region has measure one. Recall that the Pesin region $(\Sigma(T))$ is the set of points where none of the Lyapunov exponents equal to zero. Good descriptions of billiards that have Pesin region of measure one may be found in (Wojtkowski, 1986), (Markarian 1988, 1990), (Donnay, 1991).

3. Absolutely - Focusing Arcs and Invariant Manifolds

A C^4 focusing curve Γ is an *absolutely- focusing arc* if

a) any infinitesimal parallel incoming beam of trajectories focuses between each pair of collisions with Γ and focuses in Q after hitting Γ for the last time (Figure 1), and

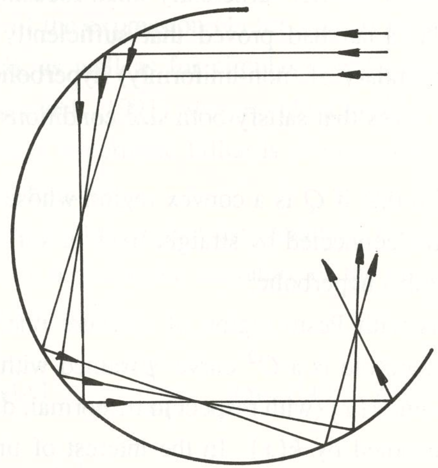

Figure 1

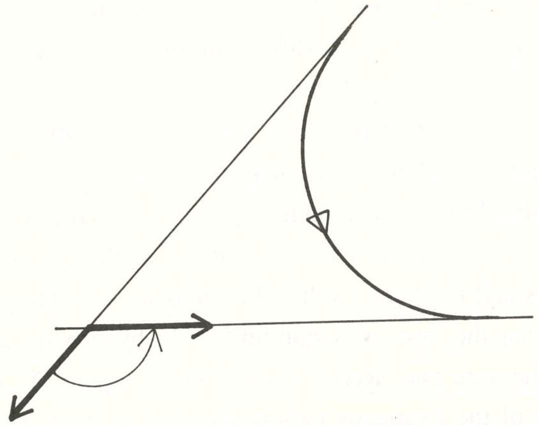

Figure 2

b) the angle between the oriented tangents at the end points of Γ is not greater than π (Figure 2).

Bunimovich (1991) realized that condition (a) is equivalent to the following simpler requirement: the beam which falls parallel upon Γ focuses after the last reflection.

Donnay (1991) proved that any sufficiently small focusing curve is absolutely-focusing. Earlier, Markarian had proved that sufficiently small focusing arcs may be parts of the boundary of (non-uniformly) hyperbolic billiards, but under different conditions. Curves that satisfy both size conditions will be called *short-focusing arcs.*

Donnay also proved that if Q is a convex region whose boundary consists of absolutely-focusing arcs connected by straight lines of sufficient length, then the billiard is (non-uniformly) hyperbolic.

In billiard systems with Pesin region of measure one, at a.e. phase point $x \in \mathcal{M}$, the unstable manifold is a C^2 curve $\tilde{\gamma}$ framed with normal vectors. The curvature of $\tilde{\gamma}$ at the point $\pi(x)$ (with respect to its normal, directed as the velocity vector of x) will be denoted by $k(x)$. In the interest of time we will call $k(x)$ the curvature of the unstable manifold (at x).

Projecting an unstable manifold to the space M, we obtain an unstable manifold for the billiard map T which is a smooth (C^1) curve γ^u in M. If $x \in M$

then it is easy to see that

$$\frac{d\varphi}{dr} = -K(x) + k(x)\cos\varphi(x)$$

is the equation of γ^u.

Consider now the countinued fraction

$$a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cfrac{1}{a_4(x) + \cfrac{1}{\dots}}}} \tag{1}$$

The analytical expression for $k(x)$ is a continued fraction of the form (1), with

$$a_{2k+1}(x) = \frac{2K(T^{-k}x)}{\cos\varphi(T^{-k}x)}, \quad a_{2k}(x) = t(T^{-k}x)$$

with $K(y)$ being the curvature of ∂Q at $\pi(y)$. Observe that

$$a_{2k+1}(T^k y)L(y) = 4,$$

if $L(y)$ is the part of the trajectory at y contained in the circle of curvature of $\pi(y)$.

The convergence of the expression (1) for $k(x)$ is proved for semi-dispersing billiards (Sinai, 1970), as well as for circular arcs (Bunimovich, 1974). Based on the work of Bunimovich (1991), we provide a proof of its convergence for a large of (non-uniformly) hyperbolic billiards (Proposition 2). We believe that it is valid for all chaotic billiards.

It is very important to observe that the continued fraction $k(x)$ may converge but may not necessarily describe the curvature of the unstable manifold. This is the case of the circle where

$$a_{2k}(x) = \frac{-2}{R\cos\varphi}, \quad a_{2k+1}(x) = 2R\cos\varphi,$$

and it results $k(x) = \dfrac{-1}{R\cos\varphi}$ (Bunimovich, 1974). In this case, the differential equation becomes $\dfrac{d\varphi}{dr} = 0$, which describes the corresponding invariant curve in the phase plane.

4. Main Results and Examples

In (Wojtkowski, 1986) it was proved that curves that verify the formula

$$|L(x)| + |L(Tx)| < 2t(x) \tag{2}$$

are absolutely-focusing. The proof is based on the fact that parallel incoming beams "containing" x focus at distance $L(x)/2$ on the trajectory of $x = (q, v)$. Condition (2) is equivalent to $d^2R/dr^2 < 0$ where $R(q) = \dfrac{1}{K(q)}$ is the radius of curvature at $q \in \partial Q$.

Proposition 1. *Curves that satisfy the formula*

$$|L(Tx)|[t(Tx) + t(x)] < 2t(Tx)t(x), \tag{3}$$

considered in Markarian (1988) are absolutely-focusing.

Theorem 1. *If a billiard map is (non-uniformly) hyperbolic and*

$$\int_M \ln^+ |k(x)| d\nu(x) < \infty$$

then the entropy of the billiard map is

$$h_\nu(T) = \int_M \ln |1 + t(x)k(x)| d\nu(x) \tag{4}$$

Recall that $\ln^+ s = \max\{0, \ln s\}$.

Theorem 2. *If the billiard map satisfies the conditions of Theorem 1, then the entropy of billiard flow* $\{S^t\}$ *is*

$$\begin{aligned} h_\mu(\{S^t\}) &= h_\nu(T) \left[\int_M t(x) d\nu(x)\right]^{-1} \\ &= h_\nu(T) c_\mu c_\nu^{-1} \\ &= h_\nu(T) |\partial Q| (\pi |Q|)^{-1}. \end{aligned} \tag{5}$$

Remark 1. In the case of billiards with concave boundary ($K > 0$), Sinai (1970) proved that

$$h_\mu(\{S^t\}) = \int_M k(x) d\mu(x).$$

In our case this integral does not exist. Indeed, each unstable manifold after reflection at a focusing arc passes a conjugate point y_0. At this point $k(y_0) = \infty$, and $k(y) \cong d^{-1}$ at the distance d to the point y_0.

Theorem 3. *Billiards that satisfy conditions of Theorem 1 can be constructed in the following ways.*

i) *Regular components of the boundary can be of any type with the exception that focusing components must satisfy (2) or be short-focusing arcs. With reference to non-adjacent components, the circles of semicurvature in each point of each focusing component do not contain points either of other regular components or of the circles of semicurvature of other focusing components. Adjacent components form interior angles greater than π. Focusing and dispersing adjacent components form angles not smaller than π, and focusing and neutral adjacent components forms angles greater than $\pi/2$.*

ii) *Regular components of the boundary can be of any type with the exception that focusing components must satisfy (3) or be short-focusing arcs. The circles of curvature of focusing components can not contain points of other components. Conditions on adjacent arcs are as in (i).*

iii) *Regular components of the boundary can be of any type with the exception that focusing ones must be absolutely-focusing arcs located sufficiently apart from each other, so that they satisfy the condition* (ii) *of Theorem 8 in Bunimovich (1991). Conditions on adjacent components are as in* (ii).

In every case almost every trajectory must have infinite hits at non-neutral components.

Proposition 2. *For billiard tables described in Theorem 3, the expression (1) is convergent. Since its billiard transformations are (non-uniformly) hyperbolic, it describes the curvature of locally unstable curves in $\mathcal{M}$.*

Remark 2. For Sinai (multidimensional) and Bunimovich-type billiards, theorem 1 was proved in Chernov (1991). The formula of Theorem 1 can also be applied if the Pesin region has positive measure but not equal to one.

Remark 3. In case (iii) of Theorem 3 and Proposition 2, focusing components are supposed to be sufficiently far apart from one another. The exact formulation of that assumption is given in Bunimovich (1991). Roughly speaking, to each focusing component Γ one can assign a positive number τ_Γ. The distance, (measured inside Q), between two such components Γ_1 and Γ_2 is supposed to be greater than $\tau_{\Gamma_1} + \tau_{\Gamma_2}$. In particular examples the value τ_Γ is rather difficult to compute. But it is scale invariant, i.e., if Γ_ε is a copy of Γ shrinked homotetically with a factor ε, then $\tau_{\Gamma_\varepsilon} = \varepsilon\tau_\Gamma$. This property is used in our illustrative example in section 6.

Examples of billiards that satisfy conditions of theorem 1 are the following:

a) The cardioid: (Wojtkowski, 1986) and Theorem 3.

b) Well-designed billiards with arcs of epicycloid, cycloid and arcs of an ellipse close to the vertices: (Wojtkowski, 1986), (Markarian, 1988), Proposition 1 and Theorem 3. If the ellipse is given by $x = a\cos\alpha$, $y = \operatorname{sen}\alpha$, $0 \leq \alpha \leq 2\pi$, $(a \geq 1)$, "good" arcs are defined by

$$\operatorname{sen}^2\alpha \leq \frac{1}{1+a^2}, \quad \text{or} \quad \frac{\pi}{4} \leq \alpha \leq \frac{3\pi}{4}.$$

c) The elliptic stadium (Figure 3) with half-ellipses that satisfy $1 \leq a < \sqrt{2}$ and with h sufficiently large: (Donnay, 1991). Numerical studies (Canale, Markarian, 1991) indicate that "good" billiards are obtained with $h > h(a)$; some values of $h(a)$ are indicated in the following table (it seems that $h(a) \to +\infty$ when $a \to \sqrt{2}$).

a	$h(a)$
1	0
1.1	0.46
1.2	0.7
1.3	1.4
1.4	7
1.41	11

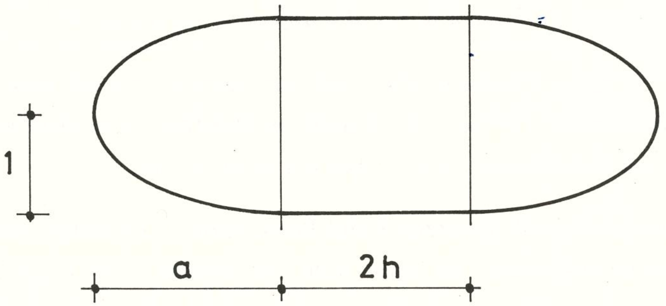

Figure 3

If $a > \sqrt{2}$, for very large h, KAM phenomena is observed.

d) The billiard with at least three ergodic components studied in Wojtkowski (1986) (Figure 4) when $h > a^2 - 1$ and F_1, F_2, F_1', F_2' are the focus of each half ellipse.

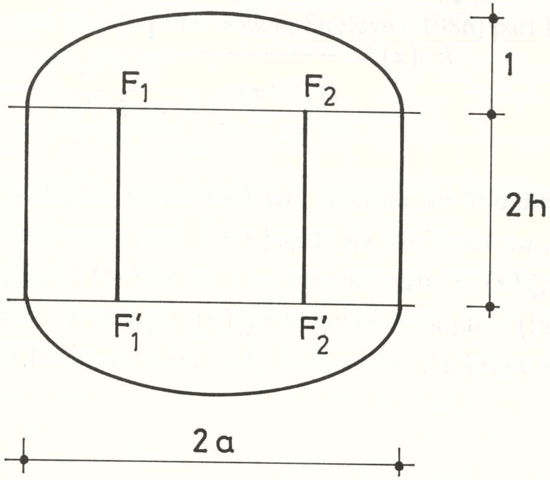

Figure 4

5. Proofs

Let us recall Theorem 11 in Bunimovich (1991) that will be used in the proofs of Proposition 1 and 2: The continued fraction of (1) is convergent if the following two conditions are satisfied:

a) all even elements are positive and their sum is infinite,

b) for any two negative elements $a_{2k'-1}, a_{2k''-1}(k' \leq k'')$ such that there are no other negative elements between them,

$$\min\{a_{2k'}, a_{2k''}\} \geq |a_{2k'-1}|^{-1}(2+\delta_{k'}) + |a_{2k''+1}|^{-1}\left(2 - \frac{\delta_{k''}}{1+\delta_{k''}}\right)$$

for some $\delta_l \geq -1$, $l = 1, 2, \ldots$,

Proof of Proposition 1. As Bunimovich pointed out to us, conditions (2) and (3) are almost the same because they can be rewritten as

$$a_{2k} \geq 2|a_{2k-1}|^{-1} + 2|a_{2k+1}|^{-1} \tag{W}$$

$$|a_{2k+1}| \geq 2a_{2k}^{-1} + 2\, a_{2k+2}^{-1}, \quad k = 1, 2, \ldots \tag{M}$$

Bunimovich's theorem allows to prove that if (1) verifies (W), then it is convergent.

But as (M) corresponds to a downward shift in the positions at the continued fraction, it also implies the convergence of $k(x)$. Recall that

$$k^-(x) = \cfrac{1}{t(T^{-1}x) + \cfrac{1}{\cfrac{2K(T^{-1}x)}{\cos\varphi(T^{-1}x)} + \cfrac{1}{\dots}}}$$

is the curvature of the unstable manifold before reflecting at x. If condition (W) is satisfied, we know that $\operatorname{sign}(k_W(x)) = \operatorname{sign}$ (first term in $k_W(x)$) is negative, and $\operatorname{sign} k_W^-(x) = \operatorname{sign}$ (second term in $k_W(x)$) is positive. This means that $\operatorname{sign}(k_M^-(x)) = \operatorname{sign}$ (first term of $k_M^-(x)$) is positive. Finally $k_M(x)$, considered as a part of $k_M(Tx)$, has the sign of its third term, and $k_M(x) < 0$.

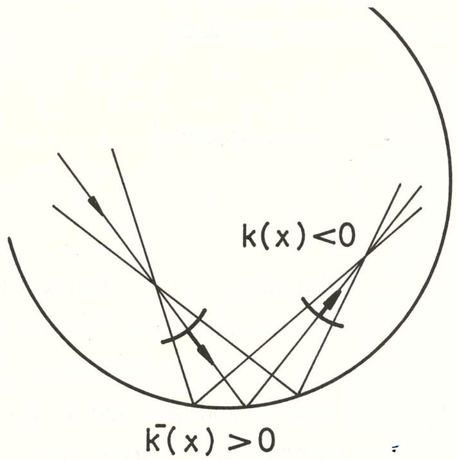

Figure 5

As all these relations are satisfied at each reflection, the conjugate point of the unstable beam $(k(x) < 0)$ happens before the next reflection $(k^-(Tx) > 0)$. If this is true for an expanding beam (before the reflection), it must be true for a parallel beam as well. □

Proof of Proposition 2. The fact that all focusing arcs are absolutely-focusing together with conditions on the relative positions of regular components of the boundary, allows us to apply Theorem 11 in Bunimovich (1991). For example, in case (i) we must apply it with $\delta_\ell = 0$. Note that for trajectories with a finite number of strikes on focusing arcs, the result follows because continuous fractions with terms of constant sign are convergent. □

Proof of Theorem 1. As it was observed in section 2, our billiard map is a smooth map with singularities. We can therefore apply the formula for entropy proved by Ledrappier and Strelcyn. See Katok, Strelcyn, (1986) part III. Prop.2.5. The metric entropy is expressed by

$$h_\nu(T) = \int_M \ln \left|T'_{|E_x^u}\right| d\nu(x) \tag{6}$$

Here E_x^u is the linear subspace corresponding to all positive Lyapunov exponents, and $T'_{|E_x^u}$ stands for the derivative of T, restricted to E_x^u, at x.

Formula (6) is valid in multidimensional case; here we have one-dimensional E_x^u, so $\left|T'_{|E_x^u}\right|$ is simply the rate of expansion of E_x^u under the action of T', in the euclidean norm of M: $dl = \sqrt{dr^2 + d\varphi^2}$. In order to evaluate this rate we consider the degenerate norm $d\rho = \cos\varphi dr$ and the jacobian

$$J(x) = \frac{d\rho(x)}{dl(x)} = \cos\varphi \left[1 + \left(\frac{d\varphi}{dr}\right)^2\right]^{-1/2}.$$

Then

$$\frac{dl(Tx)}{dl(x)} = \frac{d\rho(Tx)J(x)}{d\rho(x)J(Tx)}$$

and, denoting by $f(x) = \frac{d\varphi}{dr}(x)$ on invariant unstable curves, we obtain

$$\left|T'_{|E_x^u}\right| = |1 + t(x)k(x)| \frac{\cos\varphi(x)}{\cos\varphi(Tx)} \left[\frac{1 + f^2(Tx)}{1 + f^2(x)}\right]^{1/2}$$

Taking logarithms and integrating on M we obtain (4) due to the invariance of the measure ν. All that is required for this is the integrability of the functions $\ln\cos\varphi(x)$ and $\ln(1 + f^2(x))$. The first can be checked directly while the second was proved in Chernov (1991) for semi-dispersing billiards. In our case, since $\ln(1 + f^2(x)) = \ln(1 + (-K(x) + k(x)\cos\varphi)^2)$, and $K(x)$ is bounded, the result follows from the conditions on $k(x)$. □

Proof of Theorem 2. It is based on Abramov's formula (Abramov, 1959), and is exactly the same as that in Chernov (1991). □

Proof of Theorem 3. The corresponding billiard maps are non-uniformly hyperbolic. See Markarian (1990), Theorems A, and B for cases i), ii); and Bunimovich (1991), Theorem 8, for the other one.

The integrability of $\ln|k(x)|$ is proven in Chernov (1991) for non-focusing boundary. In our case, this proof is easily extended to the non-focusing parts of ∂Q with the aid of the proof of Proposition 1. For focusing ones, this integrability follows from the estimate

$$|k(x)| \leq |a_1(x)| = 2|K(x)|\cos^{-1}\varphi(x) \tag{7}$$

In fact, this estimate follows from the relation between $k(x)$ and $k^-(x)$:

$$k(x) = 2K(x)\cos^{-1}\varphi(x) + k^-(x).$$

The estimates $k^-(x) < 0$, $k(x) > 0$ are contained in the proof of Proposition 1 for cases (*i*) and (*ii*). Case (*iii*) may be derived from Condition ii of Theorem 8 in Bunimovich (1991), although they are not expressed explicitly therein. □

6. Illustrative Example

Let P be an arbitrary polygon and Γ an arbitrary absolutely-focusing curve. Glue identical copies of the arc Γ shrunk homotetically with a small factor $\epsilon(\Gamma_\epsilon)$ to the sides of the polygon. When $\epsilon \to 0$, one observes "ripples" on ∂P.

Erasing the initial sides of P we obtain for small ϵ a region $Q = Q_\epsilon$ with a boundary that satisfies Theorem 3 (iii). (Near the vertices of the polygon some non-admissible situations may appear. To prevent this, we can erase several adjacent copies of Γ_ϵ and restore the corresponding parts of ∂P. See Figure 6).

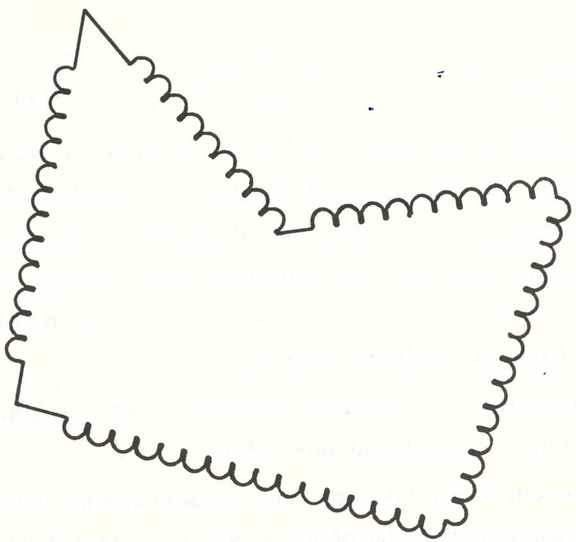

Figure 6

Proposition 3. The entropy of the billiard in Q_ϵ is

$$h_\nu(T) = -A\ln\epsilon + O(1) \tag{8}$$

for some $A = A(\Gamma) > 0$. $O(1)$ means a bounded value as $\epsilon \to 0$.

The above construction was first described in Chernov (1991). An analogous proposition was proved there for semicircles instead of arbitrary absolutely-focusing Γ.

Proof. First note that

$$c_1\epsilon^{-1} \leq |k(x)| \leq c_2[\epsilon\cos\varphi(x)]^{-1} \tag{9}$$

for some positive constants c_1, c_2. The right hand side of (9) comes from (7). To obtain the left-hand side note that the distance from any point of reflection to the nearest future conjugate point is approximately $[a_1(x)]^{-1}$. This follows from the fact that curvatures of a wave front after and before the reflection are related by $k(x) = a_1(x) + k^-(x)$, and the conjugation is obtained for $k^-(x) = 0$. But that distance is less than const. $L(x)$, and $L(x) <$ const. ϵ due to the homoteticity.

Now consider the initial arc Γ and construct a formal space

$$M_\Gamma = \{x = (q, v) : q \in \Gamma, \|v\| = 1, < v, n(q) >> 0\}$$

Let us split M_γ into two parts: M_Γ^0 consists of all the pairs (q, v) such that the ray $q + tv$ will cross Γ again for some $t > 0$, and $M_\Gamma' = M_\Gamma \backslash M_\Gamma^0$. The above partition of M_Γ induces a partition of the space M of our billiard in Q_ϵ. Each component of M, adjacent to a copy of Γ in ∂Q_ϵ is partitioned in the same way as M_Γ. We get $M = M^0 \cup M^1$.

We see that $t(x) \leq$ const $.\epsilon$ for $x \in M^0$ and that $t(x) \cong$ constant for the majority of points of $x \in M^1$. Then on M^1,

$$\ln(1 + t(x)k(x)) \cong \ln t(x) - \ln\varepsilon + \Delta(x)$$

with $|\Delta x| \leq$ const. and applying Theorem 1, we obtain (8) with $A = \nu(M_1)$. □

Note that A really depends on Γ, but not on the polygon. For instance, if Γ is half a circumference, then $A = 2/\pi$. For details of the estimates and calculus, see Chernov (1991).

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N. Chernov
Laboratory of Computing Techniques and Automation

Joint Institute for Nuclear Research
Dubna, P. O. Box 79
Head Post Office
Moscow

R. Markarian
Instituto de Matemática y Estadistica Prof. Ing. Rafael Laguardia
Facultad de Ingenieria
C. C. 30. Montevideo
Uruguay