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Expansive Flows on Seifert Manifolds and on Torus Bundles

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Abstract. We show that any expansive flow on a 3-manifold which is a Seifert fibration or a torus bundle over S^1 is topologically equivalent to a transitive Anosov flow. This is achieved by analyzing the trace of the stable foliation (with singularities) of the flow on incompressible tori embedded in such a manifold.

1. Introduction

Let M be a closed connected 3-manifold and let $\phi_t : M \to M$ be an expansive flow: ϕ_t has no fixed points and $\forall \epsilon > 0\ \exists \delta > 0$ s.t. if $x, y \in M$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) < \delta\ \forall t \in \mathbb{R}$ and for some continuous homeomorphism $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, then $y = \phi_s(x)$ for some $s \in (-\epsilon, \epsilon)$. Here $d(\cdot, \cdot)$ denotes the distance with respect to any metric on M, and the flow may be only continuous.

The aim of this paper is to prove the following result.

Theorem. *If M is a Seifert manifold or a torus bundle then ϕ_t is topologically equivalent to a transitive Anosov flow.*

We recall that a 3-manifold is said to be *Seifert* if it admits a foliation by circles with the property that every circle has an orientable tubular neighborhood; for example any circle bundle over a surface is a Seifert manifold, more in general there are "exceptional" fibres ([Hem]).

E. Ghys classified in [Ghy] up to finite covering and up to topological equivalence all Anosov flows on circle bundles, and so our theorem allows to extend such a classification to all expansive flows. Moreover, it seems that Ghys' classification holds also in the case of Seifert manifolds, the

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geodesic flows on orbifolds with constant negative curvature playing the role of models for the classification.

Recall also that J. Plante showed in [Pla] that any Anosov flow on a torus bundle is topologically equivalent to the suspension of $A : T^2 \to T^2$, where $A \in SL(2, Z)$ is hyperbolic and represents the monodromy of the fibration.

Observe that our theorem contains, as a particular case, the theorem of [Pat2]: an expansive geodesic flow on a surface is topologically equivalent to the (Anosov) geodesic flow corresponding to a metric of constant negative curvature.

M. Paternain and T. Inaba, S. Matsumoto ([Pat1], [I-M]) proved that an expansive flow on a 3-manifold possesses stable and unstable foliations with circle prong singularities (see the next section). This property is used in [Bru] to show that the theorem of D. Fried on surfaces of section ([Fri]) holds also for transitive expansive flows: any such a flow admits surfaces of section with first return maps topologically conjugate to pseudo-Anosov maps. The results of [Fri] and [Bru] imply that if the stable and unstable foliations of a transitive expansive flow have no singularities then the flow is topologically equivalent to an Anosov flow.

Hence the above theorem is a consequence of the following two propositions.

Proposition 1. *Let $\phi_t : M \to M$ be an expansive flow and let M be a Seifert manifold or a torus bundle, then the stable and unstable foliations $\mathcal{F}^s$, $\mathcal{F}^u$ have no singularities and their liftings $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$ in the universal covering $\tilde{M} = R^3$ are product foliations (i.e. topologically equivalent to the foliation by horizontal planes).*

Proposition 2. *Let $\phi_t : M \to M$ be an expansive flow such that the foliations $\mathcal{F}^s$, $\mathcal{F}^u$ satisfy the conclusion of proposition 1, then the flow ϕ_t is transitive.*

Remark. Proposition 2 is proven in [Sol2] in the Anosov case; in the expansive case it is required a little more work, due to the absence of strong stable and strong unstable foliations. On the other hand, it seems that proposition 2 is not really necessary for the proof of the theorem,

because it seems very plausible that an expansive flow whose foliations are without singularities is always topologically equivalent to an Anosov flow, even in the non-transitive case.

We conclude this introduction with two natural questions.

a) Is it true that if M supports an Anosov flow then every expansive flow on M is topologically equivalent to an Anosov one? This is true if M admits an "algebraic" Anosov flow (i.e. obtained by taking the quotient of a translation on a Lie group), because in this case M is Seifert or a torus bundle;

b) Seifert manifolds and torus bundles are examples of graph manifolds (= unions of Seifert manifolds, glued along their boundaries); is our theorem true if M is any graph manifold? Recall that there are examples of Anosov flows on manifolds which are not graph manifolds.

I thank V.V. Solodov and A. Verjovsky for useful discussions about the subject of this paper.

Stable and unstable foliations and their universal coverings.

We refer to [I-M] for definitions and notations about foliations with circle prong singularities.

Let $\phi_t : M \to M$ be an expansive flow (M= closed connected 3-manifold), define $\forall \epsilon > 0$ and $\forall x \in M$:

$$W^s_\epsilon(x)=\{y \in M | \exists h \in Homeo([0,+\infty)) s.t. \ d(\phi_t(x), \phi_{h(t)}(y)) < \epsilon \ \forall t \geq 0\}$$

$$W^u_\epsilon(x)=\{y \in M | \exists h \in Homeo((-\infty,0]) s.t. \ d(\phi_t(x), \phi_{h(t)}(y)) < \epsilon \ \forall t \leq 0\}$$

Proposition. **([I-M], [Pat1].)** *There exist two ϕ_t-invariant foliations with circle prong singularities $\mathcal{F}^s, \mathcal{F}^u$ with the following properties:*

1) *$Sing(\mathcal{F}^s) = Sing(\mathcal{F}^u) \stackrel{def}{=} \mathcal{S}$ is a finite union of closed orbits of ϕ_t and $\mathcal{F}^s, \mathcal{F}^u$ are (topologically) transverse in $M \setminus \mathcal{S}$; the number of prongs of $\mathcal{F}^s, \mathcal{F}^u$ at each singular circle $C \in \mathcal{S}$ is ≥ 3*

2) *if $\epsilon > 0$ is sufficiently small, then $\forall x \in M$ $W^s_\epsilon(x)$ ($W^u_\epsilon(x)$) is a neighborhood of x in the extended leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$) through x*

3) *every separatrix of* $\mathcal{F}^s$ *or* $\mathcal{F}^u$ *is an open cylinder, with a unique singular end; the other leaves are planes, cylinders, or Möebius strips*

4) *there are no closed transversals to* $\mathcal{F}^s$ *or to* $\mathcal{F}^u$ *homotopic to zero; every leaf injects its fundamental group in* $\pi_1(M)$.

Consider now the universal covering $\pi : \tilde{M} \to M$ and let $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u$ be the lifted foliations; $\tilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$ is a countable discrete set of lines. Let $\tilde{\mathcal{F}}^s_0$ and $\tilde{\mathcal{F}}^u_0$ be the foliations (without singularities) on $\tilde{M} \setminus \tilde{\mathcal{S}}$ obtained by restriction of $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$. The leaves of $\tilde{\mathcal{F}}^s_0$ and $\tilde{\mathcal{F}}^u_0$ are all planes, closed in $\tilde{M} \setminus \tilde{\mathcal{S}}$, by 4) of the above proposition. Remark that $\tilde{\mathcal{F}}^s_0$ and $\tilde{\mathcal{F}}^u_0$ may be not transversely orientable, for the possible existence of odd-prong singularities.

Lemma 1. *Let* $j : \mathbb{R} \to \tilde{M} \setminus \tilde{\mathcal{S}}$ *be an embedding transverse to* $\tilde{\mathcal{F}}^s_0$, *then every leaf of* $\tilde{\mathcal{F}}^s_0$ *intersects* $j(\mathbb{R})$ *in no more then one point.*

Proof. Assume that a leaf L intersects $j(\mathbb{R})$ in two points $j(q), j(p)$, $q > p$, and let $\gamma : [0,1] \to L$ be a curve joining $j(q)$ and $j(p)$. Fix a continuous co-orientation of L in $\tilde{M}$ along γ, then there are two possibilities:

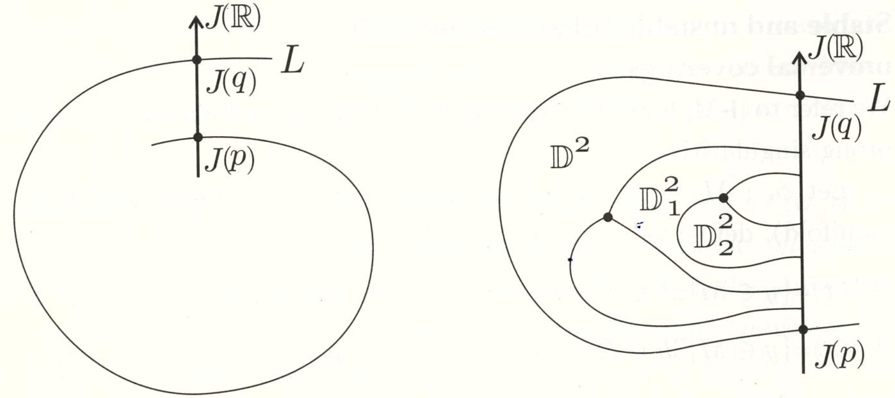


Figure 1

i) the co-orientations in $\gamma(0)$ and $\gamma(1)$ are both compatible or both incompatible with the co-orientations induced by $j(\mathbb{R})$: then it is easy to perturb the cycle $\gamma([0,1]) \cup j([p,q])$ to obtain a closed curve transverse to $\tilde{\mathcal{F}}^s_0$, but this is absurd because $\mathcal{F}^s$ does not admit closed transversals homotopic to zero.

ii) the co-orientation in $\gamma(0)$ is compatible with the one given by

$j(\mathbb{R})$ and the co-orientation in $\gamma(1)$ is incompatible (or vice versa): take a disk $\mathbb{D}^2 \hookrightarrow \tilde{M}$ with boundary $\gamma([0,1]) \cup j([p,q])$ such that (cfr. [I-M]) $\tilde{\mathcal{F}}^s$ induces on it a foliation $\mathcal{G}$ with centers, saddles, prongs (= transverse intersections with the singular set of the foliation), without connections between two prongs or a saddle and a prong, and without prong self-connections. Then a Poincarè-Hopf argument shows that $\mathcal{G}$ has at least one prong (with an odd,≥ 3, number of separatrices), and all the separatrices of this prong must intersects $j([p,q])$, because the leaves of $\mathcal{G}$ are closed in $\mathbb{D}^2 \setminus Sing(\mathcal{G})$. Two of these separatrices separate a sub-disk $\mathbb{D}^2_1 \subset \mathbb{D}^2$ with a piece of boundary on $j((p,q))$, and we may repeat the above argument for $\mathbb{D}^2_1$. An iteration of this construction would produce an infinite number of prongs on $\mathbb{D}^2$, which is absurd. □

As a consequence of the lemma we have that $V^s \stackrel{def}{=} (\tilde{M} \setminus \tilde{\mathcal{S}})/\tilde{\mathcal{F}}^s_0$ (and similarly $V^u \stackrel{def}{=} (\tilde{M} \setminus \tilde{\mathcal{S}})/\tilde{\mathcal{F}}^u_0$) is a one-dimensional connected manifold, with countable base, possibly non-Hausdorff ([God]).

The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ preserving $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{F}}^s$, and so there is an induced action on V^s. Fixed points of $\alpha \in \pi_1(M)$ on V^s correspond to leaves which project on M to leaves containing a closed orbit representing α.

Lemma 2. *Let* $\delta \in \pi_1(M)$ *be a central element, different from the identity, then* δ *acts on* V^s *without fixed points.*

Proof. It is exactly the same of lemma 2.4 of [Ghy]: the set

$$Fix^{\approx}(\delta) = \{x \in V^s | x \text{ and } \delta(x) \text{ are not separated}\}$$

is closed, countable and $\pi_1(M)$- invariant, and if it is not empty then the foliation $\mathcal{F}^s_0 = \mathcal{F}^s|_{M \setminus \mathcal{S}}$ has a closed saturated set K which is transversely countable. It is easy to see that if $\{K_\alpha\}_{\alpha \in I}$ is a collection of closed $\mathcal{F}^s_0$-saturated non-empty sets, totally ordered with respect to the inclusion, then $\cap_{\alpha \in I} K_\alpha$ is non-empty (in spite of the non-compactness of $M \setminus \mathcal{S}$: we use here the particular structure of $\mathcal{F}^s_0$ around the singular set $\mathcal{S}$). Hence we may apply Zorn lemma to deduce that K contains a minimal set, which is a closed leaf because of the transverse countability. But this is in contradiction with Inaba-Matsumoto-Paternain theorem, hence

$Fix^{\approx}(\delta) = \varnothing$. □

Remark 1. If $\beta \in \pi_1(M) \setminus \{id\}$ is such that $\beta^k \in Z(\pi_1(M))$ for some $k \neq 0$, then also β has no fixed points.

Remark 2. In the transitive case it is possible to prove the above statement as follow: if $\gamma \subset M$ is a closed orbit of ϕ_t, representing $\alpha \in \pi_1(M)$, then using the fact that the stable leaf of γ accumulates on itself it is easy to construct a closed transversal to $\mathcal{F}^s$ representing $\alpha * \beta * \alpha^{-1} * \beta^{-1}$ for some $\beta \in \pi_1(M)$, and the non-triviality of this element implies that $\alpha \notin Z(\pi_1(M))$.

Proof of proposition 1 - Seifert manifolds

Let now M be a Seifert manifold. The results of Paternain ($\pi_1(M)$ has exponential growth) and Inaba-Matsumoto (M is aspherical), together with the standard theory of Seifert manifolds ([Hem]), imply that $\tilde{M} = \mathbb{R}^3$ and that there is an exact sequence

$$1 \to Z \to \pi_1(M) \to \pi_1(\Sigma) \to 1$$

where Σ= base of the Seifert fibration = orbifold of genus ≥ 2 and Z is the cyclic infinite normal subgroup generated by a regular fibre.

It is sufficient to prove proposition 1 for some finite covering of (M, ϕ_t), so we may assume that M and Σ are orientable, and in particular Z is central in $\pi_1(M)$. Hence the regular fibres represent central elements, and the exceptional fibres roots of central elements. Lemma 2 means that there are no closed orbits of ϕ_t freely homotopic to a fibre, and consequently every closed orbit projects to Σ to a curve non homotopic to zero.

Let $\omega \subset \Sigma$ be a simple, smooth, closed curve, non homotopic to zero and disjoint from the singular points of Σ, and denote by T^2_ω the incompressible torus $p^{-1}(\omega) \subset M$, $p : M \to \Sigma$ being the canonical projection. The following lemma is obvious.

Lemma 3. *If $\gamma \subset M$ is a closed curve such that $\forall \omega \subset \Sigma$ as above γ is homotopic to a curve which does not intersect T^2_ω, then $p(\gamma) \subset \Sigma$ is homotopic to zero.*

Assume now that $\mathcal{S} \neq \varnothing$, let $\gamma \in \mathcal{S}$ be a singular closed orbit of ϕ_t and let $\omega \subset \Sigma$ be any closed curve as above. We will show that with an isotopy γ can be disjoined from T^2_ω, and the arbitrarity of ω together lemma 2 and 3 will give a contradiction and will prove that we must have $\mathcal{S} = \varnothing$.

First of all we put T^2_ω in general position with respect to the foliation $\mathcal{F}^s$ ([Sol1]): the induced foliation $\mathcal{G}$ has only a finite number of centers and saddles due to tangency points, and a finite number of prongs due to the transverse intersection $T^2_\omega \cap \mathcal{S}$; there are no connections between two different saddles, and no saddle-prong connections. An isotopy of T^2_ω will ensure that, moreover, there are no connections between two different prongs and no prong self-connections ([I-M]). We claim that, at this point, we have $T^2_\omega \cap \mathcal{S} = \varnothing$.

Consider a center $p \in T^2_\omega$ and define (cfr. [R-R]):

$$E_p = \bigcup \{\mathbb{D}^2 \subset T^2_\omega | p \in \mathbb{D}^2 \text{ and } \mathcal{G}|_{\mathbb{D}^2 \setminus p} \text{ is a foliation by circles}\}$$

as in [R-R], the absence of vanishing cycles in $\mathcal{F}^s$ implies that ∂E_p is formed by one or two separatrices and a saddle point q:

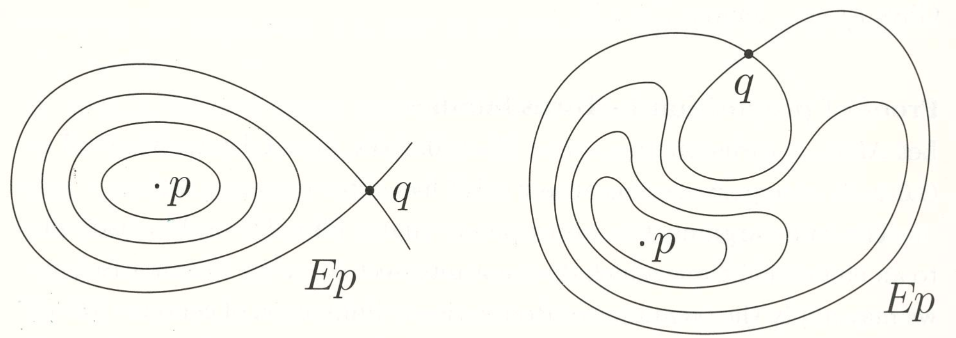


Figure 2

In this way we associate to every center p a saddle $q = q(p)$. It may happen that two centers p_1, p_2 are associated to the same saddle q:

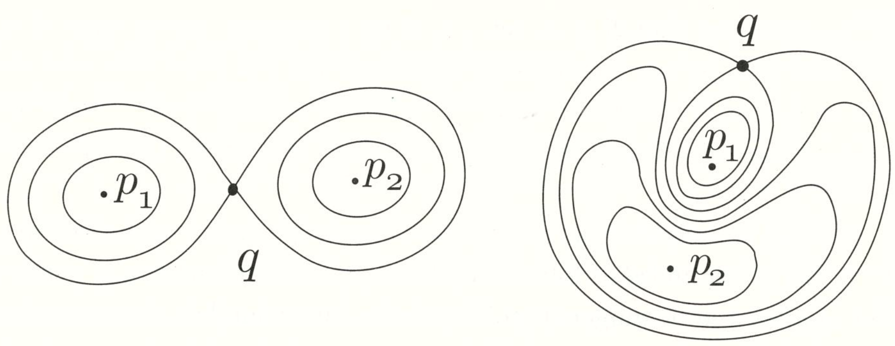


Figure 3

but the absence of vanishing cycles in $\mathcal{F}^s$ implies that there exist embeddings $\mathbb{D}^2 \overset{i}{\hookrightarrow} T^2_\omega$ such that $\overline{E_{p_1}} \cup \overline{E_{p_2}} \subset i(\mathbb{D}^2)$ and $\mathcal{G}|_{i(\mathbb{D}^2)\setminus(\overline{E_{p_1}}\cup\overline{E_{p_2}})}$ is a foliation by circles. The union of all these embeddings is again a region bounded by one or two separatrices and a saddle point.

In conclusion we see that the number of saddles must be greater or equal than the number of centers, and the Poincarè-Hopf formula shows that these two numbers are in fact equal, and the number of prongs is zero, i.e. $T^2_\omega \cap \mathcal{S} = \varnothing$. As remarked before, this means that $\mathcal{S} = \varnothing$.

It remains only to prove that $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ are product foliations. But now $\mathcal{F}^s$ and $\mathcal{F}^u$ are without singularities, so we may repeat the proof of Ghys ([Ghy], lemma 2.4-2.7). □

Proof of proposition 1 - Torus bundles

Let M be a torus bundle: $M = T^2 \times [0,1]/(x,0) \simeq (Ax,1)$, where $A \in GL(2,\mathbb{Z})$ is hyperbolic, because $\pi_1(M)$ has exponential growth.

The same argument as above shows that a fibre $T^2 \hookrightarrow M$ is isotopic to an embedded torus which does not intersect $\mathcal{S}$. After this first isotopy we may apply the technique of Roussarie of elimination of centers ([Rou]) to isotope this torus to one transverse to $\mathcal{F}^s$, and again disjoint from $\mathcal{S}$ (the fact that the foliation is only C^0 and perhaps is not transversely orientable does not give any trouble).

If we cut M along such a torus we obtain the manifold $\bar{M} = T^2 \times [0,1]$, equipped with a foliation $\bar{\mathcal{F}}$ with circle prong singularities $\bar{\mathcal{S}}$, transverse

to $\partial\bar{M} = T^2 \times \{0,1\}$. The foliations $\mathcal{G}_0 = \bar{\mathcal{F}}|_{T^2\times\{0\}}$ and $\mathcal{G}_1 = \bar{\mathcal{F}}|_{T^2\times\{1\}}$ correspond each other via A: $A_*(\mathcal{G}_0) = \mathcal{G}_1$. Because A is hyperbolic, the compact leaves of $\mathcal{G}_0$ cannot be homotopic in $\bar{M}$ to the compact leaves of $\mathcal{G}_1$.

We will pass to the double $2\bar{M} \simeq T^3$, equipped with the double foliation $2\bar{\mathcal{F}}$, and we will analyze the trace of $2\bar{\mathcal{F}}$ on incompressible tori in T^3, as in the Seifert case. However, before to do this we have to choose carefully the torus $T^2 \hookrightarrow M$ transverse to $\mathcal{F}^s$.

Let $\gamma \in \bar{\mathcal{S}}$, we will say that a separatrix $L \in \bar{\mathcal{F}}$ at γ *reaches the boundary* if $L \cap \partial\bar{M} = \Gamma \simeq S^1$ and Γ is homotopic to (a multiple of) γ on L. Then Γ must be a simple closed curve in $T^2 \times \{0\}$ or $T^2 \times \{1\}$, non homotopic to zero (and hence representing a non-multiple element of $\pi_1(\bar{M})$), and L realizes a "cobordism" between Γ and γ (and not a multiple of γ); in particular the foliation is not twisted around γ, i.e $\sharp\{$ prongs at $\gamma\} = \sharp\{$ separatrices at $\gamma\}$. Observe also that Γ is a limit cycle for $\mathcal{G}_0$ or $\mathcal{G}_1$, and that if a separatrix at γ reaches $T^2 \times \{0\}$ then no other separatrix at γ can reach $T^2 \times \{1\}$. This is essential for the proof.

A *singular half Reeb component* of $\bar{\mathcal{F}}$ is a closed saturated set $\Omega \subset \bar{M}$, homeomorphic to a solid torus, such that:

1) $\partial\Omega = A \cup L_1 \cup L_2 \cup \gamma$, where $\gamma \in \bar{\mathcal{S}}$, L_1 and L_2 are separatrices at γ reaching the boundary, $A \subset \partial\bar{M}$ is an annulus, with $\partial A = (L_1 \cap \partial\bar{M}) \cup (L_2 \cap \partial\bar{M})$

2) $int\Omega$ is foliated as the usual half Reeb component.

In particular, A is a planar Reeb component for $\bar{\mathcal{F}}|_{\partial\bar{M}}$. Remark that a singular circle $\gamma \in \bar{\mathcal{S}}$ may be in the boundary of several singular half Reeb components $\Omega_1, ..., \Omega_n$. In this case the corresponding annuli $A_1, ..., A_n$ are all on the same connected component of $\partial\bar{M}$.

We will say that $\gamma \in \bar{\mathcal{S}}$ is *bad* if all except at most one of its separatrices are in the boundary of singular half Reeb components.

Lemma 4. *We may choose $T^2 \hookrightarrow M$, isotopic to a fibre and transverse to $\mathcal{F}^s$, in such a way that the foliation $\bar{\mathcal{F}}$ on $\bar{M}$ has no half Reeb components and no bad singular circles.*

Proof. Suppose that the foliation $\bar{\mathcal{F}}$, corresponding to a choice of $T^2 \hookrightarrow M$, has a bad singular circle $\gamma \in \bar{S}$, then we isotope T^2 to another torus $\tilde{T}^2$, again transverse to $\mathcal{F}^s$, as in the picture:

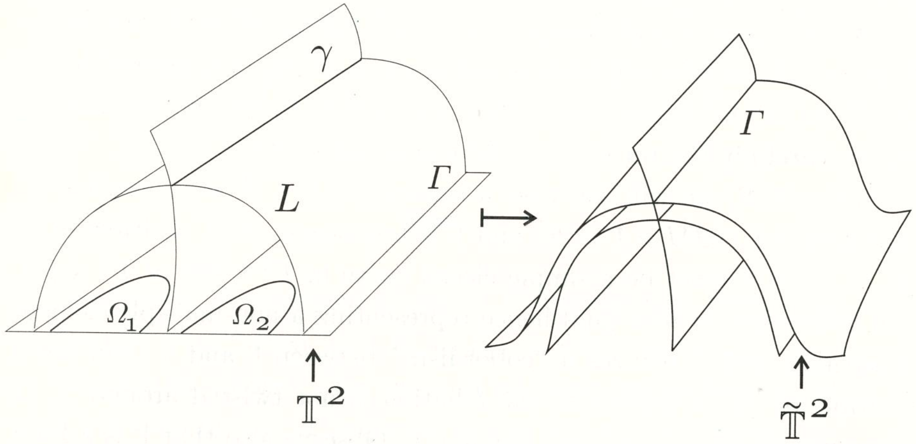


Figure 4

This is possible because $\mathcal{F}^s$ has contracting or expanding holonomy along γ. From the point of view of the foliation $\mathcal{G}_0$ or $\mathcal{G}_1$ induced by $\bar{\mathcal{F}}$ on $\partial\bar{M}$, this operation corresponds to the elimination of one or more limit cycles, bounding planar Reeb components:

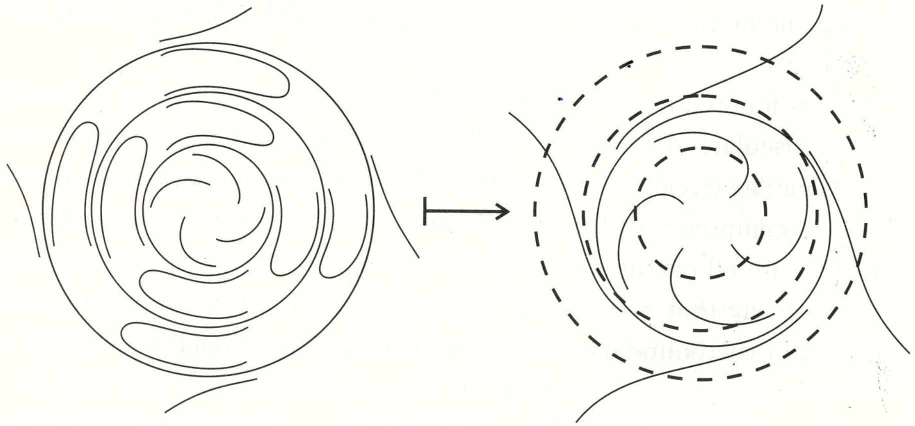
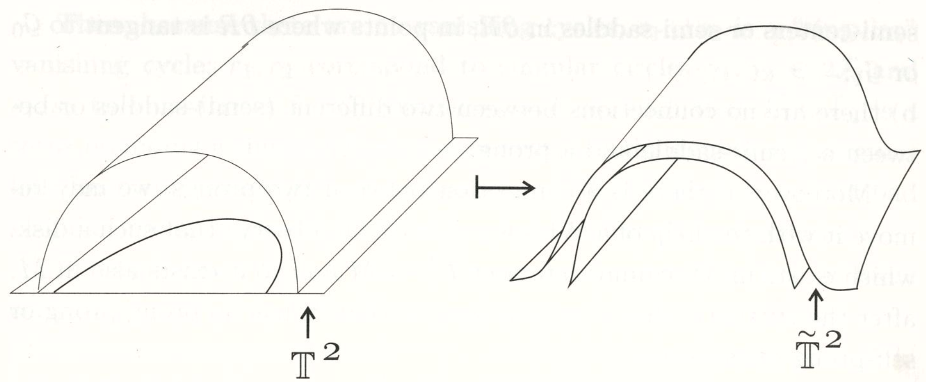


Figure 5

Similarly for the half Reeb components:



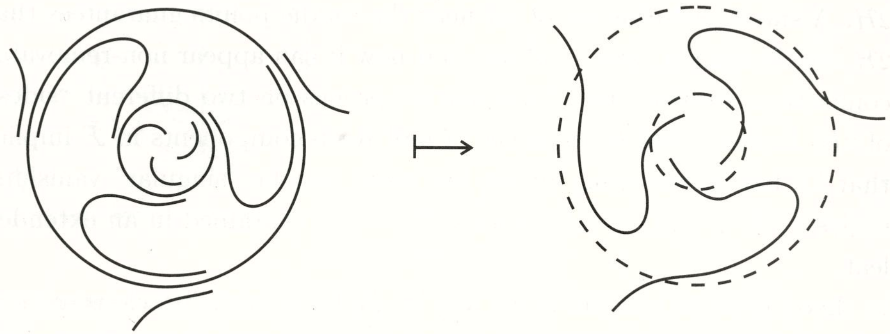


Figure 6

After such an isotopy the number of limit cycles of $\mathcal{G}_0$ or $\mathcal{G}_1$ without closed transversals (the only ones which can be in the boundary of planar Reeb components) strictly decreases. To this regard, observe that such an isotopy may produce new limit cycles, but these new cycles admit closed transversals (the axis of the old Reeb component). It follows that after a finite number of steps we have eliminated all the bad singular circles and all the half Reeb components. $\square$

Consider now $\bar{M}$, $\bar{\mathcal{F}}$ as in lemma 4. Let $R = S^1 \times [0,1]$ and let $R \hookrightarrow \bar{M}$ be an embedding with $S^1 \times \{0\} \subset T^2 \times \{0\}$, $S^1 \times \{1\} \subset T^2 \times \{1\}$. The general position argument ([Sol1]) ensures that $\mathcal{H} = \bar{\mathcal{F}}|_R$ satisfies the following properties:

a) $\mathcal{H}$ has a finite number of centers, saddles, prongs in $intR$, corresponding to tangencies of R with $\bar{\mathcal{F}}$ and transverse intersections with $\bar{S}$, and semi-centers or semi-saddles in ∂R, in points where ∂R is tangent to $\mathcal{G}_0$ or $\mathcal{G}_1$;

b) there are no connections between two different (semi)-saddles or between a (semi)-saddle and a prong.

Moreover, if there is a connection between two prongs, we may remove it with the help of a Whitney disk ([I-M]: observe that such a disk, which exists in M, cannot intersect $T^2 \hookrightarrow M$ and so it exists also in $\bar{M}$, after the cutting). Hence we will assume that $\mathcal{H}$ has no prong-prong or self-prong connections.

Now we pass to the double $2\bar{M} \simeq T^2 \times S^1$, equipped with the foliation with circle prong singularities $2\bar{\mathcal{F}}$ which induces on $2R \simeq T^2$ a foliation $2\mathcal{H}$. A small perturbation of $2R$ near the saddle points guarantees that $2\mathcal{H}$ satisfy again a) and b) above, but now it can appear non-removable connections between two "symmetric" prongs on two different "sides" of $2R$. Remark that the absence of half Reeb components in $\bar{\mathcal{F}}$ implies that $2\bar{\mathcal{F}}$ has no vanishing cycles, but there may be "singular" vanishing cycles $c : S^1 \times [0,1) \to 2\bar{M}$, with $c(S^1 \times \{0\})$ contained in an extended leaf.

Let $p \in 2R$ be a center of $2\mathcal{H}$ and let E_p be as in the previous section. Then ∂E_p is again formed by a saddle q together 1 or 2 separatrices, or by two "symmetric" prongs r_1, r_2 connected by two separatrices s_1, s_2:

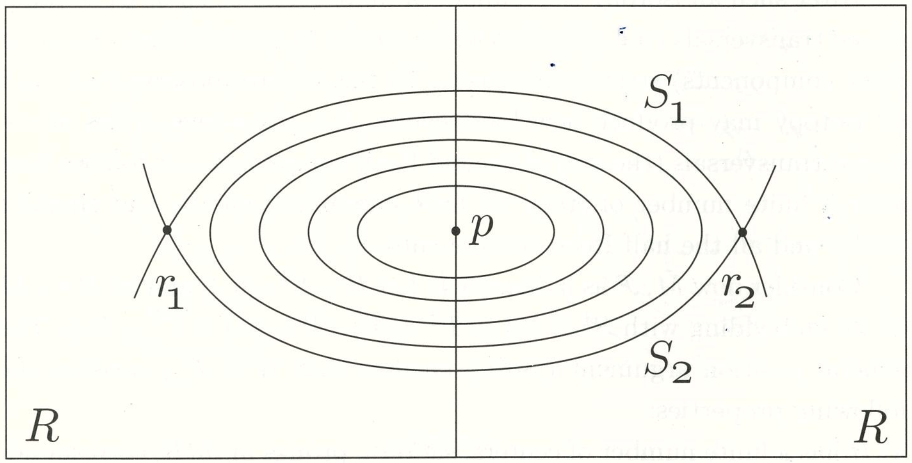


Figure 7

Then, because there are no vanishing cycles, $s_1 \cup s_2$ is a "singular" vanishing cycle; r_1, r_2 correspond to singular circles $\gamma_1, \gamma_2 \in 2\bar{\mathcal{F}}$ and s_1, s_2 to separatrices L_1, L_2 joining γ_1 and γ_2, such that $\overline{L_1} \cup \overline{L_2}$ is a torus bounding a singular Reeb component, i.e. the double of a singular half Reeb component (cfr. the appendix of [R-R]). The absence of bad singular circles implies that r_1 and r_2 have at least 4 separatrices.

It may happen that a pair of prongs is associated to two or more centers:

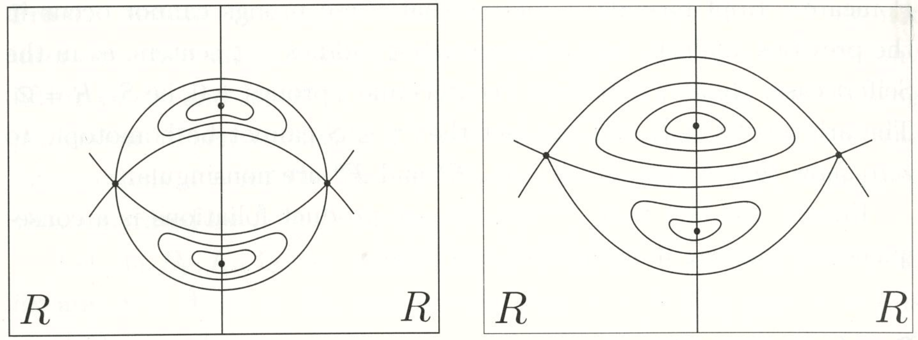


Figure 8

These centers must belong to the same connected component of $\partial\bar{M} \subset 2\bar{M}$, because a singular circle $\gamma \in \bar{\mathcal{F}}$ cannot have two separatrices reaching two different components of $\partial\bar{M}$. If k is the number of such centers, r_1 and r_2 have at least $k+3$ separatrices (because there are no bad singular circles), hence they (together) contribute to the Poincarè-Hopf index of $2\mathcal{H}$ with at most $-(k+1)$, while the contribute of the k centers is $+k$; so that the total contribute of r_1, r_2 and the k centers is at most -1.

It may happen also that a saddle q is associated to two centers p_1, p_2, as in the previous section. Then, as there, we have again a foliation by circles in an annulus around $\overline{E_{p_1}} \cup \overline{E_{p_2}}$, and this family of circles must die on a saddle or on a pair of symmetric prongs. In the second case, as before, the two prongs are connected by two separatrices which form a singular vanishing cycle, contained in the boundary of a singular half Reeb component; each prong has at least 4 separatrices. We shall say that the two prongs are associated to the triple (p_1, p_2, q).

A pair of prongs may be associated to several centers and several triples, but then the absence of bad singular circles implies that the total contribute to the Poincarè - Hopf index of the two prongs, the centers and the saddles and centers forming the triples will be at most -1 (a triple contribute as it were a center).

A repetition of these arguments (whose formalization is left to the reader) shows that the total index of $2\mathcal{H}$ is ≤ 0, and strictly less than 0 if some prongs is associated to some center (or triple, quintuple,...). From Poincarè - Hopf formula it follows that these prongs cannot occur in the previous analysis, and consequently $\sharp$ saddles $\geq \sharp$ centers, as in the Seifert case. Hence $\sharp$ saddles $= \sharp$ centers and $\sharp$ prongs $= 0$, i.e $\bar{\mathcal{S}} \cap R = \emptyset$. The arbitrarity of R and the fact that $\gamma \in \bar{\mathcal{S}}$ cannot be homotopic to zero show that $\bar{\mathcal{S}} = \emptyset$, i.e. $\mathcal{S} = \emptyset$, $\mathcal{F}^s$ and $\mathcal{F}^u$ are nonsingular.

Finally, the fact that $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ are product foliations is a consequence of theorem 3 or theorem 4 of [Mat].

Proof of proposition 2

We return to the general case of an expansive flow $\phi_t : M \to M$ on any 3-manifold, and assume that $\mathcal{F}^s, \mathcal{F}^u$ are without singularities and that the liftings $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u$ in $\tilde{M} = \mathbb{R}^3$ are product foliations.

It is sufficient to prove the transitivity of some finite covering of (M, ϕ_t), so we may assume that $\mathcal{F}^s$ and $\mathcal{F}^u$ are orientable and transversely orientable. A transverse orientation of $\mathcal{F}^u$ allows to distinguish between a "positive side" $\mathcal{F}^{s+}_x$ and a "negative side" $\mathcal{F}^{s-}_x$ of every leaf $\mathcal{F}^s_x \in \mathcal{F}^s$: $\mathcal{F}^s_x = \mathcal{F}^{s+}_x \cup [\phi_{\mathbb{R}}(x)] \cup \mathcal{F}^{s-}_x$. Similarly we decompose $\mathcal{F}^u_x = \mathcal{F}^{u+}_x \cup [\phi_{\mathbb{R}}(x)] \cup \mathcal{F}^{u-}_x$.

Define (cfr. [Ver]):

$$C_+ = \{x \in M | \mathcal{F}^{s+}_x \cap \mathcal{F}^u_x = \emptyset\}$$

as in [Ver], C_+ is a finite union of closed orbits of ϕ_t. If $x \in C_+$ and if $\tilde{x} \in \mathbb{R}^3$ is a lifting of x, then $\tilde{\mathcal{F}}^{s+}_{\tilde{x}}$ (obviously defined) does not intersects $\pi_1(M)(\tilde{\mathcal{F}}^u_{\tilde{x}})$; but any other point $\tilde{y} \in \tilde{\mathcal{F}}^u_{\tilde{x}}$ which is not in the orbit of $\tilde{\phi}_t$ through $\tilde{x}$ does not project to a closed orbit of ϕ_t, hence $\tilde{\mathcal{F}}^{s+}_{\tilde{y}}$

must intersects some $\alpha_{\tilde{y}}(\tilde{\mathcal{F}}^u_{\tilde{x}})$, for some $\alpha_{\tilde{y}} \in \pi_1(M)$. The fact that $\tilde{\mathcal{F}}^u$ is a product foliation (i.e. $\mathbb{R}^3/\tilde{\mathcal{F}}^u = \mathbb{R}$ is totally ordered) implies easily that if $\tilde{y}_1, \tilde{y}_2$ are two such points, then there exists $\alpha \in \pi_1(M)$ s.t. $\tilde{\mathcal{F}}^{s+}_{\tilde{y}_1}$ and $\tilde{\mathcal{F}}^{s+}_{\tilde{y}_2}$ both intersects $\alpha(\tilde{\mathcal{F}}^u_{\tilde{x}})$. Choosing $\tilde{y}_1 \in \tilde{\mathcal{F}}^{u+}_{\tilde{x}}$ and $\tilde{y}_2 \in \tilde{\mathcal{F}}^{u-}_{\tilde{x}}$ we see that also $\tilde{\mathcal{F}}^{s+}_{\tilde{x}}$ must intersects $\alpha(\tilde{\mathcal{F}}^u_{\tilde{x}})$, i.e. $x \notin C_+$. This means that $C_+ = \emptyset$. Similarly $C_- = \{x \in M | \mathcal{F}^{s-}_x \cap \mathcal{F}^u_x = \emptyset\}$ is empty.

Lemma 5. *Every leaf of $\mathcal{F}^s$ intersects every leaf of $\mathcal{F}^u$.*

Proof. Assume by contradiction that $L^s \in \mathcal{F}^s$ and $L^u \in \mathcal{F}^u$ do not intersect, then also L^s and $\overline{L^u}$ do not intersect. Let $\tilde{L}^s \in \tilde{\mathcal{F}}^s$ be any lifting of L^s and let $K = \pi^{-1}(\overline{L^u})$, $\pi : \mathbb{R}^3 \to M$ being the covering projection. Then $\tilde{L}^s$ divide $\mathbb{R}^3$ into two open subset O_1, O_2, and the intersections $A = K \cap O_1$, $B = K \cap O_2$ are two disjoint, closed, nonempty sets, separated by $\tilde{L}^s$ and saturated by $\tilde{\mathcal{F}}^u$.

Let A_0, B_0, K_0 be the projections of A, B, K to $V^u = \mathbb{R}^3/\tilde{\mathcal{F}}^u \simeq \mathbb{R}$, fix any $a \in A_0$, $b \in B_0$, and assume that $a < b$ (the case $a > b$ is similar), then define

$$a_0 = Max\{r \in A_0 | r < b\} \qquad b_0 = Min\{s \in B_0 | s > a_0\}$$

a_0 and b_0 are well defined, because A_0 and B_0 are closed and disjoint, moreover $a_0 < b_0$ and $(a_0, b_0) \cap K_0 = \emptyset$. On the other hand, the property $C_+ = \emptyset$ means that there exists a leaf $L_0 \in \tilde{\mathcal{F}}^s$ which intersects the leaf corresponding to a_0 and a leaf corresponding to some $k_0 \in K_0$, $k_0 > a_0$. Such a leaf L_0 must intersect also the leaf corresponding to b_0, and this contradicts the fact that $\tilde{L}^s$ separates A and B. □

To complete the proof of proposition 2 we observe that (thanks to the existence of stable and unstable foliations) Smale's spectral decomposition theorem holds also for expansive flows on 3-manifolds: the nonwandering set $\Omega(\phi_t)$ is the closure of the set of closed orbits and is a finite union of closed, invariant, transitive sets $\Omega_1, ..., \Omega_N$, pairwise disjoint. Then

$$M = \bigcup_{j=1}^{N} \bigcup_{x \in \Omega_j} \mathcal{F}^s_x = \bigcup_{j=1}^{N} \bigcup_{x \in \Omega_j} \mathcal{F}^u_x$$

and if $\Omega(\phi_t) \neq M$ there exist a source Ω_j $(\cup_{x\in\Omega_j}\mathcal{F}_x^s = \Omega_j)$ and a sink

$$\Omega_k(\cup_{x\in\Omega_k}\mathcal{F}_x^u = \Omega_k).$$

But lemma 5 guarantees that such a situation cannot arise. □

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