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On the Poles of Regular Differentials of Singular Curves

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Abstract. We describe the pole behaviour of the regular differentials of projective algebraic curves in terms of discrete invariants of the singular points.

Introduction

The regular differentials of a projective algebraic curve may have poles on its non-singular model. In this paper we describe their pole behaviour in terms of discrete invariants of the local rings (Section 2), and apply this to study Weierstrass points of singular curves (Section 3) and to analyze how the Hasse-Witt invariant and the zeta-function change under desingularization (Section 4). In Section 1 while introducing our notation we indicate – influenced by Roquette's analysis [**R2**] of Gorenstein's theorem – how Weil's approach [**W**] to the Riemann-Roch theorem for function fields generalizes nearly literally to curves with singularities.

1. Singular Curves

Let X be a complete irreducible algebraic curve with constant field k and let K be the field of rational functions on X. This means that K is a function field in one variable with the constant field k and that X is (the index set of) a set $\{\mathcal{O}_P\}_{P\in X}$ of local k-algebras, properly contained in K with quotient field K, satisfying the two properties:

i) For almost all $P \in X$, the local ring $\mathcal{O}_P$ is a discrete valuation ring.

ii) For each discrete valuation ring B of $K|k$ there is an unique $P \in X$

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such that $\mathcal{O}_P \subseteq B$.
(In the language of schemes, X has one more point namely its generic point whose local ring is the function field K.) The first condition means that the number of singular points of X is finite. By the second condition we have a morphism $\pi: \tilde{X} \to X$ where $\tilde{X}$ is the *non-singular model* of X defined to be the set of all discrete valuation rings of $K|k$. For each $P \in X$ the elements of the fiber $\pi^{-1}(P)$ are called the *branches of X centered at P*. By the extension theorem of valuation theory there is at least one branch centered at P. On the other hand the branches at P (are zeros of each rational function vanishing at P and so they) are finite in number.

By a *divisor* of X we mean (a *coherent fractional ideal sheaf* of X or equivalently) a formal product

$$\mathbf{a} = \prod_{P \in X} \mathbf{a}_P$$

where $\mathbf{a}_P$ is a (non-zero fractional) ideal of $\mathcal{O}_P$ for each $P \in X$ and $\mathbf{a}_P = \mathcal{O}_P$ for almost all P. A divisor $\mathbf{a}$ is called *locally principal* (or a *Cartier divisor*) if each component $\mathbf{a}_P$ is a principal ideal. For two divisors $\mathbf{a}$ and $\mathbf{b}$ we define the *product* $\mathbf{a} \cdot \mathbf{b}$ and the *quotient* $\mathbf{a}\colon\mathbf{b}$ by setting:

$(\mathbf{a} \cdot \mathbf{b})_P := \mathbf{a}_P \cdot \mathbf{b}_P = \mathcal{O}_P$-ideal generated by the products ab with $a \in \mathbf{a}_P$ and $b \in \mathbf{b}_P$

$(\mathbf{a}\colon\mathbf{b})_P := \mathbf{a}_P\colon\mathbf{b}_P = \{z \in K | z\mathbf{b}_P \subseteq \mathbf{a}_P\}$

Note that the locally principal divisors form a multiplicative group whose neutral element is the *structure divisor*

$$\mathcal{O} := \prod_{P \in X} \mathcal{O}_P.$$

We define

$$\mathbf{a} \geq \mathbf{b} :\iff \mathbf{a}_P \supseteq \mathbf{b}_P \quad \text{for each} \quad P \in X$$

and call a divisor $\mathbf{a}$ *positive* when $\mathbf{a} \geq \mathcal{O}$. It is common in the literature but it would be inconvenient in our approach, to invert the ordering and consider as positive the divisors $\mathbf{a}$ with $\mathbf{a}_P \subseteq \mathcal{O}_P$ for each P (which

correspond bijectively to the *zero-dimensional closed subschemes* of X). The *degree* of a divisor is defined by the properties $\deg(\mathcal{O}) = 0$ and

$$\deg(\mathbf{a}) - \deg(\mathbf{b}) = \sum_{P \in X} \dim \mathbf{a}_P/\mathbf{b}_P \quad \text{whenever} \quad \mathbf{a} \geq \mathbf{b}.$$

For each non-zero rational function $z \in K^*$ we define the *principal divisor*:

$$\operatorname{div}(z) := \prod_{P \in X} z^{-1}\mathcal{O}_P$$

Let

$$L(\mathbf{a}) := \bigcap_{P \in X} \mathbf{a}_P = \{z \in K | \operatorname{div}(z) \cdot \mathbf{a} \geq \mathcal{O}\}$$

be the *k-vector space of global sections* of $\mathbf{a}$ (also denoted by $H^0(X, \mathbf{a})$).

To get the link with Rosenlicht's paper [**R**] we assign to each element $\Sigma n_P P$ of the free abelian group generated by the non-singular points of X the locally principal divisor whose P-component is equal to $\mathbf{m}_P^{-n_P}$ (respectively, $\mathcal{O}_P$) when P is non-singular (respectively, singular), where $\mathbf{m}_P$ denotes the maximal ideal of $\mathcal{O}_P$.

Since K is a function field in one variable with constant field k, each integral k-algebra with quotient field K has finite k-codimension in its integral closure $\tilde{A}$ (cf. [**R**, Theorem 1], [**R1**, Satz 3]). The integer

$$\delta_P := \dim \tilde{\mathcal{O}}_P/\mathcal{O}_P$$

is called the *singularity degree* of P. Denoting by $Q_1, \ldots, Q_m \in \tilde{X}$ the branches centered at P, the integral closure

$$\tilde{\mathcal{O}}_P = \mathcal{O}_{Q_1} \cap \cdots \cap \mathcal{O}_{Q_m}$$

of $\mathcal{O}_P$ is a principal ideal domain whose maximal ideals correspond bijectively to the branches $Q_1, \ldots, Q_m$. Thus the divisors of the non-singular model $\tilde{X}$ correspond bijectively to the $\tilde{\mathcal{O}}$-divisors of X defined as the divisors whose P-components are $\tilde{\mathcal{O}}_P$-ideals. The structure divisor of $\tilde{X}$ corresponds to the $\tilde{\mathcal{O}}$-divisor

$$\tilde{\mathcal{O}} := \prod_{P \in X} \tilde{\mathcal{O}}_P.$$

Since $\delta_P < \infty$, by the Artin-Rees lemma the topology of $\mathcal{O}_P$ is induced by the topology of $\tilde{\mathcal{O}}_P$, and so the completion $\hat{\mathcal{O}}_P$ is a closed subring of $\hat{\tilde{\mathcal{O}}}_P$ of codimension δ_P. By applying the chinese remainder theorem to the residue rings of $\tilde{\mathcal{O}}_P$ and passing to the projective limit one obtains:

$$\hat{\tilde{\mathcal{O}}}_P = \hat{\mathcal{O}}_{Q_1} \times \cdots \times \hat{\mathcal{O}}_{Q_m}.$$

Thus the completion $\hat{\mathbf{a}}_P$ of the P-component $\mathbf{a}_P$ of a divisor $\mathbf{a}$ is contained in the product $\hat{K}_{Q_1} \times \cdots \times \hat{K}_{Q_m}$, and thus the *paralleletope* of $\mathbf{a}$, that is, the cartesian product

$$\Lambda(\mathbf{a}) := \prod_{P \in X} \hat{\mathbf{a}}_P$$

is contained in the *k-algebra $A = A_{K|k}$ of adeles of $K|k$* defined to be the restricted product of the local fields $\hat{K}_Q$ of the branches $Q \in \tilde{X}$. The two dimensions

$$\ell(\mathbf{a}) := \dim L(\mathbf{a}) = \dim \Lambda(\mathbf{a}) \cap K$$

and

$$i(\mathbf{a}) := \dim A/(\Lambda(\mathbf{a}) + K)$$

are finite. In fact, since $\mathbf{a}\colon \tilde{\mathcal{O}} \leq \mathbf{a} \leq \mathbf{a} \cdot \tilde{\mathcal{O}}$ we can assume that $\mathbf{a}$ is an $\tilde{\mathcal{O}}$-divisor and thus corresponds to a divisor $\mathcal{A}$ of the non-singular model $\tilde{X}$, hence $\Lambda(\mathbf{a}) = \Lambda(\mathcal{A}), \ell(\mathbf{a}) = \ell(\mathcal{A})$ and $i(\mathbf{a}) = i(\mathcal{A})$, and we can apply the non-singular case (cf. e.g. [**L**, Ch. I, §2]). Since we have an exact sequence

$$0 \to L(\mathbf{a})/L(\mathbf{b}) \to \Lambda(\mathbf{a})/\Lambda(\mathbf{b}) \to (\Lambda(\mathbf{a}) + K)/(\Lambda(\mathbf{b}) + K) \to 0$$

whenever $\mathbf{a} \geq \mathbf{b}$, we conclude that $\ell(\mathbf{a}) - \deg(\mathbf{a}) - i(\mathbf{a})$ does not depend on the divisor $\mathbf{a}$ and so we obtain (cf. [**G1**, Theorem 5.4]):

(1.1) Riemann-Roch theorem for singular curves. *Each divisor* $\mathbf{a}$ *of* X *satisfies*

$$\ell(\mathbf{a}) = \deg(\mathbf{a}) + 1 - g + i(\mathbf{a})$$

where $g := i(\mathcal{O})$ is called the arithmetic genus of X.

In particular the degree of $\mathbf{a}$ only depends on the *linear equivalence* class $\{\operatorname{div}(z) \cdot \mathbf{a} | z \in K^*\}$ of the divisor $\mathbf{a}$, and so the *product formula* extends to the singular case:

$$\deg \operatorname{div}(z) = 0 \quad \text{for each} \quad z \in K^*. \tag{1.2}$$

In particular $L(\mathbf{a}) = 0$ whenever $\deg(\mathbf{a}) < 0$. If $\mathbf{a}$ is an $\tilde{\mathcal{O}}$-divisor of X corresponding to the divisor $\mathcal{A}$ of $\tilde{X}$ then

$$\deg(\mathbf{a}) = \deg(\mathcal{A}) + \deg(\tilde{\mathcal{O}}) = \deg(\mathcal{A}) + \sum_{P \in X} \delta_P$$

and so by the Riemann-Roch theorem we get the *genus formula*

$$g = \tilde{g} + \sum_{P \in X} \delta_P \tag{1.3}$$

(cf. Hironaka [**H**, Theorem 2]) where $\tilde{g}$ is the *geometric genus* of X defined to be the genus of the non-singular model $\tilde{X}$.

By a (*Weil*) *differential* of X we mean a k-linear functional $A_{K|k} \to k$ vanishing on $\Lambda(\mathbf{a}) + K$ for some divisor $\mathbf{a}$ of X. Since $\Lambda(\mathbf{a}\colon \tilde{\mathcal{O}}) \subseteq \Lambda(\mathbf{a}) \subseteq \Lambda(\mathbf{a} \cdot \tilde{\mathcal{O}})$ this notion only depends on the non-singular model $\tilde{X}$. Note that $i(\mathbf{a}) = \dim\ \Omega(\mathbf{a})$ where $\Omega(\mathbf{a})$ stands for the k-vector space of all differentials vanishing on $\Lambda(\mathbf{a})$.

Let λ be a non-zero differential say $\lambda \in \Omega(\mathbf{a}) \setminus \{0\}$ for some divisor $\mathbf{a}$. As in the non-singular case one proves *Riemann's theorem* which says that $\deg(\mathbf{a}) \leq 2g - 2$. (In fact, if $\mathbf{b}$ is a locally principal divisor with $\deg(\mathbf{b}) > \deg(\mathbf{a})$, then $\deg(\mathbf{a}\colon \mathbf{b}) = \deg(\mathbf{a}) - \deg(\mathbf{b}) < 0$ and therefore $\ell(\mathbf{a}\colon \mathbf{b}) = 0$, and if $z_1, \ldots, z_n$ form a basis of $L(\mathbf{b})$ then $z_1\lambda, \ldots, z_n\lambda \in \Omega(\mathbf{a}\colon \mathbf{b})$ are linearly independent and therefore $\ell(\mathbf{b}) \leq i(\mathbf{a}\colon \mathbf{b})$, and by applying the Riemann-Roch theorem to the divisors $\mathbf{b}$ and $\mathbf{a}\colon \mathbf{b}$ we obtain $\deg(\mathbf{a}) \leq 2g - 2$.) Thus among the paralleletopes where λ vanishes there is a largest one, say $\Lambda(\mathbf{c})$. We put $\operatorname{div}(\lambda) := \mathbf{c}$. Note that the divisor of λ on the non-singular model $\tilde{X}$ corresponds to the divisor $\mathbf{c}\colon \tilde{\mathcal{O}}$ of X which is the largest $\tilde{\mathcal{O}}$-divisor smaller than or equal to $\mathbf{c}$.

Since the space $\Omega_{K|k}$ of differentials is a one-dimensional vector space over the function field K (cf. [**L**, Theorem 6]), the class of the divisor $\mathbf{c} = \operatorname{div}(\lambda)$ does not depend on the choice of the non-zero differential λ, and it is called the *canonical class*. Moreover we deduce $\Omega(\mathbf{a}) = L(\mathbf{c}\colon \mathbf{a})\lambda$ and therefore

$$i(\mathbf{a}) = \ell(\mathbf{c}\colon \mathbf{a}).$$

In particular we get

$$\dim \Omega(\mathcal{O}) = \ell(\mathbf{c}) = g.$$

Now by applying the Riemann-Roch theorem to the divisor $\mathbf{c}$ we obtain

$$\deg(\mathbf{c}) = 2g - 2.$$

(We will see in a moment that $\mathbf{c}\colon\mathbf{c} = \mathcal{O}$, but it is enough to observe $\mathcal{O} \leq \mathbf{a}\colon\mathbf{a} \leq \tilde{\mathcal{O}}$ in order to deduce $\ell(\mathbf{a}\colon\mathbf{a}) = 1$ for each divisor $\mathbf{a}$.) Conversely, each divisor $\mathbf{c}$ satisfying $\ell(\mathbf{c}) \geq g$ and $\deg(\mathbf{c}) = 2g - 2$ is a canonical divisor.

The *P-component* λ_P of the differential λ is defined to be the composition homomorphism:

$$\lambda_P\colon K \hookrightarrow \bigoplus_{i=1}^{m} \hat{K}_{Q_i} \hookrightarrow A_{K|k} \xrightarrow{\lambda} k.$$

Since the P-component $\mathbf{c}_P$ of $\mathbf{c} = \operatorname{div}(\lambda)$ is the largest $\mathcal{O}_P$-ideal on which completion $\hat{\mathbf{c}}_P$ the homomorphism $\oplus_{i=1}^{m} \hat{K}_{Q_i} \to A_{K|k} \to k$ vanishes, and since by the approximation theorem the field K is dense in $\oplus_{i=1}^{m} \hat{K}_{Q_i}$, we conclude that $\mathbf{c}_P$ is the largest $\mathcal{O}_P$-ideal where λ_P vanishes, or equivalently $\mathbf{c}_P = \{z \in K | \lambda_P(\mathcal{O}_P z) = 0\}$. More generally, for each $\mathcal{O}_P$-ideal $\mathbf{a}_P$ we have:

$$\mathbf{c}_P\colon\mathbf{a}_P = \{z \in K | \lambda_P(\mathbf{a}_P z) = 0\}. \tag{1.4}$$

(In fact, λ_P vanishes on $\mathbf{a}_P z = \mathcal{O}_P \mathbf{a}_P z$ if and only if $\mathbf{a}_P z \subseteq \mathbf{c}_P$ that is $z \in \mathbf{c}_P\colon\mathbf{a}_P$.) For each $P \in X$ we define the $\mathcal{O}_P$-module

$$\Omega(\mathbf{a}_P) := \{\mu \in \Omega_{K|k} | \operatorname{div}(\mu)_P \supseteq \mathbf{a}_P\} = \{\mu | \mu_P(\mathbf{a}_P) = 0\}.$$

Note that

$$\Omega(\mathbf{a}_P) = (\mathbf{c}_P\colon\mathbf{a}_P)\lambda$$

and

$$\Omega(\mathbf{a}) = \bigcap_{P \in X} \Omega(\mathbf{a}_P) = L(\mathbf{c}\colon\mathbf{a})\lambda.$$

Theorem 1.5. (**Local Duality.**) *If $\mathbf{a}_P \supseteq \mathbf{b}_P$ then there is an isomorphism of k-vector spaces*

$$\Omega(\mathbf{b}_P)/\Omega(\mathbf{a}_P) \xrightarrow{\sim} \operatorname{Hom}_k(\mathbf{a}_P/\mathbf{b}_P, k)$$

defined by $\mu + \Omega(\mathbf{a}_P) \mapsto (a + \mathbf{b}_P \mapsto \mu_P(a))$. Equivalently, there is a k-isomorphism

$$(\mathbf{c}_P\colon\mathbf{b}_P)/(\mathbf{c}_P\colon\mathbf{a}_P) \xrightarrow{\sim} \operatorname{Hom}_k(\mathbf{a}_P/\mathbf{b}_P, k)$$

defined by $\overline{c} \mapsto (\overline{a} \mapsto \lambda_P(ac))$.

Proof. By (1.4) we have an injective homomorphism $K/(\mathbf{c}_P\colon\mathbf{a}_P) \hookrightarrow \operatorname{Hom}_k(\mathbf{a}_P, k)$ and in particular $(\mathbf{c}_P\colon\mathbf{b}_P)/(\mathbf{c}_P\colon\mathbf{a}_P) \hookrightarrow \operatorname{Hom}_k(\mathbf{a}_P/\mathbf{b}_P, k)$. Thus $\dim(\mathbf{c}_P\colon\mathbf{b}_P)/(\mathbf{c}_P\colon\mathbf{a}_P) \leq \dim \mathbf{a}_P/\mathbf{b}_P$, and it is enough to prove the equality of the dimensions. Since $\mathbf{b}_P\colon\tilde{\mathcal{O}}_P \subseteq \mathbf{b}_P \subseteq \mathbf{a}_P \subseteq \mathbf{a}_P \cdot \tilde{\mathcal{O}}_P$, we can assume that $\mathbf{a}_P$ and $\mathbf{b}_P$ are $\tilde{\mathcal{O}}_P$-ideals. Then $\mathbf{c}_P\colon\mathbf{a}_P = \mathbf{d}_P\colon\mathbf{a}_P$ and $\mathbf{c}_P\colon\mathbf{b}_P = \mathbf{d}_P\colon\mathbf{b}_P$ where $\mathbf{d}_P := \mathbf{c}_P\colon\tilde{\mathcal{O}}_P$ is also an $\tilde{\mathcal{O}}_P$-ideal. Since $\tilde{\mathcal{O}}_P$ is a principal ideal domain, $(\mathbf{d}_P\colon\mathbf{b}_P)/(\mathbf{d}_P\colon\mathbf{a}_P)$ is isomorphic to $\mathbf{a}_P/\mathbf{b}_P$. □

It follows from the local duality that $\deg_P(\mathbf{a}_P) + \deg_P(\mathbf{c}_P\colon\mathbf{a}_P)$ does not depend on the ideal $\mathbf{a}_P$ and therefore

$$\deg_P(\mathbf{c}_P\colon\mathbf{a}_P) = \deg_P(\mathbf{c}_P) - \deg_P(\mathbf{a}_P)$$

where the *local degree function* $\deg_P$ is defined by the properties $\deg_P(\mathcal{O}_P) = 0$ and $\deg_P(\mathbf{a}_P) - \deg_P(\mathbf{b}_P) = \dim \mathbf{a}_P/\mathbf{b}_P$ whenever $\mathbf{a}_P \supseteq \mathbf{b}_P$. By summing up we obtain for each divisor $\mathbf{a}_P$ of X the formula:

$$\deg(\mathbf{c}\colon\mathbf{a}) = \deg(\mathbf{c}) - \deg(\mathbf{a}). \tag{1.6}$$

In particular $\deg(\mathbf{c}\colon(\mathbf{c}\colon\mathbf{a})) = \deg(\mathbf{a})$, and since $\mathbf{c}\colon(\mathbf{c}\colon\mathbf{a}) \geq \mathbf{a}$ we deduce:

(1.7) Reciprocity. $\mathbf{c}\colon(\mathbf{c}\colon\mathbf{a}) = \mathbf{a}$.

In the special case where $\mathbf{a} = \mathcal{O}$ we obtain:

$$\mathbf{c}\colon\mathbf{c} = \mathcal{O}. \tag{1.8}$$

By the reciprocity, the assignment

$$\mathbf{a}_P \longmapsto \mathbf{c}_P\colon\mathbf{a}_P$$

defines an anti-monotonous permutation between the $\mathcal{O}_P$-ideals. In particular we deduce:

$$\mathbf{c}_P\colon(\mathbf{a}_P + \mathbf{b}_P) = (\mathbf{c}_P\colon\mathbf{a}_P) \cap (\mathbf{c}_P\colon\mathbf{b}_P)$$
$$\mathbf{c}_P\colon(\mathbf{a}_P \cap \mathbf{b}_P) = (\mathbf{c}_P\colon\mathbf{a}_P) + (\mathbf{c}_P\colon\mathbf{b}_P).$$

It now follows from [**HK**, Satz 2.8] that a divisor $\mathbf{d}$ satisfies $\mathbf{d}\colon(\mathbf{d}\colon\mathbf{a}) = \mathbf{a}$ for each divisor $\mathbf{a}$ if and only if $\mathbf{d} = \mathbf{b}\cdot\mathbf{c}$ for some locally principal divisor $\mathbf{b}$.

Theorem 1.9. *For any two divisors* $\mathbf{a}$ *and* $\mathbf{b}$ *of* X, *for each* $P \in X$ *and each function* $z \in K$ *we have:*

$$z\Omega(\mathbf{a}_P) \subseteq \Omega(\mathbf{b}_P) \Longleftrightarrow z\mathbf{b}_P \subseteq \mathbf{a}_P$$

If $\mathbf{b}_P \supseteq \mathbf{a}_P$ and $z \in \mathcal{O}_P$ then this follows by noting that by the local duality (1.5) the $\mathcal{O}_P$-modules $\mathbf{b}_P/\mathbf{a}_P$ and $\Omega(\mathbf{a}_P)/\Omega(\mathbf{b}_P)$ have the same annihilator (cf. Serre [**S2**, Ch. IV, §3.11]). The theorem simply says that

$$(\mathbf{c}\colon\mathbf{b})\colon(\mathbf{c}\colon\mathbf{a}) = \mathbf{a}\colon\mathbf{b}.$$

Since $(\mathbf{c}\colon\mathbf{b})\colon(\mathbf{c}\colon\mathbf{a}) \geq \mathbf{a}\colon\mathbf{b}$ and $(\mathbf{c}\colon\mathbf{a}')\colon(\mathbf{c}\colon\mathbf{b}') \geq \mathbf{b}'\colon\mathbf{a}'$ where $\mathbf{a}' := \mathbf{c}\colon\mathbf{a}$ and $\mathbf{b}' := \mathbf{c}\colon\mathbf{b}$, the theorem follows from the reciprocity (1.7). Taking $\mathbf{b} = \mathcal{O}$ we even see that it is equivalent to the reciprocity.

2. The pole behaviour of the regular differentials

A differential $\mu \in \Omega_{K|k}$ is called *regular* at a point $P \in X$ when $\mu \in \Omega(\mathcal{O}_P)$ that is $\mu_P(\mathcal{O}_P) = 0$ or equivalently $\operatorname{div}(\mu)_P \supseteq \mathcal{O}_P$. Note that

$$\Omega(\mathcal{O}_P) \supseteq \Omega(\tilde{\mathcal{O}}_P)$$

where $\Omega(\tilde{\mathcal{O}}_P)$ is the space of the differentials regular at the branches centered in P. It may happen that a regular differential on X has poles at branches centered in singular points. In order to analyze the pole behaviour, we study the space $\Omega(\mathcal{O})$ of regular differentials on X modulo the space $\Omega(\tilde{\mathcal{O}})$ of regular differentials on $\tilde{X}$.

Theorem 2.1. *There is a canonical isomorphism*

$$\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}}) \xrightarrow{\sim} \bigoplus \Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)$$

where P *varies over the singular points of* X.

Thus it is enough to study the pole behaviour of the differentials $\mu \in \Omega(\mathcal{O}_P)$ at the branches centered at P. To prove the theorem

we simply observe that the inclusions $\Omega(\mathcal{O}) \subseteq \Omega(\mathcal{O}_P)$ induce an injective homomorphism of $\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})$ into the direct sum of the quotients $\Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)$, and that by the Riemann-Roch theorem the dimension of $\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})$ is equal to $g - \tilde{g} = \sum \delta_P$, and so it remains to note that

$$\dim \Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P) = \delta_P.$$

More generally, by the local duality (1.5) we have an isomorphism:

$$\Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P) \cong \operatorname{Hom}_k(\tilde{\mathcal{O}}_P/\mathcal{O}_P, k)$$

We denote by $\mathbf{f}$ the *conductor divisor* of X, that is, the largest $\tilde{\mathcal{O}}$-divisor of X smaller than or equal to the structure divisor $\mathcal{O}$. Note that

$$\Omega(\mathcal{O}_P) \subseteq \Omega(\mathbf{f}_P) = (\tilde{\mathcal{O}}_P\colon\mathbf{f}_P)\Omega(\tilde{\mathcal{O}}_P).$$

Thus the pole orders of the regular differentials of X are not larger than the corresponding exponents of the conductor. Since $\mathbf{f} = \mathcal{O}\colon\tilde{\mathcal{O}}$, by Theorem 1.9 we have

$$\{z \in K | z\Omega(\mathcal{O}_P) \subseteq \Omega(\tilde{\mathcal{O}}_P)\} = \mathbf{f}_P$$

or equivalently $(\mathbf{c}\colon\tilde{\mathcal{O}})\colon\mathbf{c} = \mathbf{f}$ where, as in Section 1, $\mathbf{c}$ is the divisor of some non-zero differential λ. From now on we will assume in this section that the constant field k is infinite.

Proposition 2.2. $\Omega(\tilde{\mathcal{O}}_P) = \mathbf{f}_P \cdot \Omega(\mathcal{O}_P)$. *This means that* $\mathbf{c}\colon\tilde{\mathcal{O}} = \mathbf{f}\cdot\mathbf{c}$, *and holds trivially when the ideals* $\mathbf{c}_P$ *are principal.*

Proof. For each branch Q centered at P let c_Q be an element of $\mathbf{c}_P$ such that and $\operatorname{ord}_Q(c_Q) = \min\{\operatorname{ord}_Q(c) \mid c \in \mathbf{c}_P\}$. Since the constant field k is infinite, by taking a suitable linear combination of the functions c_Q we obtain an element $z = z_P \in \mathbf{c}_P$ such that $\operatorname{ord}_Q(z) = \operatorname{ord}_Q(c_Q)$ for each Q that is $\tilde{\mathcal{O}}_P \cdot \mathbf{c}_P = z\tilde{\mathcal{O}}_P$ or equivalently

$$z\mathcal{O}_P \subseteq \mathbf{c}_P \subseteq z\tilde{\mathcal{O}}_P.$$

Since as just observed $(\mathbf{c}\colon\tilde{\mathcal{O}})\colon\mathbf{c} = \mathbf{f}$ and since $(\mathbf{a}\colon\tilde{\mathcal{O}})\colon\mathbf{a} = \mathbf{a}\colon(\mathbf{a}\cdot\tilde{\mathcal{O}})$ for each divisor $\mathbf{a}$ we have

$$\mathbf{c}\colon(\mathbf{c}\cdot\tilde{\mathcal{O}}) = \mathbf{f}$$

and therefore $\mathbf{c}_P\colon\tilde{\mathcal{O}}_P = z\mathbf{f}_P = \mathbf{c}_P\cdot\mathbf{f}_P$. □

Theorem 2.3. (**Rosenlicht [R, Theorem 10].**) *We have* $\dim \mathcal{O}_P/\mathbf{f}_P \leq \delta_P$, *and equality holds if and only if the* $\mathcal{O}_P$*-module* $\Omega(\mathcal{O}_P)$ *is free (of rank 1) or equivalently the ideal* $\mathbf{c}_P$ *is principal.*

Proof. We keep the notation of the preceding proof. Since $z\mathcal{O}_P \subseteq \mathbf{c}_P \subseteq z\tilde{\mathcal{O}}_P$ and since $\mathcal{O}_P$ is the only principal ideal between $\mathcal{O}_P$ and its integral closure $\tilde{\mathcal{O}}_P$, the ideal $\mathbf{c}_P$ is principal if and only if it is equal to $z\mathcal{O}_P$. Since

$$\mathcal{O}_P/\mathbf{f}_P \cong z\mathcal{O}_P/z\mathbf{f}_P \subseteq \mathbf{c}_P/z\mathbf{f}_P = \mathbf{c}_P/(\mathbf{c}_P : \tilde{\mathcal{O}}_P) \cong \Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)$$

we have $\dim \mathcal{O}_P/\mathbf{f}_P \leq \delta_P$, and equality holds if and only if $\mathbf{c}_P = z\mathcal{O}_P$.□

The one-dimensional local ring $\mathcal{O}_P$ is called a *Gorenstein ring* when $\dim \mathcal{O}_P/\mathbf{f}_P = \delta_P$.

Corollary 2.4. *The curve* X *is a Gorenstein curve (that is, all its local rings are Gorenstein rings) if and only if its canonical divisors are locally principal.*

For the remainder of this section we will assume that the constant field k is algebraically closed. As usual by the residue formula the (Weil) differentials are identified with the *differential forms*. Thus the P-component of a differential form μ of $K|k$ is given by

$$\mu_P(z) = \sum_{i=1}^{m} \operatorname{res}_{Q_i}(z\mu) \quad \text{for each} \quad z \in K$$

where $Q_1, \dots, Q_m \in \tilde{X}$ are the branches centered at P.

For each $i = 1, \dots, m$ let $\mathbf{p}_i$ be the maximal ideal of the principal ideal domain $\tilde{\mathcal{O}}_P = \mathcal{O}_{Q_1} \cap \dots \cap \mathcal{O}_{Q_m}$ corresponding to the branch Q_i, that is, $\mathbf{p}_i := \{z \in \tilde{\mathcal{O}}_P | \operatorname{ord}_{Q_i}(z) \geq 1\}$. We write

$$\mathbf{f}_P = \mathbf{p}_1^{f_1} \cdot \dots \cdot \mathbf{p}_m^{f_m}$$

where the exponents $f_1, \dots, f_m$ are non-negative integers. In order to study the pole behaviour of the regular differentials we consider the set

$$R := \{(\operatorname{ord}_{Q_1}(\mu), \dots, \operatorname{ord}_{Q_m}(\mu)) | \mu \in \Omega(\mathcal{O}) \setminus \{0\}\} \subset \mathbb{Z}^m.$$

This set is finite, because for each $(r_1, \dots, r_m) \in R$ we have

$$r_i \geq -f_i \quad (i = 1, \dots, m)$$

and $r_1 + \dots + r_m \leq 2g - 2 + \sum_{Q \in X \setminus \{P\}} \dim \tilde{\mathcal{O}}_Q/\mathbf{f}_Q$. The set R is non-empty whenever $g > 0$ or equivalently when X is not isomorphic to the projective line. Since for each $i = 1, \dots, m$ the function $\operatorname{ord}_{Q_i} : K \to \mathbb{Z} \cup \{\infty\}$ is a valuation with the infinite residue field k and since $\Omega(\mathcal{O})$ is a vector space over k, the set R satisfies the two properties:

(2.5) If $\mathbf{r} = (r_1, \dots, r_m)$ and $\mathbf{s} = (s_1, \dots, s_m)$ are elements of R then the vector with the coordinates $\min\{r_i, s_i\}$ also belongs to R.

(2.6) If $\mathbf{r}, \mathbf{s} \in R$ and $r_i = s_i$ for some i, then there is a vector $\mathbf{t} \in R$ such that $t_i > r_i$, $t_j \geq \min\{r_j, s_j\}$ for each j and $t_j = \min\{r_j, s_j\}$ whenever $r_j \neq s_j$.

We compare the set R with the additive semigroup

$$S_P := \{(\operatorname{ord}_{Q_1}(z), \dots, \operatorname{ord}_{Q_m}(z)) | z \in \mathcal{O}_P \setminus \{0\}\} \subseteq \mathbb{N}^m$$

which has been studied by several authors (cf. [**Wa**], [**G**], [**B**], [**D1**], [**D2**] and [**GL**]). Since the conductor $\mathbf{f}_P$ is the largest $\tilde{\mathcal{O}}_P$-ideal contained in $\mathcal{O}_P$, the vector $(f_1, \dots, f_m)$ is the smallest vector (with respect to the product ordering of $\mathbb{N}^m$) such that

$$(f_1, \dots, f_m) + \mathbb{N}^m \subseteq S_P.$$

Since S_P also satisfies properties of type (2.5) and (2.6), one deduces (cf. Garcia [**G**]):

(2.7) A vector $(s_1, \dots, s_m) \in \mathbb{N}^m$ belongs to S_P if and only if the vector with the coordinates $\min\{s_i, f_i\}$ belongs to S_P.

In particular, it follows from the minimality of the conductor vector $(f_1, \dots, f_m)$ that the vectors $(r_1, \dots, r_m)$ with $r_i = f_i - 1$ for some i and $r_j \geq f_j$ for each $j \neq i$ do not belong to S_P. Note that the zero-vector $(0, \dots, 0)$ is the only point of S_P having at least one coordinate equal to zero. If P is a singular point of X then $f_i > 0$ for each $i = 1, \dots, m$.

Theorem 2.8. *Let* $(n_1, \dots, n_m)$ *be a vector of non-negative integers. There is a regular differential* μ *on* X *such that*

$$\operatorname{ord}_{Q_i}(\mu) = -n_i \quad \textit{when} \quad n_i > 0$$
$$\operatorname{ord}_{Q_i}(\mu) \geq 0 \quad \textit{when} \quad n_i = 0$$

if and only if for each $i = 1,\dots,m$ the vectors $(r_1,\dots r_m)$ with $r_i = n_i - 1$ and $r_j \geq n_j$ for each $j \neq i$ do not belong to the semigroup S_P associated to the point P.

Furthermore, by Theorem 2.1 we can impose that $\operatorname{ord}_Q(\mu) \geq 0$ for each $Q \in \tilde{X} \setminus \{Q_1,\dots,Q_m\}$.

Corollary 2.9. *If $g > 0$ then there is a regular differential μ on X such that*

$$\operatorname{ord}_{Q_i}(\mu) = -f_i \quad \text{for each} \quad i = 1,\dots,m$$

and $\operatorname{ord}_Q(\mu) \geq 0$ for each $Q \in \tilde{X} \setminus \{Q_1,\dots,Q_m\}$.

Corollary 2.10. *Let $i \in \{1,\dots,m\}$ and let n be a positive integer. There is a regular differential on X having at Q_i a pole of order n if and only if the vector $(f_1,\dots,f_{i-1},n-1,f_{i+1},\dots,f_m)$ does not belong to the semigroup S_P.*

In fact, if there is a regular differential μ on X with $\operatorname{ord}_{Q_i}(\mu) = -n$, then by Corollary 2.9 and Property 2.5 we have $(-f_1,\dots,-f_{i-1},-n,-f_{i+1},\dots,-f_n) \in R$ and this by Theorem 2.8 and Property 2.7 means that $(f_1,\dots,f_{i-1},n-1,f_{i+1},\dots,f_m) \notin S_P$.

Proof of Theorem 2.8. For each vector $\mathbf{n} = (n_1,\dots,n_m)$ we abbreviate

$$\mathbf{p}^{\mathbf{n}} := \mathbf{p}_1^{n_1} \cdot \ldots \cdot \mathbf{p}_m^{n_m}$$

and consider the divisor

$$\mathbf{a}_{\mathbf{n}} := (\mathcal{O}_P + \mathbf{p}^{\mathbf{n}}) \cdot \prod_{Q \in X \setminus \{P\}} \mathcal{O}_Q.$$

Note that

$$\Omega(\mathbf{a}_{\mathbf{n}}) = \{\mu \in \Omega(\mathcal{O}) | \operatorname{ord}_{Q_1}(\mu) \geq -n_1,\dots,\operatorname{ord}_{Q_m}(\mu) \geq -n_m\}.$$

Thus $w_{\mathbf{n}} := \dim \Omega(\mathbf{a}_{\mathbf{n}})$ is equal to $w_{n_1,\dots,n_i-1,\dots,n_m} + 1$ (respectively, $w_{n_1,\dots,n_i-1,\dots,n_m}$) if and only if there is a differential (respectively, no differential) μ_i, with $\operatorname{ord}_{Q_i}(\mu_i) = -n_i$ and $\operatorname{ord}_{Q_j}(\mu_i) \geq -n_j$ for each $j \neq i$.

Let I be a non-empty subset of $\{1,\dots,m\}$. Now, by Property 2.5, there is a differential $\mu \in \Omega(\mathcal{O})$ with $\operatorname{ord}_{Q_j}(\mu) = -n_i$ for each $i \in I$ and

$\operatorname{ord}_{Q_j}(\mu) \geq -n_j$ for each $j \in \{1,\dots,m\} \setminus I$ if and only if

$$w_{\mathbf{n}} = w_{n_1,\dots,n_i-1,\dots,n_m} + 1 \quad \text{for each} \quad i \in I.$$

On the other hand by the Riemann-Roch theorem (1.1) we have

$$\ell(\mathbf{a}_{\mathbf{n}}) = \deg(\mathbf{a}_{\mathbf{n}}) + 1 - g + w_{\mathbf{n}}.$$

Note that

$$\begin{aligned} \deg(\mathbf{a}_{\mathbf{n}}) &= \dim(\mathcal{O}_P + \mathbf{p}^{\mathbf{n}})/\mathcal{O}_P \\ &= \dim \mathbf{p}^{\mathbf{n}}/(\mathcal{O}_P \cap \mathbf{p}^{\mathbf{n}}) \\ &= \ell_{\mathbf{n}} + \delta_P - (n_1 + \cdots + n_m), \end{aligned}$$

where

$$\ell_{\mathbf{n}} := \dim \mathcal{O}_P/(\mathcal{O}_P \cap \mathbf{p}^{\mathbf{n}})$$

is the codimension of the valuation ideal

$$\mathcal{O}_P \cap \mathbf{p}^{\mathbf{n}} = \{z \in \mathcal{O}_P | \operatorname{ord}_{Q_1}(z) \geq n_1,\dots,\operatorname{ord}_{Q_m}(z) \geq n_m\}.$$

Now we will use the assumption $n_i \geq 0$ for each $i = 1,\dots,m$. Since $\mathcal{O} \leq \mathbf{a}_{\mathbf{n}} \leq \tilde{\mathcal{O}}$ we have $L(\mathbf{a}_{\mathbf{n}}) = k$. Taking

$$I := \{i \in \{1,\dots,m\} | n_i > 0\}$$

we deduce

$$\ell(\mathbf{a}_{\mathbf{n}}) = \ell(\mathbf{a}_{n_1,\dots,n_i-1,\dots,n_m}) = 1 \quad \text{for each} \quad i \in I.$$

Thus we have proved that there is a differential $\mu \in \Omega(\mathcal{O})$ with $\operatorname{ord}_{Q_i}(\mu) = -n_i$ for each $i \in I$ and $\operatorname{ord}_{Q_j}(\mu) \geq 0$ for each $j \in \{1,\dots,m\} \setminus I$ if and only if, for each $i \in I$, $\ell_{\mathbf{n}} = \ell_{n_1,\dots,n_i-1,\dots,n_m}$ or equivalently there is no function $z_i \in \mathcal{O}_P$ with $\operatorname{ord}_{Q_i}(z_i) = n_i - 1$ and $\operatorname{ord}_{Q_j}(z_i) \geq n_j$ for each $j \neq i$. □

Alternatively, by introducing the set

$$R_P := \{(\operatorname{ord}_{Q_1}(\mu),\dots,\operatorname{ord}_{Q_m}(\mu)) | \mu \in \Omega(\mathcal{O}_P) \setminus \{0\}\}$$

the proof of Theorem 2.8 can be divided in two parts. In fact we have $R \subset R_P$ and conversely, since by Theorem 2.1

$$\Omega(\mathcal{O}_P) = \Omega(\mathcal{O}) + \Omega(\tilde{\mathcal{O}}_P)$$

there is for each $(r_1, \dots, r_m) \in R_P$ a differential $\mu \in \Omega(\mathcal{O})$ such that

$$\operatorname{ord}_{Q_i}(\mu) = r_i \quad \text{when} \quad r_i < 0$$

and

$$\operatorname{ord}_{Q_i}(\mu) \geq 0 \quad \text{when} \quad r_i \geq 0.$$

Now, Theorem 2.8 is a consequence of the first part of the following theorem which expresses R_P in terms of the semigroup S_P.

Theorem 2.11. *Let* $(n_1, \dots, n_m) \in \mathbb{Z}^m$.

(i) *There is a differential* $\mu \in \Omega(\mathcal{O}_P)$ *with* $\operatorname{ord}_{Q_i}(\mu) = -n_i$ *for each* $i = 1, \dots, m$ *if and only if for each* $i = 1, \dots, m$ *the vectors* $(r_1, \dots r_m)$ *with* $r_i = n_i - 1$ *and* $r_j \geq n_j$ *for each* $j \neq i$ *do not belong to the semigroup* S_P *associated to the point* P.

(ii) *The vector* $(n_1, \dots, n_m)$ *belongs to the semigroup* S_P *if and only if for each* $i = 1, \dots, m$ *the vectors* $(r_1, \dots, r_m)$ *with* $r_i = -n_i - 1$ *and* $r_j \geq -n_j$ *for each* $j \neq i$ *do not belong to* R_P.

Proof. For each vector $\mathbf{n} = (n_1, \dots, n_m)$ we consider the $\mathcal{O}_P$-ideal

$$\mathbf{d_n} := \mathcal{O}_P + \mathbf{p^n} \quad \text{where} \quad \mathbf{p^n} := \mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_m^{n_m}.$$

Note that

$$\Omega(\mathbf{d_n}) = \{\mu \in \Omega(\mathcal{O}_P) | \operatorname{ord}_{Q_1}(\mu) \geq -n_1, \dots, \operatorname{ord}_{Q_m}(\mu) \geq -n_m\}.$$

Thus $\dim \Omega(\mathbf{d_n})/\Omega(\mathbf{d}_{n_1,\dots,n_i-1,\dots,n_m}) = 1$ if and only if there is a differential $\mu_i \in \Omega(\mathcal{O}_P)$ with $\operatorname{ord}_{Q_i}(\mu_i) = -n_i$ and $\operatorname{ord}_{Q_j}(\mu_i) \geq -n_j$ for each j. If such a differential μ_i exists for each i, then by the Property (2.5) there is even a differential $\mu \in \Omega(\mathcal{O}_P)$ such that $\operatorname{ord}_{Q_i}(\mu) = -n_i$ for each i or equivalently $(-n_1, \dots, -n_m) \in R_P$.

On the other hand by the local duality we have:

$$\dim \frac{\Omega(\mathbf{d_n})}{\Omega(\mathbf{d}_{(n_1,\dots,n_i-1,\dots,n_m)})} = \dim \frac{\mathcal{O}_P + \mathbf{p}^{(n_1,\dots,n_i-1,\dots,n_m)}}{\mathcal{O}_P + \mathbf{p^n}}$$
$$= \dim \frac{\mathbf{p}^{(n_1,\dots,n_i-1,\dots,n_m)}}{\mathbf{p^n}} - \dim \frac{\mathcal{O}_P \cap \mathbf{p}^{(n_1,\dots,n_i-1,\dots,n_m)}}{\mathcal{O}_P \cap \mathbf{p^n}}$$
$$= 1 - \dim \frac{\mathcal{O}_P \cap \mathbf{p}^{(n_1,\dots,n_i-1,\dots,n_m)}}{\mathcal{O}_P \cap \mathbf{p^n}}.$$

Thus the vector $(-n_1, \dots, -n_m)$ belongs to R_P if and only if, for each i, $\mathcal{O}_P \cap \mathbf{p}^{(n_1,\dots,n_i-1,\dots,n_m)} = \mathcal{O}_P \cap \mathbf{p^n}$ or equivalently there is no $z_i \in \mathcal{O}_P$ with $\operatorname{ord}_{Q_i}(z_i) = n_i - 1$ and $\operatorname{ord}_{Q_j}(z_i) \geq n_j$ for each j. This proves Part (i).

The proof of Part (ii) is similar.□

Theorem 2.12. *If* $\mathcal{O}_P$ *is a Gorenstein ring then*

$$R_P = S_P - (f_1, \dots, f_m).$$

Proof. Since $\mathcal{O}_P$ is a Gorenstein ring, by Theorem 2.3 there is a differential λ such that $\Omega(\mathcal{O}_P) = \mathcal{O}_P\ \lambda$ and therefore

$$R_P = S_P + (\operatorname{ord}_{Q_1}(\lambda), \dots, \operatorname{ord}_{Q_m}(\lambda)),$$

and it follows from Corollary 2.9 that $\operatorname{ord}_{Q_i}(\lambda) = -f_i$ for each $i = 1, \dots, m$.□

By combining Theorem 2.11 and 2.12 we get symmetry properties for S_P and R_P:

Corollary 2.13. *Assume that* $\mathcal{O}_P$ *is a Gorenstein ring.*

(i) *A vector* $(s_1, \dots, s_m) \in \mathbb{Z}^m$ *belongs to the semigroup* S_P *if and only if for each* $i = 1, \dots, m$ *the vectors* $(r_1, \dots, r_m)$ *with* $r_i = f_i - 1 - s_i$ *and* $r_j \geq f_j - s_j$ *for each* $j \neq i$ *do not belong to* S_P.

(ii) *A vector* $(r_1, \dots, r_m) \in \mathbb{Z}^m$ *belongs to* R_P *if and only if for each* $i = 1, \dots, m$ *the vectors* $(n_1, \dots, n_m)$ *with* $n_i = -r_i - f_i - 1$ *and* $n_j \geq -r_j - f_j$ *for each* $j \neq i$ *do not belong to* R_P.

The symmetry of the semigroup S_P has been discussed by Kunz [**K**] in the case of one branch, by Waldi [**Wa**] and Garcia [**G**] in the case of two branches, and by Delgado ([**D1**], [**D2**]) in the general case.

For each $i = 1, \dots, m$ we denote by $S_P^{(i)} \subseteq \mathbb{N}$ the semigroup of the algebroid curve corresponding to the branch Q_i. Since $S_P^{(i)}$ is the projection of S_P on the i-th coordinate axis, we obtain:

Corollary 2.14. *Assume that* $\mathcal{O}_P$ *is a Gorenstein ring. Let* $i \in \{1, \dots, m\}$ *and let* n *be a positive integer. There is a regular differential on* X *having at* Q_i *a pole of order* n *if and only if* $f_i - n \in S_P^{(i)}$.

Proof. Since as already noted $\Omega(\mathcal{O}_P) = \Omega(\mathcal{O}) + \Omega(\tilde{\mathcal{O}}_P)$ there is a regular differential μ with $\mathrm{ord}_{Q_i}(\mu) = -n$ if and only if

$$(r_1, \ldots, r_{i-1}, -n, r_{i+1}, \ldots, r_m) \in R_P$$

for some integers r_j, or equivalently by Theorem 2.12,

$$(s_1, \ldots, s_{i-1}, f_i - n, s_{i+1}, \ldots, s_m) \in S_P$$

where $s_j = r_j + f_j$ for each j. □

If $\mathcal{O}_P$ is a Gorenstein ring, then by comparing the Corollaries 2.10 and 2.14 or by applying the symmetry (2.13) of S_P we obtain the equivalence:

$$f_i - n \in S_P^{(i)} \Longleftrightarrow (f_1, \ldots, f_{i-1}, n-1, f_{i+1}, \ldots, f_m) \notin S_P.$$

If $m > 1$ then f_i is not the conductor of the semigroup $S_P^{(i)}$.

3. Projective representations and Weierstrass points of singular curves

Let X be a complete irreducible algebraic curve defined over an algebraically closed field k. Let $x_0, \ldots, x_n$ be k-linear independent elements of the function field K. If $n \geq 1$ then there is a morphism defined on the non-singular model

$$(x_0 \colon \ldots \colon x_n) \colon \tilde{X} \to \mathbb{P}^n(k)$$

whose image by the extension theorem of valuation theory is a projective irreducible non-degenerated algebraic curve in $\mathbb{P}^n(k)$ (cf. [**SV1**, §1]). There is a morphism $X \to \mathbb{P}^n(k)$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbb{P}^n(k) \\ & \searrow^{\pi} \quad \nearrow & \\ & X & \end{array}$$

commutes if and only if for each $P \in X$ there is an integer $i(P) \in \{0, \ldots, n\}$ such that $\frac{x_i}{x_{i(P)}} \in \mathcal{O}_P$ for each i or equivalently the $\mathcal{O}_P$-ideal $\sum_{i=0}^{n} \mathcal{O}_P x_i$ is principal. In this case for each hyperplane in $\mathbb{P}^n(k)$, say $\sum_{i=0}^{n} c_i X_i = 0$, its *intersection theoretic inverse image* under the morphism $X \to \mathbb{P}^n(k)$ is defined to be the locally principal positive divisor $\mathrm{div}(\Sigma c_i x_i) \cdot \prod_{P \in X} (\Sigma \mathcal{O}_P x_i)$ of X, whose support – as easily seen – is the inverse image of the hyperplane. By the product formula (1.2) the degree of this divisor is equal to

$$\sum_{P \in X} \deg_P(\mathcal{O}_P x_0 + \cdots + \mathcal{O}_P x_n)$$

and so it does not depend on the choice of the hyperplane; and it will be called the *degree of the morphism* $X \to \mathbb{P}^n(k)$. It coincides with the degree of the image curve if and only if the morphism is birational.

More generally, if $h(X_0, \ldots, X_n)$ is a homogeneous polynomial of degree d say, then the *intersection theoretic inverse image* of the corresponding hypersurface of degree d under the morphism $X \to \mathbb{P}^n(k)$ is the locally principal positive divisor

$$\mathrm{div}(h(x_0, \ldots, x_n)) \cdot \prod_{P \in X} (\mathcal{O}_P x_0 + \cdots + \mathcal{O}_P x_n)^d$$

whose degree is equal to d times the degree of the morphism $X \to \mathbb{P}^n(k)$. This recovers the classical *theorem of Bezout*.

Theorem 3.1. *Let* $\mathbf{a}$ *be a locally principal divisor of* X *and let* $x_0, \ldots, x_n$ *be a basis of* $L(\mathbf{a})$.

(i) *If* $\deg(\mathbf{a}) \geq 2g$ *then*

$$\mathbf{a}_P = \mathcal{O}_P x_0 + \cdots + \mathcal{O}_P x_n$$

for each $P \in X$ *and so* $(x_0 \colon \ldots \colon x_n)$ *induces a morphism* $X \to \mathbb{P}^n(k)$.

(ii) *If* $\deg(\mathbf{a}) \geq 2g + 1$ *then the morphism* $X \to \mathbb{P}^n(k)$ *is injective and birational.*

(iii) *If* $\deg(\mathbf{a}) \geq 2g + \dim \mathbf{m}_P/\mathbf{m}_P^2$ *for each* $P \in X$ *then the morphism* $X \to \mathbb{P}^n(k)$ *is an isomorphism onto the image curve.*

Recall that by Nakayama's lemma the dimension of Zariski's cotangent space $\mathbf{m}_P/\mathbf{m}_P^2$ is equal to one if and only if P is non-singular. By the theorem, the complete curve X is already projective (see also

[**R**, Theorem 5]). In the non-singular case the theorem is proved in Hartshorne's book (cf. [**Ha**, Ch. IV, Cor. 3.2]).

Proof. (i) Let $P \in X$ and let $\mathbf{b}$ be the divisor of X with $\mathbf{b}_P = \mathbf{m}_P \cdot \mathbf{a}_P$ and $\mathbf{b}_Q = \mathbf{a}_Q$ for each $Q \in X \backslash \{P\}$. Since $\deg(\mathbf{b}) = \deg(\mathbf{a}) - 1 > 2g-2$, by Riemann's theorem we have $\ell(\mathbf{b}) = \ell(\mathbf{a}) - 1$, and so there exists $x \in L(\mathbf{a})$ with $x \notin \mathbf{m}_P \cdot \mathbf{a}_P$ and therefore $\mathbf{a}_P = \mathcal{O}_P x$.

(ii) Assume by way of contradiction that there are points $P_1, P_2 \in X$ with $P_1 \neq P_2$ whose images in $\mathbb{P}^n(k)$ coincide. After a projective coordinate transformation we can assume that the common image point is equal to $(1:0:\ldots:0)$ that is $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \in \mathbf{m}_{P_i}$ $(i = 1, 2)$. By the first part of the theorem we deduce that $\mathbf{a}_{P_i} = \mathcal{O}_{P_i} x_0$. Let $\mathbf{d}$ be the divisor of X defined by $\mathbf{d}_{P_i} = \mathbf{m}_{P_i} \cdot \mathbf{a}_{P_i}$ $(i = 1, 2)$ and $\mathbf{d}_Q = \mathbf{a}_Q$ for each $Q \in X \setminus \{P_1, P_2\}$. Then $x_1, \ldots, x_n \in L(\mathbf{d})$ and therefore $\ell(\mathbf{d}) \geq n$. On the other hand, since $\deg(\mathbf{d}) = \deg(\mathbf{a}) - 2 > 2g - 2$, by Riemann's theorem we obtain $\ell(\mathbf{d}) = \ell(\mathbf{a}) - 2 = n - 1$. This is a contradiction. The birationality will follow from the proof of (iii) applied to a non-singular point P of X.

(iii) Let Y be the image of the injective morphism $X \to \mathbb{P}^n(k)$, let $P \in X$ and Q its image in Y. We can identify $k(Y) \subseteq k(X)$ and so $\mathcal{O}_Q \subseteq \mathcal{O}_P$. We have to show $\mathcal{O}_Q = \mathcal{O}_P$ and this will also imply the birationality $k(Y) = k(X)$. Since by the second part P is the only point in the inverse image of Q, the local ring $\mathcal{O}_P$ is contained in the integral closure of $\mathcal{O}_Q$ in $k(X)$, and so $\mathcal{O}_P$ is finitely generated as $\mathcal{O}_Q$-module. Thus by Nakayama's lemma it is enough to show that $\mathcal{O}_P \subseteq \mathcal{O}_Q + \mathbf{m}_Q \cdot \mathcal{O}_P$. We can assume that $Q = (1:0\ldots:0)$ and so $\mathbf{a}_P = \mathcal{O}_P x_0$ by (i). Let $\mathbf{d}$ be the divisor of X defined by $\mathbf{d}_P = \mathbf{m}_P^2 \cdot \mathbf{a}_P$ and $\mathbf{d}_R = \mathbf{a}_R$ for each $R \in X \setminus \{P\}$. Since $\deg(\mathbf{d}) = \deg(\mathbf{a}) - \dim \mathcal{O}_P/\mathbf{m}_P^2 > 2g - 2$, by Riemann's theorem we have $\ell(\mathbf{a}) - \ell(\mathbf{d}) = \dim \mathcal{O}_P/\mathbf{m}_P^2$. Thus the injective homomorphism $L(\mathbf{a})/L(\mathbf{d}) \to \mathbf{a}_P/\mathbf{d}_P = x_0\mathcal{O}_P/x_0\mathbf{m}_P^2$ is surjective that is $x_0\mathcal{O}_P = L(\mathbf{a}) + x_0\mathbf{m}_P^2$ and therefore

$$\mathcal{O}_P = \sum_{i=0}^{n} k\frac{x_i}{x_0} + \mathbf{m}_P^2 \subseteq \mathcal{O}_Q + \mathbf{m}_P^2.$$

In particular $\mathbf{m}_P \subseteq \mathbf{m}_Q + \mathbf{m}_P^2$ and thus by Nakayama's lemma $\mathbf{m}_P \subseteq \mathbf{m}_Q \cdot \mathcal{O}_P$. Since

$$\mathcal{O}_P/\mathbf{m}_P \cong \mathcal{O}_Q/\mathbf{m}_Q \quad (\cong k)$$

this implies $\mathcal{O}_P \subseteq \mathcal{O}_Q + \mathbf{m}_Q \cdot \mathcal{O}_Q$. $\square$

For the remainder of this section we assume that X is not isomorphic to the projective line or equivalently that the arithmetic genus g of the complete curve X is different from zero. Let $\lambda_1, \ldots, \lambda_g$ be a basis of the space of regular differentials on X. Let λ be a non-zero differential of X and $\mathbf{c}$ its divisor. Then we can write

$$\lambda_i = z_i \lambda \quad (i = 1, \ldots g)$$

where $z_1, \ldots, z_g$ is a basis of $L(\mathbf{c})$. In particular we have $(\lambda_1: \ldots : \lambda_g) = (z_1: \cdots : z_g)$.

Theorem 3.2. *For each $P \in X$ we have*

$$\mathbf{c}_P = \mathcal{O}_P z_1 + \cdots + \mathcal{O}_P z_g$$

or equivalently $\Omega(\mathcal{O}_P) = \mathcal{O}_P\lambda_1 + \cdots + \mathcal{O}_P\lambda_g$. The morphism $(\lambda_1: \ldots : \lambda_g) : \tilde{X} \to \mathbb{P}^{g-1}(k)$ induces a morphism $X \to \mathbb{P}^{g-1}(k)$ if and only if X is a Gorenstein curve.

Proof. Since by Corollary 2.4 the canonical divisor $\mathbf{c}$ is locally principal if and only if X is a Gorenstein curve, it is enough to show that $\mathbf{c}_P = \Sigma \mathcal{O}_P z_i$ for each $P \in X$. $\square$

We will first assume that P is a singular point of X. Then $\mathbf{f}_P \subseteq \mathbf{m}_P$ and so by Nakayama's lemma it is enough to observe that $\mathbf{c}_P \subseteq \Sigma\, k z_i + \mathbf{f}_P \cdot \mathbf{c}_P$ or equivalently $\Omega(\mathcal{O}_P) \subseteq \Omega(\mathcal{O}) + \mathbf{f}_P \cdot \Omega(\mathcal{O}_P)$, and this holds by Proposition 2.2 and Theorem 2.1.

Thus we can assume that P is a non-singular point of X. Then $\mathcal{O}_P$ is a discrete valuation ring and $\mathbf{c}_P$ is a principal ideal. Let $\mathbf{b}$ be the divisor of X defined by $\mathbf{b}_P = \mathbf{m}_P \cdot \mathbf{c}_P$ and $\mathbf{b}_Q = \mathbf{c}_Q$ for each $Q \in X \setminus \{P\}$. If there is a function $y \in L(\mathbf{c}) \setminus L(\mathbf{b})$ then $\mathbf{c}_P = \mathcal{O}_P\, y$ and we are done. Thus it its enough to show that $\ell(\mathbf{b}) < \ell(\mathbf{c})$. By the Riemann-Roch theorem, since $\deg(\mathbf{b}) = \deg(\mathbf{c}) - 1$ this means that $\ell(\mathbf{c}\colon \mathbf{b}) \leq 1$. Note that

$$\mathbf{c}\colon \mathbf{b} = \mathbf{m}_P^{-1} \prod_{Q \neq P} \mathcal{O}_Q \geq \mathcal{O}$$

and therefore $L(\mathbf{c}\colon \mathbf{b}) \supseteq k = L(\mathcal{O})$. Assume by way of contradiction that there is a non-constant function $z \in L(\mathbf{c}\colon \mathbf{b})$. Note that z has a simple pole at P, and does not have any pole in $\tilde{X} \setminus \{P\}$. Thus $K = k(z)$ and in particular the genus $\tilde{g}$ of $\tilde{X}$ is zero. Let $Q \in X \setminus \{P\}$ and let c be the value of z at Q. Since $z - c$ vanishes at Q and since P is the only pole of $z - c$, by the product formula (1.2) we conclude $\dim \mathcal{O}_Q/(z-c) = 1$, and therefore Q is non-singular. Thus $X = \tilde{X}$ has genus 0, and this is excluded by hypothesis. □

Note that the morphisms of X onto the projective line $\mathbb{P}^1(k)$ correspond bijectively to the non-constant functions $z \in K$ satisfying $z \in \mathcal{O}_P$ or $\frac{1}{z} \in \mathcal{O}_P$ for each $P \in X$. The degree of such a morphism $X \longrightarrow \mathbb{P}^1(k)$ is equal to the degree of the field extension $K|k(z)$. A curve X is called *hyperelliptic* when there is a morphism $X \longrightarrow \mathbb{P}^1(k)$ of degree 2. By applying theorems of Rosenlicht (cf.[**R**, Theorem 15 and 17]) we obtain:

Theorem 3.3. *Assume that X is a Gorenstein curve. Then the morphism $X \longrightarrow \mathbb{P}^{g-1}(k)$ is an isomorphism onto the image curve if and only if X is non-hyperelliptic.*

Weierstrass points of singular curves have been studied by several authors (cf. [**K1**], [**K2**], [**WL**], [**LW**], [**F**] and [**GL**]). Our approach consists in applying the theory of Weierstrass points of linear systems on non-singular curves to the morphism $\tilde{X} \longrightarrow \mathbb{P}^{g-1}(k)$ (cf. [**SV1**, §1]). For each $Q \in \tilde{X}$ let $\varepsilon_0(Q) < \varepsilon_1(Q) < \cdots < \varepsilon_{g-1}(Q)$ be the *hermitian Q-invariants* that is the intersection multiplicities of the corresponding parametrized branch with the hyperplanes in $\mathbb{P}^{g-1}(k)$. If Q is (centered at) a non-singular point of X then it follows from the Riemann-Roch theorem (1.1) that

$$\varepsilon_n(Q) = \ell_{n+1}(Q) - 1$$

for each $n = 0, \ldots, g-1$ where $\ell_1(Q), \ldots, \ell_g(Q)$ are the Weierstrass gaps

of X at Q. There are integers ε_n $(n = 0, \ldots, g-1)$ such that

$$\varepsilon_n(Q) = \varepsilon_n \quad \text{for almost all} \quad Q \in \tilde{X}$$

and $\varepsilon_n(Q) \geq \varepsilon_n$ for all $Q \in \tilde{X}$. The curve X is called *classical* when $\varepsilon_n = n$ for each n; this always happens when the characteristic p of k is zero or larger than $2g - 2$. Non-classical Gorenstein curves have been first studied by Freitas [**F**], who established a complete classification when the arithmetic genus is three or four.

Theorem 3.4. *Let $Q_1, \ldots, Q_m$ be the branches centered at a point $P \in X$, and let $f_1, \ldots, f_m$ be the exponents of the conductor of $\mathcal{O}_P$. For each $i = 1, \ldots, m$ an integer h with $0 \leq h < f_i$ is a hermitian Q_i-invariant if and only if the vector $(f_1, \ldots, f_{i-1}, f_i - h - 1, f_{i+1}, \ldots, f_m)$ does not belong to the semigroup S_P associated to the point P. Moreover if the local ring $\mathcal{O}_P$ is a Gorenstein ring this means that h belongs to the semigroup $S_P^{(i)}$ of the algebroid curve corresponding to the branch Q_i.*

In the special case where P is a one-branched point of a Gorenstein curve this has also been noted by C.F. de Carvalho (IMPA-seminar 1991) and by Garcia-Lax [**GL**].

Proof. Let $r_0 < r_1 < \cdots < r_{g-1}$ be the orders of the non-zero regular differentials at Q_i. By Corollary 2.9 we have $r_0 = -f_i$ and therefore

$$\varepsilon_n(Q_i) = r_n + f_i$$

for each $n = 0, \ldots, g-1$. Now, the theorem follows by applying the Corollaries 2.10 and 2.14.□

Remark 3.5. There is an hermitian Q_i-invariant larger than or equal to f_i if and only if $m > 1$ or $\tilde{g} > 0$ or $X \setminus \{P\}$ is singular. In this case f_i is a hermitian Q_i-invariant, as follows by applying the Riemann-Roch theorem to the divisors

$$\mathbf{p}_i^n \prod_{j \neq i} \mathbf{p}_j^{f_i} \prod_{Q \in X \setminus \{P\}} \mathcal{O}_Q$$

with $n = 0$ and $n = -1$. □

Denoting by $w(Q)$ the Weierstrass weight of a branch $Q \in \tilde{X}$, and calling Q a *Weierstrass branch* when $w(Q)$ is positive, the number of

Weierstrass branches counted with their weights is given by the formula

$$\sum_{Q\in\tilde{X}} w(Q) = g(2g-2) + (\Sigma\varepsilon_n)(2\tilde{g}-2)$$

(cf. [**SV 1**, p.6]). Now by applying the genus formula (1.3) we obtain

$$\sum_{P\in X} W(P) = (g+\Sigma\varepsilon_n)(2g-2)$$

(cf. Garcia-Lax [**GL**]), where the Weierstrass weight of $P \in X$ is defined by the formula

$$W(P) := \sum_{i=1}^{m} w(Q_i) + 2\delta_P \Sigma\varepsilon_n .$$

Remind that, for each $Q \in \tilde{X}$,

$$w(Q) \geq \Sigma(\varepsilon_n(Q) - \varepsilon_n)$$

and equality holds if and only if $\det(\binom{\varepsilon_n(Q)}{\varepsilon_r}) \not\equiv 0 \pmod p$. Thus Theorem 3.4 provides a lower estimate of the Weierstrass weights of the (branches centered at) singular points of a classical curve.

4. Zeta-function

Let X be a complete irreducible algebraic curve defined over a perfect field k of positive characteristic p. Recall that by a theorem of Tate an operator C, called the *Cartier operator*, is well defined on the space of differential forms by setting

$$C(zdx) := -\left(\frac{d^{p-1}z}{dx^{p-1}}\right)^{1/p} dx$$

for each $z \in K$ and each separating variable x of $K|k$. If Q is a point of the non-singular model $\tilde{X}$, k_Q its residue field, t a local parameter at Q, and $\Sigma c_i t^{i-1}dt \in k_Q((t))dt$ the local expansion of a differential form at Q, then it follows that

$$C(\Sigma c_i t^{i-1}\, dt) = \Sigma c_{pi}^{1/p}\, t^{i-1}\, dt.$$

Since the P-component of a differential form μ satisfies the formula

$$\mu_P(z) = \sum_{i=1}^{m} tr_{k_{Q_i}|k}\ \mathrm{res}_{Q_i}(z\mu)$$

for each $z \in K$ where $Q_1, \ldots, Q_m \in \tilde{X}$ are the branches centered at P, we obtain

$$(C\mu)_P(z) = \left(\mu_P(z^p)\right)^{1/p}$$

for each $z \in K$. Thus if $\mathbf{a}$ is a divisor of X satisfying $a^p \in \mathbf{a}_P$ for each $a \in \mathbf{a}_P$ and $P \in X$ then the Cartier operator C acts on the spaces $\Omega(\mathbf{a}_P)$ and in particular on $\Omega(\mathbf{a})$.

It follows from the Hasse-Witt normal forms of p-linear algebra that the rank of the r-th power $C^r_{\Omega(\mathcal{O})}$ does not depend on r when $r \geq g$ (cf. [**HW**, Satz 11]). This rank, denoted by $\sigma(X)$, is called the *Hasse-Witt invariant* of X. If X is non-singular and k algebraically closed then $p^{\sigma(X)}$ is the number of the p-torsion points of the divisor class group of X (cf. Serre [**S1**, §2.11]). While the Hasse-Witt invariant may decrease under desingularization, the number of the p-torsion points of the divisor class group remains invariant.

In order to study the difference $\sigma(X) - \sigma(\tilde{X})$ we analyze the action of the Cartier operator on the quotient space $\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})$. Since by Theorem 2.1.

$$\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}}) \cong \bigoplus_{P\in X} \Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)$$

we are reduced to study the action on the quotients $\Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)$. Let

$$\mathbf{r}_P := \prod_{i=1}^{m} \mathbf{p}_i = \bigcap_{i=1}^{m} \mathbf{p}_i$$

be the *Jacobson radical* of $\tilde{\mathcal{O}}_P$. Note that:

$$\Omega(\mathcal{O}_P + \mathbf{r}_P) = \{\mu \in \Omega(\mathcal{O}_P) |\ \mathrm{ord}_{Q_i}(\mu) \geq -1 \text{ for each } i = 1, \ldots, m\}.$$

It follows from the above description of the Cartier operator in terms of local expansions that

$$C^r\Omega(\mathcal{O}_P) \subseteq \Omega(\mathcal{O}_P + \mathbf{r}_P) \quad \text{when} \quad r >> 0$$

or more precisely when $p^r \geq f_i$ for $i = 1, \ldots, m$ where $f_1, \ldots, f_m$ are the exponents of the conductor of $\mathcal{O}_P$. Thus the action of the Cartier operator on $\Omega(\mathcal{O}_P)/\Omega(\mathcal{O}_P + \mathbf{r}_P)$ is nilpotent. Now we describe its action on $\Omega(\mathcal{O}_P + \mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P)$.

Theorem 4.1. *There is an isomorphism*

$$\Omega(\mathcal{O}_P + \mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P) \simeq \{(a_1, \ldots, a_m) \in \bigoplus_{i=1}^{m} k_{Q_i} \mid \sum_{i=1}^{m} tr_{k_{Q_i}|k_P}(a_i) = 0\}$$

defined by

$$\mu + \Omega(\tilde{\mathcal{O}}_P) \mapsto \Big(\operatorname{res}_{Q_1}(\mu), \ldots, \operatorname{res}_{Q_m}(\mu)\Big).$$

The Cartier operator induces on the vectors of the right hand side the action

$$(a_1, \ldots, a_m) \mapsto (a_1^{1/p}, \ldots, a_m^{1/p}).$$

Proof. Recall that a differential form μ is regular at P if and only if

$$\mu_P(z) := \sum_{i=1}^{m} tr_{k_{Q_i}|k} \operatorname{res}_{Q_i}(z\mu) = 0 \text{ for each } z \in \mathcal{O}_P.$$

If $\operatorname{ord}_{Q_i}(\mu) \geq -1$ for each $i = 1, \ldots, m$ then

$$\mu_P(z) = tr_{k_P|k}(z(P) \cdot \sum_{i=1}^{m} tr_{k_{Q_i}|k_P} \operatorname{res}_{Q_i}(\mu)) \text{ for each } z \in \mathcal{O}_P$$

and so the regularity of μ at P means that

$$\sum_{i=1}^{m} tr_{k_{Q_i}|k_P} \operatorname{res}_{Q_i}(\mu) = 0\ .$$

Thus the homomorphism of the theorem is well defined and injective. Now the surjectivity follows from a dimension count, which uses the local duality (1.5):

$$\begin{aligned}
\dim \Omega(\mathcal{O}_P + \mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P) &= \dim \tilde{\mathcal{O}}_P/(\mathcal{O}_P + \mathbf{r}_P) \\
&= \dim \tilde{\mathcal{O}}_P/\mathbf{r}_P - \dim(\mathcal{O}_P + \mathbf{r}_P)/\mathbf{r}_P \\
&= \sum_{i=1}^{m} \dim \tilde{\mathcal{O}}_P/\mathbf{p}_i - \dim \mathcal{O}_P/(\mathcal{O}_P \cap \mathbf{r}_P) \\
&= \sum_{i=1}^{m} \deg(Q_i) - \deg(P).
\end{aligned}$$

Finally, the action of the Cartier operator on the vector $(a_1, \ldots, a_m)$ is obtained from its action on the local expansions of the differentials at the branches $Q_1, \ldots, Q_m$. □

Note that $\Omega(\mathcal{O} + \mathbf{r})$ is the space of regular differentials on X whose poles are all of order 1.

Corollary 4.2.

$$\sigma(X) - \sigma(\tilde{X}) = \dim \Omega(\mathcal{O} + \mathbf{r})/\Omega(\tilde{\mathcal{O}}) = \sum_{P \in X} \Big(\sum_{Q|P} \deg(Q) - \deg(P)\Big)$$

where the symbol "$Q|P$" indicates that Q ranges over the branches centered at P.

Proof. Since the action of the Cartier operator on $\Omega(\mathcal{O})/\Omega(\mathcal{O} + \mathbf{r})$ is nilpotent we conclude:

$$\operatorname{rank}\ C^r_{\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})} = \operatorname{rank}\ C^r_{\Omega(\mathcal{O}+\mathbf{r})/\Omega(\tilde{\mathcal{O}})} \quad \text{for each} \quad r \geq g - \tilde{g}.$$

The isomorphism of Theorem 2.1 induces an isomorphism

$$\Omega(\mathcal{O} + \mathbf{r})/\Omega(\tilde{\mathcal{O}}) \cong \bigoplus_{P \in X} \Omega(\mathcal{O}_P + \mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P)$$

and so we obtain for each r:

$$\operatorname{rank}\ C^r_{\Omega(\mathcal{O}+\mathbf{r})/\Omega(\tilde{\mathcal{O}})} = \sum_{P \in X} \operatorname{rank}\ C^r_{\Omega(\mathcal{O}_P+\mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P)}.$$

Finally, by Theorem 4.1 we have for each r:

$$\begin{aligned}
\operatorname{rank}\ C^r_{\Omega(\mathcal{O}_P+\mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P)} &= \dim \Omega(\mathcal{O}_P + \mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P) \\
&= \sum_{i=1}^{m} \deg(Q_i) - \deg(P). \quad \square
\end{aligned}$$

In particular, if k is algebraically closed then $\sigma(X) = \sigma(\tilde{X})$ if and only if each point of X admits only one branch.

For the remainder of this paper we will assume that the constant field of the complete irreducible curve X is a finite field with p^n elements that is

$$k = \mathbb{F}_q \quad \text{where} \quad q = p^n.$$

Then the n-th power C^n of the Cartier operator is k-linear. We calculate its characteristic polynomial on $\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})$.

Corollary 4.3.

$$\det\left(Id - tC^n_{\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})}\right) = \prod_{P\in X_{\rm sing}} \frac{\prod_{Q|P}(1-t^{\deg(Q)})}{(1-t^{\deg(P)})}$$

where P varies in the finite set $X_{\rm sing}$ of singular points of X and Q ranges over the branches centered at P.

Proof. Since $\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}}) \cong \bigoplus_{P\in X_{\rm sing}} \Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)$ we have:

$$\det\left(Id - tC^n_{\Omega(\mathcal{O})/\Omega(\tilde{\mathcal{O}})}\right) = \prod_{P\in X_{\rm sing}} \det\left(Id - tC^n_{\Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)}\right).$$

Since $C_{\Omega(\mathcal{O}_P)/\Omega(\mathcal{O}_P+\mathbf{r}_P)}$ is nilpotent we deduce:

$$\det\left(Id - t\, C^n_{\Omega(\mathcal{O}_P)/\Omega(\tilde{\mathcal{O}}_P)}\right) = \det\left(Id - t\, C^n_{\Omega(\mathcal{O}_P+\mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P)}\right).$$

By Theorem 4.1 we have a commutative diagram with exact lines

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(\mathcal{O}_P+\mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}) & \longrightarrow & \bigoplus_{i=1}^m k_{Q_i} & \longrightarrow & k_P & \longrightarrow & 0\\
 & & \downarrow C^n & & \downarrow F^{-1} & & \downarrow F^{-1} & & \\
0 & \longrightarrow & \Omega(\mathcal{O}_P+\mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P) & \longrightarrow & \bigoplus_{i=1}^m k_{Q_i} & \longrightarrow & k_P & \longrightarrow & 0
\end{array}$$

where F is the Frobenius automorphism over k. Thus we obtain

$$\det\left(Id - tC^n_{\Omega(\mathcal{O}_P+\mathbf{r}_P)/\Omega(\tilde{\mathcal{O}}_P)}\right) = \frac{\prod_{i=1}^m \det\left(Id - tF^{-1}_{k_{Q_i}|k}\right)}{\det\left(Id - tF^{-1}_{k_P|k}\right)}.$$

If k_r is an extension field of k of a finite degree r then $k_r = \bigoplus_{i=0}^{r-1} k\, a^{q^i}$ for some $a \in k^r$ and therefore

$$\det\left(Id - tF^{-1}_{k_r|k}\right) = \det\begin{pmatrix} 1 & 0 & \cdot & 0 & -t\\ -t & 1 & 0 & \cdot & 0\\ 0 & -t & 1 & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & 0\\ 0 & \cdot & 0 & -t & 1\end{pmatrix} = 1 - t^r. \quad \Box$$

The *zeta-function* of X is defined to be the eulerian product

$$\zeta(X,s) := \prod_{P\in X} \frac{1}{1-q^{-s\deg(P)}}$$

when the real part $R(s)$ of $s \in \mathbf{C}$ is larger than 1. Thus abbreviating $t = q^{-s}$ we obtain:

$$\frac{\zeta(X,s)}{\zeta(\tilde{X},s)} = M(X,t) := \prod_{P\in X_{\rm sing}} \frac{\prod_{Q|P}(1-t^{\deg(Q)})}{(1-t^{\deg(P)})}.$$

Formally this is the same expression as in Corollary 4.3; but now we are in characteristic zero. Since $\deg(P)$ divides $\deg(Q)$ whenever Q is a branch centered at P, $M(X,t)$ is a polynomial in $t = q^{-s}$ with integer coefficients whose roots are on the unit circle $|t| = 1$ (or equivalently on the imaginary axis in the s-plane). Denoting by d the degree of $M(X,t)$, by Corollary 4.2 we have:

$$d := \deg\ M(X,t) = \dim \Omega(\mathcal{O}+\mathbf{r}) - \tilde{g} = \sigma(X) - \sigma(\tilde{X}).$$

Similar to the number field case (cf. Jenner [**J**]), the zeta-function $\zeta(X,s)$, except for possible new zeros on the imaginary axis $R(s) = 0$, has the same zeros as $\zeta(\tilde{X},s)$.

By the Riemann hypothesis for non-singular curves we can write

$$\zeta(\tilde{X},s) = \frac{L(\tilde{X},t)}{(1-t)(1-qt)}$$

where $L(\tilde{X},t)$ is a polynomial with integer coefficients in $t = q^{-s}$ of degree $2\tilde{g}$ whose zeros are on the circle $|t| = q^{-1/2}$ (or equivalently on the line $R(s) = \frac{1}{2}$ in the s-plane). Thus we obtain

$$\zeta(X,s) = \frac{L(X,t)}{(1-t)(1-tq)}$$

where

$$L(X,t) := L(\tilde{X},t)M(X,t) \in \mathbb{Z}[t]$$

and

$$\deg\ L(X,t) = \tilde{g} + \dim \Omega(\mathcal{O}+\mathbf{r}).$$

The importance of the zeta-function in algebraic geometry is based on the well-known identity

$$\zeta(X,s) = \exp\Big(\sum_{r=1}^{\infty} \frac{1}{r} N_r t^r\Big)$$

where $N_r := \text{card } X(k_r)$ is the number of rational points of X over the extension field $k_r = \mathbb{F}_{q^r}$ of $k = \mathbb{F}_q$ of degree r. By taking logarithmic derivatives this means

$$N_r = q^r + 1 - \sum_{i=1}^{2\tilde{g}} \alpha_i^r - \sum_{i=1}^{d} \beta_i^r$$

where we factorize:

$$L(\tilde{X},t) = \prod_{i=1}^{2\tilde{g}} (1-\alpha_i t) \quad \text{with} \quad |\alpha_i| = q^{1/2}$$

$$M(X,t) = \prod_{i=1}^{d} (1-\beta_i t) \quad \text{with} \quad |\beta_i| = 1.$$

In particular, abbreviating $\tilde{N}_r = \text{card } \tilde{X}(k_r)$ we obtain

$$\tilde{N}_r - N_r = \sum_{i=1}^{d} \beta_i^r$$

and therefore we have proved the following estimate:

Proposition 4.4. $\left|N_r - \tilde{N}_r\right| \leq \sigma(X) - \sigma(\tilde{X})$ *for each* r.

We denote by $\overline{L}(X,t) \in \mathbb{F}_p[t]$ the reduction of $L(X,t) \in \mathbb{Z}[t]$ modulo p.

Proposition 4.5. $\deg \overline{L}(X,t) = \sigma(X)$

This is known in the non-singular case (cf. Stichtenoth [**St**]) and the general result now follows by observing $\deg \overline{M}(X,t) = d = \sigma(X) - \sigma(\tilde{X})$.

Theorem 4.6. $\overline{L}(X,t) = \det\big(Id - tC^n_{\Omega(\mathcal{O})}\big)$

In particular by comparing the coefficients of order 1, one deduces:

$$\text{card } X(k) \equiv 1 - trC^n_{\Omega(\mathcal{O})} \mod p$$

This theorem has been obtained by Manin [**M**] in the non-singular case. By Corollary 4.3 the theorem holds for X if and only if it holds for the non-singular model $\tilde{X}$. Thus we are reduced to the non-singular case proved by Manin. Alternatively to get an elementary proof we may reduce to plane (possibly singular) curves where the Cartier operator has an explicit description (cf. [**SV2**]). A generalization of the theorem to complete varieties of any dimension has been given by Katz [**Ka**]. This is well understood by applying the Lefschetz fixed point formula in a suitable cohomology theory to the Frobenius endomorphism.

We compare the zeta-function $\zeta(X,s)$ with the Dirichlet series

$$\zeta(\mathcal{O},s) := \sum_{\mathbf{a}\geq\mathcal{O}} q^{-s\deg(\mathbf{a})}$$

where $\mathbf{a}$ ranges over the positive divisors of X. Since the number of positive divisors which are linearly equivalent to a divisor $\mathbf{a}$ is equal to $\big(q^{\ell(\mathbf{a})}-1\big)/(q-1)$ and since $\ell(\mathbf{a})$ and $\deg(\mathbf{a})$ only depend on the linear equivalence class $\bar{\mathbf{a}}$ of $\mathbf{a}$ we have

$$\zeta(\mathcal{O},s) = \frac{1}{q-1}\sum (q^{\ell(\bar{\mathbf{a}})}-1)q^{-s\deg(\bar{\mathbf{a}})}$$

where $\bar{\mathbf{a}}$ ranges over the linear equivalence classes of the divisors of non-negative degree. Since by Riemann's theorem

$$\ell(\mathbf{a}) = \deg(\mathbf{a}) + 1 - g \quad \text{when} \quad \deg(\mathbf{a}) > 2g-2$$

one deduces as in the non-singular case that the subseries of the terms with $\deg(\bar{\mathbf{a}}) > 2g-2$ converges absolutely in the semi-plane $R(s) > 1$ to a rational function in $t = q^{-s}$ satisfying the functional equation below. By applying the reciprocity (1.7), the degree formula (1.6) and the Riemann-Roch formula

$$\ell(\mathbf{a}) = \deg(\mathbf{a}) + 1 - g + \ell(\mathbf{c}\colon\mathbf{a})$$

to the finitely many divisor classes $\bar{\mathbf{a}}$ with $0 \leq \deg(\bar{\mathbf{a}}) \leq 2g-2$ we obtain:

(4.7) Functional Equation. *The function* $q^{s(g-1)}\zeta(\mathcal{O},s)$ *is invariant under the substitution* $s \mapsto 1-s$.

This identity has been obtained by F.K. Schmidt [**Sch**] when X is non-singular, by Galkin [**Ga**] when X is a Gorenstein curve and by Green [**G2**] in the general case. (To indicate the connection with Green's paper we note that by the reciprocity (1.7) the assignment $\mathbf{a} \mapsto \mathbf{c}\colon\mathbf{a}$ defines a bijection between the set of positive divisors and the set of divisors smaller than or equal to the canonical divisor **c**.) Furthermore there is an eulerian product expansion

$$\zeta(\mathcal{O}, s) = \prod_{P \in X} \zeta(\mathcal{O}_P, s)$$

where we define

$$\zeta(\mathcal{O}_P, s) := \sum_{\mathbf{a}_P \supseteq \mathcal{O}_P} q^{-s \dim(\mathbf{a}_P/\mathcal{O}_P)}, \quad R(s) > 1,$$

where $\mathbf{a}_P$ ranges over the $\mathcal{O}_P$-ideals containing $\mathcal{O}_P$. Local factors of this type have been studied by Green [**G2**]. Since

$$\zeta(\mathcal{O}_P, s) = \frac{1}{1 - q^{-s \deg(P)}} \quad \text{when} \quad P \text{ is non-singular}$$

we have

$$\frac{\zeta(\mathcal{O}, s)}{\zeta(X, s)} = \prod_{P \in X_{\text{sing}}} (1 - q^{-s \deg(P)}) \zeta(\mathcal{O}_P, s).$$

By applying the functional equation (4.7) and the genus formula (1.3) to $\tilde{X}$ and the curve obtained from X by resolving all singular points except P, one obtains:

(4.8) Local Functional Equation. *The function*

$$q^{s\delta_P} \zeta(\mathcal{O}_P, s) \prod_{Q|P} (1 - q^{-s \deg(Q)})$$

is invariant under the substitution $s \mapsto 1 - s$.

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