Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 24, N. 2, 201-210

Linearizable Circle Diffeomorphisms in One-Parameter Families

Jacek Graczyk

Abstract. We consider smooth families of diffeomorphisms of the circle. We prove that the set of parameter values which correspond to non-linearizable maps with irrational rotation numbers is of Hausdorff dimension 0.

1. Preliminaries

1.1 Introduction. We study one-parameter families of smooth diffeomorphisms of the circle which increase with respect to the parameter value.

The objective of our presentation is to determine the size of the set of parameter values which correspond to non-linearizable maps with irrational rotation numbers (We define the rotation number a little later). We prove that this set is of Hausdorff dimension zero and, in particular, it is of Lebesgue measure zero. This gives an affirmative answer to a conjecture posed by J.Palis during a workshop in ICTP-Trieste in June 1992 (see also [**9**]). Palis' conjecture was motivated by Herman's result [**5**] (see [**1**] for the case of the sine family below), that the complementary set of parameter values corresponding to linearizable maps is of positive Lebesgue measure.

In this paper a circle map is represented by a real monotone function $F(x)$ obeying

$$F(x+1) = F(x) + 1.$$

Unless otherwise stated we assume that all derivatives of F exist.

Partially supported by Polish KBN grant No 2109001
Received 24 October 1992.

A standard example of smooth families considered here is the sine family:

$$x \longmapsto x + t - (c/2\pi)\sin(2\pi x) \pmod 1,$$

where $0 < c < 1$ is a constant.

Assumptions. Here is a detailed list of our assumptions:

1. The smooth diffeomorphisms $F_t(x)$ considered are infinitely many times differentiable.
2. The family F_t is at least twice continuously differentiable with respect to the parameter value t.
3. The family is monotone with respect to $t \in T \subset \mathbb{R}$, that is $\frac{\partial F}{\partial t}$ is positive everywhere in $T \times \mathbb{R}$.

The dynamics of a typical aperiodic diffeomorphism F is easily understood by the fact that, for most irrational rotation numbers, F can be obtained from a pure rotation by a C^∞ change of coordinates (Herman [4]). We will need, however, a more precise version of this theorem which is due to Yoccoz [10] and in a slightly more general form to Katznelson and Ornstein (see [6] and also [7] and [9]).

Let us introduce first a class of diophantine numbers by putting some restrictions on the speed of approximating irrationals by rational numbers with small denominators. For a real β we say that irrational number ρ is diophantine with exponent β if there is a constant C so that

$$\mid \rho - \frac{p}{q} \mid \geq \frac{C}{q^{2+\beta}} .$$

for all rationals p/q.

Theorem 1 (Katznelson-Ornstein). *If $F \in C^k$, $k \in \mathbb{R}$, $k > \beta + 2$, then for any $\varepsilon > 0$ there exists diffeomorphism $\varphi \in C^{k-1-\beta-\varepsilon}$ so that F is conjugated by φ to a rotation, $\phi \circ \rho = F \circ \phi$.*

On the other hand, in [1] Arnold proved that the set of rotation numbers for which there exists a smooth conjugacy is strictly contained in the set of irrational numbers. He also gave an example of C^∞ (in fact real analytic) diffeomorphism which is not even absolutely continuously conjugate to a rotation.

We will show that generically this situation can happen very rarely:

Main Theorem. *The set of parameter values corresponding to maps with irrational rotation numbers which are not smoothly conjugated to a rotation has Hausdorff dimension* 0.

The proof of the theorem is based on the results of [2], especially on the harmonic scaling rules stated there.

Generalization. Actually, it is enough to assume that F_t depends C^1 on the parameter value. However, all results in [2] were proved under the stronger C^2 assumption. If we consider C^k mappings $(k > 2)$, instead of smooth ones, then still still the Hausdorff dimension of the set of parameters corresponding to non-linearizable maps with irrational rotation numbers[1] is not greater than $2/1 + k$.

1.2. Description of the Dynamics

Uniform Constants. By a uniform constant we will mean a function on the above defined class of families of circle diffeomorphisms which continuously depends on the quasi symmetric norm of the maps, the lower bounds of the derivative both with respect to the parameter value t and the argument x on the real line and the C^2 norm of a family. Uniform constants will be always denoted by the letter K. Whenever confusion can arise, we specify uniform constants by adding subscripts.

Rotation numbers. The rotation number $\rho(t)$ of the map F_t, measures the rate at which the orbit of a point wraps around the circle and is usually given by

$$\rho(t) = \lim_{n \to \infty} \frac{F^n(x) - x}{n},$$

where x is any point of the realm.

There are a few methods of organizing rotation numbers in order to expose in a good way, the underlying dynamics. One of them, relies on the concept of the so-called Farey trees and harmonic coordinates. In the present paper we are not going to introduce the whole formalism

[1] Here, we consider only C^1 changes of coordinates

of harmonic coordinates, but only concentrate on the most important elements of it.

Farey trees. We assume that all rationals p/q are written in their simplest form.

The Farey mediant interpolates between two rationals

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'},$$

yielding the fraction with the smallest denominator lying between p/q and p'/q'. The Farey tree is constructed by starting with the endpoints of the unit interval written as $0/1$ and $1/1$, and interpolating by means of Farey mediants. The first level of the Farey tree is $1/2$, the second $1/3$, $2/3$, the third $1/4$, $2/5$, $3/4$ and so forth. It is well known that all rationals can be obtained in this construction in exactly one way. By definition, each rational p/q has exactly two immediate neighbors in the next level. The left is a smaller number called the "left daughter" and the other is a greater number called the "right daughter". Join any rational to its daughters with two edges. As a result we obtain a binary connected tree with vertices in all rationals. This tree is called the *Farey tree*.

Continued Fractions and Dynamics. Let $\rho = [a_1, \cdots, a_n, \cdots]$ be the expansion of ρ into continued fraction, a_i are positive integers. This representation can be finite or infinite depending whether ρ is rational or not. For any rotation number we can define the sequence of the closest returns. We cut off the portion of the continued fraction beyond the n-*th* position, and write the resulting fraction in the lowest terms as p_n/q_n . Those numbers have a transparent dynamical interpretation. Namely, the number q_n is that iterate of the rotation by ρ for which the orbit of any point makes the closest return so far to the point itself. Because of this interpretation the numbers q_n are called the *closest returns* for ρ.

Take an arbitrary point of the circle. Consider its orbit under F, i.e. $z = F^i(z)$, $i \in \mathbb{N}$. By classical Theory of Poincaré, homeomorphisms with irrational rotation numbers are topologically semi-conjugate to rotations. Thus, the order of the points of the orbit is exactly the same

as the order of orbits of the rotation.

The Distortion Lemma. Here is one way of stating the Distortion Lemma for diffeomorphisms of the circle:

Lemma 1.1. *If the intervals*

$$(a,b), \ldots, (F_t^m(a), F_t^m(b))$$

are disjoint, then for arbitrary points $x_1, x_2 \in (a,b)$, the uniform estimate

$$|\log \frac{\partial F_t^m(x_1)}{\partial x} - \log \frac{F_t^m(x_2)}{\partial x}| \le K$$

holds.

1.3 Scaling Rules in Parameter Space

Harmonic subdivision.

Definition 1.1 The interval from P/Q to P'/Q' is called a Farey domain regardless of the order of these two points if and only if either there is an edge between P/Q and P'/Q' in the Farey tree or it is one of the three: $(0,1)$, $(0,1/2)$, $(1/2,1)$.

For every Farey domain, we define a sequence u_n, where n ranges over all integers.

- If n is positive,

$$u_n = \frac{(n+1)P + P'}{(n+1)Q + Q'}.$$

- For n non-positive:

$$u_n = \frac{P + P'(1-n)}{Q + Q'(1-n)}$$

It can be easily checked that the intervals (u_n, u_{n+1}) are all Farey domains. The collection of the intervals (u_n, u_{n+1}) will be called the *harmonic subdivision* of the Farey domain $(P/Q, P'/Q')$.

Fact 1.1. *If P/Q and P'/Q' are endpoints of a Farey domain, then the following relations hold:*

1. $|PQ' - P'Q| = 1$ *and* $1/2 \le Q/Q' \le 2$.
2. $|P/Q - u_n| \ge 1/3nQ^2$

We will use harmonic subdivisions to construct nested sequences of Farey domains which are in one-to-one correspondence to irrationals from the unit interval.

The intervals obtained by harmonic subdivision of the unit interval will be called the *fundamental domains* of level 1. Next, harmonically subdivide every domain of the previous subdivision and so on. The Farey domains obtained on the k-th level will be called *fundamental domains* of level k.

Fact 1.2. *All numbers in the same fundamental domain of level k have the same closest returns up to* q_k.

Harmonic scaling. Without loss of generality we may assume that the parameter space T is equal to $[0, 1]$.

It is well known that the rotation number $\rho(t)$ is continuous (in fact, ρ is both absolutely continuous [4] and Hölder continuous [**2**]) and, as a consequence of assumption 2., non-decreasing in T. Hence, the organization of irrationals given by nested sequences of fundamental domains can be carried over to the parameter space by the rotation function ρ. So, we will talk of fundamental domains and harmonic subdivision in the parameter space as well. However, there is an important difference. Pre-images of rationals are usually non-degenerate intervals and form the set of positive but not full Lebesgue measure in the parameter space (see [**4**]).

In the physical terminology the intervals on which the rotation function ρ makes stops are called "mode lockings" and will be denoted in this paper by $M_{p/q} = \rho^{-1}(p/q)$.

The rotation function is strictly increasing on the set of all preimages of irrationals numbers.

We fix a *point* z of the real line. A simple verification shows that in any "mode locking" $M_{P/Q} \in T$ there is a unique parameter value $t_{P/Q}$, called the center of $M_{P/Q}$, so that $F^Q(z) = z + P$.

Denote by $t(n)$ the centers of "mode lockings" M_{un}, which constitute the harmonic subdivision of the fundamental domain $(t_{P/Q}, t_{P'/Q'})$. Moreover, we will write t_∞ and $t_{-\infty}$ for $t_{P/Q}$ and $t_{P'/Q'}$ respectively. By

the *harmonic scaling* in the fundamental domain $(t_{-\infty}, t_\infty)$ we mean:

$$\frac{|t_{n+1} - t_n|}{|t_\infty - t_{-\infty}|}.$$

Now we are in position to formulate one of the main facts in [**2**].

Theorem 2 (Harmonic Scaling). *The harmonic scaling is bounded, independently from a fundamental domain, by two sequences: from below by* K_1/n^3 *and from above by* K_2/n^2.

2. Proof of the Main Theorem

2.1 Estimates in Parameter Space

Lemma 2.1. *There is a uniform constant* K *for which the following inequality*

$$|t_\infty - t_{-\infty}| \leq \frac{K}{Q^2}$$

holds.

Proof. We can assume that $P/Q > P'/Q'$ and $Q < Q'$, for if it not, we could consider instead of our family F_t a family F'_t given by

$$F'_t(x) = 1 - F_{1-t}(x).$$

The new family F'_t still satisfies our assumptions, but because the rotation numbers have been flipped around one half, so has been the Farey tree.

We fix the counterclockwise orientation of the circle. Denote by f_t the projection of F_t to the circle

$$f_t \circ \Pi = \Pi \circ F_t$$

by the natural projection $\Pi(z) = \exp(2\pi i z)$. To simplify notation, we continue to write z for the projection of the point z from the realm to the circle.

By the mean value theorem, we can find a point $\xi \in (t_{-\infty}, t_\infty)$ so that

$$|t_\infty - t_{-\infty}| \frac{\partial F_\xi^Q}{\partial t}(z) = |f_{-\infty}^Q(z) - z|,$$

Using the chain rule we rewrite the time derivative in the form:

$$\frac{\partial F_\xi^Q}{\partial t}(z) = \sum_{i=1}^{Q} \frac{\partial F_\xi^{Q-i}}{\partial x}(F_\xi^i(z)) \cdot \frac{\partial F_\xi}{\partial t}(F_\xi^{i-1}(z)).$$

By the Distortion Lemma, we can replace the space derivatives by the ratios of small intervals lying nearby with a bounded error. The time derivative of F is bounded, as can be seen using a compactness argument. Hence,

$$|t_\infty - t_{-\infty}| \leq K \frac{|f_{-\infty}^Q(z) - z|}{\sum_{i=1}^{Q} \frac{|f_\xi^{Q-q}(z) - f_\xi^Q(z)|}{|f_\xi^{-q+i}(z) - f_\xi^i(z)|}},$$

where $q = Q' - Q$ is the *closest return* for all maps f_t, $t \in (t_{-\infty}, t_\infty)$, immediately preceding Q (see Fact 1.2).

By the monotonicity of the family f_t and the assumption that $P/Q > P'/Q'$,

$$f_\xi^{Q-q}(z) < f_{-\infty}^Q(z) < f_\xi^Q < z < f_\xi^{q+Q}(z).$$

The intervals $(f_\xi^{Q-q}(z), f_\xi^Q)$ and $(f_\xi^Q(z), f_\xi^{Q+q}(z))$ are uniformly comparable as a immediate consequence of the Distortion Lemma. Therefore,

$$\frac{|f_{-\infty}^Q(z) - z|}{|f_\xi^{Q-q}(z) - f_\xi^Q(z)|} < K_1.$$

To conclude the Lemma we will need the following elementary fact:

Fact 2.1. *The function $g(x_1, \ldots, x_n) = ((1/x_1) + \cdots + (1/x_n))^{-1}$ limited to the simplex $\{x_1 \geq 0, \ldots, x_n \geq 0 : x_1 + \cdots + x_n \leq P\}$ is dominated by P/n^2.*

Since the intervals $(f_\xi^{-q+i}(z), f_\xi^i(z))$, $1 \leq i \leq Q$, are disjoint we immediately get the desired conclusion.

A cover. Let $I_{P/Q}$ be an open interval with an endpoint in P/Q and the length $1/Q^{2+\beta}$. By definition, diophantine numbers with an exponent β can belong only to finitely many intervals of the form $I_{P/Q}$. We transport the interval $I_{P/Q}$ by ρ into the parameter space.

Lemma 2.2. *There is a uniform constant K so that the following estimate*

$$|\rho^{-1}(I_{P/Q})| \leq K/Q^{2+\beta}$$

holds.

Proof. By Fact 1.1, the interval $\rho^{-1}(I_{P/Q})$ is contained in the fundamental domain $(t_{-\infty}, t_\infty)$ so that only one endpoint of $\rho^{-1}(I_{P/Q})$ coincides with an endpoint of the fundamental domain. We will estimate the length of $\rho^{-1}(I_{P/Q})$ using the scaling rules. Let the other endpoint of $\rho^{-1}(I_{P/Q})$ belongs to the interval (t_n, t_{n+1}). Then by Theorem 2, we get that

$$\frac{|\rho^{-1}(I_{P/Q})|}{|t_{-\infty} - t_\infty|} \leq K_1/(n-1).$$

We will dispose of n in the estimate above. By the choice of n, $I_{P/Q} \supset (u_{n+1}, P/Q)$. Therefore, by Fact 1.1,

$$1/Q^{2+\beta} = |I_{P/Q}| \geq K_2/nQ^2,$$

and consequently

$$|\rho^{-1}(I_{P/Q})| \leq K_2 |t_{-\infty} - t_\infty| / Q^\beta.$$

Finally, from Lemma 2.1 the desired conclusion follows. □

Hausdorff Dimension

Lemma 2.3. *The set of parameters values which is covered by the collection of intervals $\rho^{-1}(I_{P/Q})$ infinitely many times has Hausdorff dimension less than $2/(2+\beta)$.*

Proof. For a given $\alpha < 1$ we want to bound from the above

$$\sum_{P/Q} |\rho^{-1}(I_{P/Q})|^\alpha, \tag{1}$$

where the sum is over all rational numbers P/Q. Substituting the estimate from Lemma 2.2 to (1) and summarizing with respect to all rationals with the same numerator Q we obtain the following bound from above on (1).

$$K_1 \sum_{Q=1}^{\infty} \frac{1}{Q^{(2+\beta)\alpha-1}} \leq \infty,$$

provided α is greater than $2/(2+\beta)$. □

The Final Conclusion. The Katznelson-Ornstein Theorem implies that the set of parameter values for which diffeomorphisms F_t are not smoothly conjugated to a rotation is covered infinitely many times by the collection of intervals $\rho^{-1}(I_{P/Q})$ of size $1/Q^{2+\beta}$, where β is arbitrary large. From Lemma 2.3 it follows that this set has Hausdorff dimension zero. This completes the proof of the main theorem.

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Jacek Graczyk
Institute of Mathematics
Warsaw University
ul.Banacha 2, 02-097 Warszawa
Poland