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Local Normal Forms for Constrained Systems on 2-Manifolds

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Abstract. We give a complete classification in the smooth category of local phase portraits (near the origin) of generic constrained systems of the form $A(x)\dot{x} = f(x)$, where $x \in \mathbb{R}^2$, $A(x)$ is a 2×2 matrix-valued function, f is a vector field, the origin is an impasse point ($detA(0) = 0$, and the existence and the uniquiness of solutions breakes down). In the last section we discuss some of the main difficulties to obtain a similar classification in higher dimensions.

1. Introduction and main results.

There exist several definitions of constrained equations (see [1], [2] and refs. in [2]). We will follow the one given in [2], and begin with a coordinate definition of a constrained system in $\mathbb{R}^n$. All objects considered below (vector fields, functions, differential forms, diffeomorphisms, matrix-valued functions, sections, etc.) are assumed to be smooth (C^∞).

Definition 1.1. A system of the form

$$A(x)\dot{x} = v(x), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $A(x)$ is a matrix-valued function and $v(x)$ is a vector field in $\mathbb{R}^n$ is said to be a constrained system (or constrained vector field).

Constrained vector fields generalize vector fields, i.e. systems

$$\dot{x} = \mu(x), \tag{1.2}$$

since if any $x \in \mathbb{R}^n$ is a *regular point* (i.e. $detA(x) \neq 0$), then (1.1) can be written in the form (1.2), where $\mu(x) = A^{-1}(x)v(x)$. A point which

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is not regular is said to be an impasse point of system (1.1). At impasse points existence or/and uniqueness of solutions breakes down.

We understand in the usual way the notion of a solution and of a phase curve of a constrained system.

Definition 1.2. A solution of a constrained system (1.1) is a differentiable map $t \to x(t)$ from an open interval $I \in \mathbb{R}$ to $\mathbb{R}^n$, such that $A(x(t))x\prime(t) = v(x(t))$ for any $t \in I$. The graph of a solution is said to be a trajectory and the projection of a trajectory to the phase space $\mathbb{R}^n$ along the time-axis is said to be a phase curve.

Example 1.1. The system

$$\dot{x}_1 = 0, \quad x_2\dot{x}_2 = 1/2 \tag{1.3}$$

has a solution

$$x_1(t) = x_1(0), \quad x_2(t) = \sqrt{t + x_2^2(0)}, \quad -x_2^2(0) < t < \infty \tag{1.4}$$

and a solution

$$x_1(t) = x_1(0), \quad x_2(t) = -\sqrt{t + x_2^2(0)} \quad - x_2^2(0) < t < \infty. \tag{1.5}$$

Solution (1.4) starts from the point $(x_1(0), x_2(0) \geq 0)$ and solution (1.5) starts from the point $(x_1(0), x_2(0) \leq 0)$. The phase portrait of system (1.3) is shown in Fig.1,a. The directions of the phase curves correspond to the increase of time. The bold line denotes the set of impasse points (the x_1-axis).

Example 1.2. The phase portrait of the system

$$\dot{x}_1 = 0, \quad x_2\dot{x}_2 = -1/2 \tag{1.6}$$

is different (see Fig.1,b): in a finite time each trajectory reaches the x_1-axis (the set of impasse points).

Example 1.3. The system

$$\dot{x}_1 = 1, \quad x_2\dot{x}_2 = -x_1 \tag{1.7}$$

has two solutions

$$x_1 = x_1(0) + t, \quad x_2 = \pm \sqrt{x_1^2(0) - (x_1(0) + t)^2}, \quad t \in (0, 2|x_1(0)|)$$

starting from an impass point $(x_1(0) < 0, 0)$. Both solutions in a finite time $t = 2|x_1(0)|$ reach another impass point $(|x_1(0)|, 0)$. There are no solutions starting from the origin. Oriented phase curves are shown in Fig.1,g. It is important to note that the "formal integration" of (1.7) gives the first integral $x_1^2 + x_2^2$, but none of the curves

$$x_1^2 + x_2^2 = \text{const}$$

is a phase curve of (1.7).

The phase portrait of systems (1.3) and (1.6) are typical, while that of (1.7) is not. The main result of the present paper is the following one.

Theorem 1.1. *Let E be a generic constrained system on a 2-manifold M, $\alpha \in M$ be an impass point of E. The germ at α of the phase portrait of E is equivalent to the phase portrait of one and only one of the following 5 systems of normal forms:*

$$\text{Normal form 1:} \quad \dot{x}_1 = 0, \ x_2\dot{x}_2 = 1 \tag{1.8}$$

$$\text{Normal form 2:} \quad \dot{x}_1 = 0, \ x_2\dot{x}_2 = -1 \tag{1.9}$$

$$\text{Normal form 3:} \quad \dot{x}_1 = 0, \ (x_1 + x_2^2)\dot{x}_2 = 1 \tag{1.10}$$

$$\text{Normal form 4:} \quad \frac{1}{\lambda}\dot{x}_2 - \dot{x}_1 = 1, \quad (x_2 - x_1)\dot{x}_2 = \lambda x_2, \ |\lambda| > 1 \tag{1.11}$$

$$\text{Normal form 5:} \quad \lambda\dot{x}_1 - \dot{x}_2 = 1 + \lambda^2, x_2\dot{x}_2 = -\lambda x_1 - x_2, \ \lambda > 0 \tag{1.12}$$

The phase portraits of the normal forms are shown in Fig.1, a-f; the set of impass points is shown by a bold line.
a-f: phase portraits of typical singularities of constrained systems on a 2-manifold
g: non-typical phase portarit (of system (1.7)).

Remarks: 1. The precise meaning of the notions of "generic constrained system on a manifold" and "equivalence of phase portraits" will be given in Sections 2 and 3, respectively.
2. The parameter λ is the modulus of normal forms 4 and 5.
3. Classification results for the constrained systems (1.1), under the condition $rank A(x) = const$, have been obtained in [2].

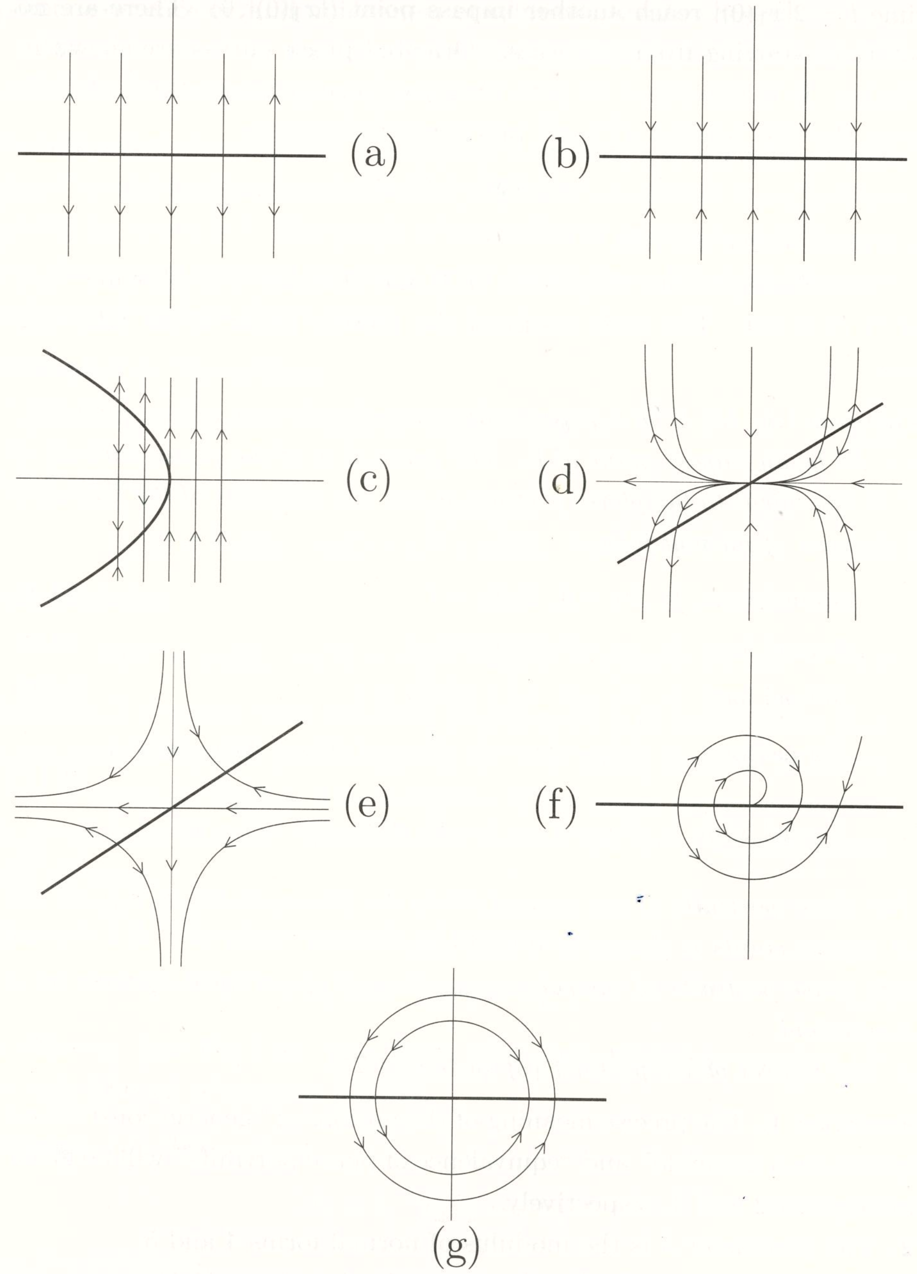

Fig. 1

The paper is organized as follows. In Section 2, we give two invariant definitions of constrained systems - in terms of bundle endomorphisms and in terms of differential 1-forms. In Section 3, the notions of equivalence and of phase equivalence of constrained systems are introduced. Some preliminary normal forms for constrained systems, to be used in the next sections, are given in Section 4. In Section 5, we show that the set of impass points of a generic constrained system on a 2-manifold is a smooth curve and it intersects the phase curves at isolated points. The proof of Theorem 1.1 is based on the notion and properties of extension of a constrained system (Section 6). We show that the phase classification of constrained systems can be reduced to the classification of pairs consisting of a vector field on a manifold and a hypersurface. This problem is related also with the study of vector fields near the boundary of a manifold and of discontinuous vector fields (Refs. [1,3]). In these references a topological classification of generic singularities of the pairs can be found. We give, in proving Theorem 1.1, a smooth classification for 2-dimensional case (Section 8). The notion of extension of a constrained system also allows us to define all typical singularities (Section 7). We complete the proof of Theorem 1.1 in Section 9.

In Section 10, we point out some essential differences to obtain classification results for constrained systems in $\mathbb{R}^n$.

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2. Constrained systems on a manifold

In this section we give two (invariant) definitions of constrained systems, in terms of bundle endomorphisms and in terms of Pfaffian systems, and define the phase equivalence of germs of constrained systems.

Let M be an n-dimensional manifold. By the endomorphism bundle we understand the smooth bundle on M whose fiber at each point $x \in M$ consists of all linear mappings $T_xM \to T_xM$. By a bundle endomorphism we understand a smooth section of the endomorphism bundle.

Definition 2.1. (see [2]). A constrained system on a manifold M is a pair (A, v) consisting of a bundle endomorphism A and a vector field v on M.

Given a constrained system $E = (A, v)$ and $x \in M$ we denote by $A|_x$ the linear mapping $T_xM \to T_xM$ corresponding to A at x, and by $v|_x$ the value of the vector field v at x. A solution of E is a differentiable mapping $t \to x(t)$ of an open interval $I \subset R$ to M such that $A|_{x(t)}\dot{x}(t) = v(x(t))$ for any $t \in I$. Definitions of the trajectories and phase curves are similar to those for the case $M = \mathbb{R}^n$. The set of points $\alpha \in M$ at which $A|_\alpha$ is degenerated is said to be the set of impasse points ($Imp(E)$). A point $\alpha \notin Imp(E)$ is said to be a regular point.

We will use one more definition of a constrained system. Take n independent differential 1-forms $\mu_1, \mu_2, ..., \mu_n$ on M. Let $\mu = (\mu_1, \mu_2, ..., \mu_n)^T$. Apply μ to both right and left hand side of a constrained system

$$A\dot{x} = v, \tag{2.1}$$

where A is a bundle endomorphism and v is a smooth vector field. We obtain the equation

$$\mu(A\dot{x}) = f, \tag{2.2}$$

where $f = (f_1, f_2, ..., f_n)^T$ is a vector function on M, $f = \mu(v)$. For any $\alpha \in M$ and $i \in 1, ..., n$ the operator $\xi \to \mu_i(A\xi), \xi \in T_\alpha M$ is a 1-form on $T_\alpha M$. Therefore, we can define a tuple $\omega = (\omega_1, ..., \omega_n)$ of differential 1-forms:

$$\omega|_\alpha(\xi) = \mu(A|_\alpha(\xi)), \qquad \alpha \in M, \xi \in T_\alpha M \tag{2.3}$$

Now (2.2) can be written in the form

$$\omega(\dot{x}) = f \tag{2.4}$$

This gives sense to the following definition.

Definition 2.2. A constrained system on an n-dimensional manifold M is a pair (ω, f) consisting of an n-tuple ω of differentiable 1-forms on M and a vector function f on M.

A solution of a constrained system (2.4) is a differentiable mapping $t \to x(t)$ of an open interval $I \subset \mathbb{R}$ to M such that $\omega|_{x(t)}(\dot{x}(t)) = f(x(t))$ for any $t \in I$. The set of impasse points can be defined as $\alpha : (\omega_1 \cap ... \cap \omega_n)|\alpha = 0$.

If the tuple ω is related to a bundle endomorphism A by (2.3) and $f = \mu(v)$, then the solutions of (2.4) coincide with those of (2.1) for any tuple μ. The transition from (2.1) to (2.4) in local coordinates $x = (x_1, ..., x_n)$ is as follows. Let $A(x)$ be the matrix of $A|_x$, $v|_x = f_1(x)\partial/\partial x_1 + ... + f_n(x)\partial/\partial x_n$. Take $\mu = (dx_1, ..., dx_n)$. Then (2.1) can be written in the form (2.4), where $f(x) = (f_1(x), ..., f_n(x))$, $\omega = A(x)dx$, $dx = (dx_1, ..., dx_n)^T$.

Example 2.1. The normal form 4 can be written in the form (2.4) where $\omega = (\omega_1, \omega_2), f = (f_1, f_2)^T, \omega_1 = -dx_1 + \lambda^{-1}dx_2$, $\omega_2 = (x_2 - x_1)dx_21$, $f_1 = 1, f_2 = \lambda x_2$.

We will make use the Whitney topology in the set of all constrained systems on a manifold M (see [4]) and say that some property holds for a generic constrained system if there exists a countable number of open sets $A_1, A_2, ...$ of constrained systems such that their intersection A is everywhere dense and the property holds for any constrained system in A.

Throughout the paper we shall use some basic notions of singularity theory and the following variant (or corollary) of the transversality theorem (see [4]).

Proposition 2.1. *Let E be a generic constrained system on a manifold M, S be a singularity class of germs of constrained systems, M_S be the set of points $\alpha \in M$ such that the germ of E at α belongs to S. We have*
a) *if $codimS > n$, then the set M_S is empty,*
b) *if $codimS = n$, then the set M consists of isolated points.*

3. Equivalence of constrained systems

Note that the multiplication of both right and left side of (1.1) by a non-singular matrix-valued function $T(x)$ does not change the trajectories of constrained system (1.1). Another possibility to reduce (1.1) to a simple form is to introduce new coordinates y such that $y = \Phi(x)$ where Φ is a local diffeomorphism. This leads to the following definition (we use form (2.4) for constrained systems).

Definition 3.1. Two germs at $0 \in \mathbb{R}^n$ of constrained systems $\omega(\dot{x}) = f(x)$ and $\widetilde{\omega}(\dot{x}) = \widetilde{f}(x)$ are said to be equivalent if there exists a local diffeomorphism Φ and a smooth matrix-valued function $T = T(x), detT(0) \neq 0$, such that

$$\widetilde{\omega} = T\Phi^*\omega, \quad \widetilde{f} = Tf(\Phi).$$

If two germs of constrained systems are equivalent, then there exists a diffeomorphism transforming the germ of the set of impasse points of the first system into that of the second, and transforming every trajectory of the first system into a trajectory of the second. If we are only interested in the phase portraits then we can make use a weaker equivalence.

Definition 3.2. Two germs at $0 \in \mathbb{R}^n$ of constrained systems E_1 and E_2 are said to be phase-equivalent if there exists neighbourhoods U and V of $0 \in \mathbb{R}^n$ and a diffeomorphism $\Phi : U \to V$ such that

a) Φ transforms orientated phase curves of E_1 belonging to U into orientated phase curves of E_2. Φ^{-1} transforms orientated phase curves of E_2 belonging to V into orientated phase curves of E_1,

b) $\Phi(Imp(E_1) \cap U) = Imp(E_2) \cap V$.

This definition is similar to the definition of orbital equivalence of vector fields (see [5]).

Definitions of equivalence and phase equivalence of germs of constrained systems on a manifold can be reduced to Definitions 3.1 and 3.2 in a standard way. Let M be an n-dimensional manifold, $\alpha \in M$. The space of germs at α of constrained systems on M can be identified with the space of germs at $0 \in \mathbb{R}^n$ of constrained systems on $\mathbb{R}^n$. Using this identification we will denote by $\pi_\alpha E$ the germ at $0 \in \mathbb{R}^n$ corresponding to the germ at α of a constrained system E.

Definition 3.3. Let E_1 and E_2 be constrained systems on a manifgold M, $\alpha, \beta \in M$. The germ at α of E_1 is equivalent (phase-equivalent) to the germ at β of E_2 if the germs $\pi_\alpha(E_1)$ and $\pi_\beta(E_2)$ are equivalent (phase-equivalent).

By equivalence of local phase portraits we mean, in the formulation of Theorem 1.1, the phase-equivalence of germs of constrained systems.

4. Local classification of Pfaffian systems on a 2-manifold. Preliminary normal forms

The possibility to present a constrained system in form (2.4) shows that the problem of smooth classification of germs of constrained systems on 2-manifolds includes the problem of smooth classification of germs at $0 \in \mathbb{R}^2$ of Pfaffian systems $\omega_1 = 0, \omega_2 = 0$, where ω_1 and ω_2 are differential 1-forms. In other words, we have to classify germs (ω_1, ω_2) with respect to the following equivalence: (ω_1, ω_2) is equivalent to $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ if there exists a local diffeomorphism Φ and a matrix-valued function H such that $detH(0) \neq 0$ and

$$H(\Phi_*\omega_1, \Phi_*\omega_2)^T = (\widetilde{\omega}_1, \widetilde{\omega}_2)^T.$$

Given differential 1-forms ω_1 and ω_2 on a 2-manifold M, we introduce the set

$$S_{\omega_1,\omega_2} = \{\alpha \in M : \omega_1 \cap \omega_2)|_\alpha = 0\}.$$

Theorem 4.1. *Let ω_1 and ω_2 be generic smooth differential 1-forms on a 2-manifold M. Then*

1) *if $\alpha \in S_{\omega_1,\omega_2}$, then $Ker\omega_1|_\alpha \cap Ker\omega_2|_\alpha$ is a 1-dimensional subspase of $T|_\alpha M$,*

2) *the set $S_{j^1_\alpha\omega_1, j^1_\alpha\omega_2}$ is a smooth curve in a neighbourhood of any point $\alpha \in S_{\omega_1,\omega_2}$,*

3) *a germ of (ω_1, ω_2) at a point $\alpha \in M$ is equivalent to one of the following germs:*

$$(dx_1, dx_2) \tag{4.1}$$

if $\alpha \in S_{\omega_1,\omega_2}$;

$$(dx_1, x_2dx_2) \tag{4.2}$$

if $\alpha \in S_{\omega_1,\omega_2}$ and S_{ω_1,ω_2} is transversal to $Ker\omega_1|_\alpha \cap Ker\omega_2|_\alpha$ at α;

$$(dx_1, (x_1 - x_2^2)dx_2) \tag{4.3}$$

if $\alpha \in S_{\omega_1,\omega_2}$ and S_{ω_1,ω_2} is tangent at α to $Ker\omega_1|_\alpha \cap Ker\omega_2|_\alpha$.

The same result in terms of the presentation of constrained systems in form (2.1) is as follows.

Theorem 4.2. *Let E be a generic smooth constrained system on a 2-manifold M. Assume that E has the form (2.1). Then*

1) *if α is an impasse point of E, then $dimKerA|_\alpha = 1$,*
2) *the set of impasse points of a constrained system $(j^1_\alpha A)\dot{x} = v$ (or, equivalently, the set $\alpha : det(j^1_\alpha A) = 0$) is a smooth curve in a neighbourhood of any impass point α,*
3) *a germ of E at a point $\alpha \in M$ is equivalent to one of the following germs:*

$$\dot{x}_1 = f_1(x), \quad \dot{x}_2 = f_2(x) \tag{4.4}$$

if α is a regular point of E;

$$\dot{x}_1 = f_1(x), \quad x_2\dot{x}_2 = f_2(x) \tag{4.5}$$

if α is an impasse point and $Imp(E)$ is transversal to $KerA|_\alpha$ at α;

$$\dot{x}_1 = f_1(x), \quad (x_1 - x_2^2)\dot{x}_2 = f_2(x) \tag{4.6}$$

if α is an impasse point of E and $Imp(E)$ is tangent at α to $KerA|_\alpha$.

Proof of Theorem 4.1. **1.** The degeneration $\omega_1|_\alpha = \omega_2|_\alpha = 0$ has codimension 4 and by the transversality theorem it takes place at no points $\alpha \in M$.

2. Let $\alpha \in S_{\omega_1,\omega_2}$. We can assume that $\omega_1|_\alpha \neq 0$. Then there exist local coordinates $x = (x_1, x_2)$ and a function $R = R(x), R(\alpha \neq 0$, such that the germ at α of (ω_1, ω_2) is equivalent to the germ $(dx_1, g(x)dx_2)$,where $g(0) \neq 0$. The set S_{ω_1,ω_2} is given locally by the equation $j^1 g = 0$. The degeneration $\frac{\partial g}{\partial x_1}(\beta) = \frac{\partial g}{\partial x_2}(\beta) = g(\beta) = 0$ has codimension 3 and by the

transversality theorem it takes place at no points $\beta \in M$. The second statement of the theorem follows.

3. We have shown that at any point a germ of (ω_1, ω_2) is reducible to

$$(\omega_1 = dx_1, \omega_2 = g(x)dx_2). \tag{4.7}$$

If $\alpha \notin S_{\omega_1,\omega_2}$, then $g(\alpha) \neq 0$ and (ω_1, ω_2) is equivalent to (dx_1, dx_2). Consider the case $\alpha \in S_{\omega_1,\omega_2}$, i.e. $g(0) = 0$. Note that locally near α, $S_{\omega_1,\omega_2} = x : g(x) = 0$ and $Ker\omega_1|_\alpha \cap Ker\omega_2|_\alpha$ is generated by $\frac{\partial}{\partial x_2}$. If S_{ω_1,ω_2} is transversal to $Ker\omega_1|_\alpha \cap Ker\omega_2|_\alpha$ at α then $\frac{\partial g}{\partial x_2}(\alpha) \neq 0$. In this case we can, preserving ω_1, reduce $g(x)$ to x_2 and ω_2 to $\widetilde{\omega}_2 = r_1(x)x_2dx_2 + r_2(x)dx_1$, where $r_1(\alpha) \neq 0$. It is clear that the germ $(\omega_1, \widetilde{\omega}_2)$ is equivalent to the germ (4.2).

It remains to consider the case where the transversality condition is violated, i.e. $\frac{\partial g}{\partial x_2}(\alpha) = 0$. Using once again the transversality theorem we can prove that the tangency of S_{ω_1,ω_2} and $Ker\omega_1|_\alpha \cap Ker\omega_2|_\alpha$ is simple, i.e. $\frac{\partial^2 g}{\partial x_2^2}(\alpha) \neq 0$. This condition allows us to introduce new local coordinates (we denote them by the same letters x_1 and x_2) in which $\omega_1 = dx_1, \omega_2 = (\gamma(x_1)^+_- x_2^2)r_1(x)dx_2 + r_2(x)dx_1$, where $r_1(0) \neq 0$, $\gamma(0) = 0$. The set $S_{j^1_\alpha \omega_1, j^1_\alpha \omega_2}$ is given by the equation $\gamma(x_1) = 0$, therefore $\gamma\prime(\alpha) \neq 0$. Using this condition we can reduce (ω_1, ω_2) to $(q_1(x)dx_1, q_2(x)(x_1 +_- x_2^2)dx_2 + q_3(x)dx_1)$, where $q_1(0) \neq 0$, $q_2(0) \neq 0$. It is clear that this germ is equivalent either to the germ (4.3) or to the germ $(dx_1, (x_1 + x_2^2)dx_2)$. Changing x_1 by $-x_1$ in the latter normal form and multiplying both ω_1 and ω_2 by -1 we obtain the germ (4.3).

5. Generic constrained systems

In this Section we prove that a generic constrained system E on a 2-manifold satisfies the following 2 conditions:

Condition A. The set $Imp(E)$ is a smooth curve;

Condition B. Any phase curve of E either does not intersect $Imp(E)$ or intersects $Imp(E)$ at isolated points.

Condition A follows from the second statement of Theorem 4.2. To prove condition B, assume that α is a point of intersection of a phase curve of system (2.1) and the set of its impasss points. Then $rankA|_\alpha < 2$ and $v|_\alpha \in ImA|_\alpha$. This degeneration has codimension 2 and condition B follows from the transversality theorem.

6. Extensions of constrained systems

In this section we show that phase classification of constrained systems can be reduced to the classification of pairs consisting of a vector field and a manifold.

Let E be a constrained system on a manifold M satisfying condition A (Section 5). Take an impasse point $\alpha \in M$ and a germ U of its neighbourhood. Define germs U_E^+ and U_E^- of open sets such that

$$U_E^+ \cap U_E^- =, \quad U = (U \cap Imp(E)) \cup U_E^+ \cup U_E^-.$$

The germs U_E^+ and U_E^- are defined up to their order. Note that the restrictions of E to U_E^+ and U_E^- are germs of smooth vector fields on U_E^+ and U_E^-, respectively.

Definition 6.1. We say that a germ at $\alpha \in M$ of a vector field μ is an extension of the germ at α of E if the following conditions are valid:
a) orientated phase curves of $\mu|U_E^+$ coincide with those of $E|U_E^+$,
b) orientated phase curves of $(-\mu)|U_E^-$ coincide with those of $E|U_E^-$.

Example 6.1. The germ at $0 \in \mathbb{R}^2$ of the vector field

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 1$$

is an extension of the normal form 1 ($U^+ = \{(x_1, x_2) : x_2 > 0\}$, $U^- = \{(x_1, x_2) : x_2 < 0\}$).

Proposition 6.1. Any germ at an impasse point $\alpha \in M$ of a generic constrained system on M has an extension.

Proof. Take local coordinates in which the germ has the form (1.1). Let $\xi(x) = detA(x)$. Define a vector field $\widetilde{\mu}(x)$ on the set of regular points: $\widetilde{\mu}(x) = \xi(x)A^{-1}(x)v(x)$. It is clear that there exists a smooth extension of $\widetilde{\mu}(x)$ to the set of impasse points, i.e. a smooth vector field $\mu(x)$ defined in a neighbourhood of α and such that $\mu(x) = \widetilde{\mu}(x)$ at each

regular point. Let U be a sufficiently small neighbourhood of α. Using the second statement of Theorem 4.2, we can define the sets U^+ and U^- as follows:

$$U^+ = \{(x_1, x_2) \in U : \xi(x) > 0\}, \quad U^- = \{(x_1, x_2) \in U : \xi(x) < 0\}.$$

Since the multiplication of a vector field by a positive function does not change its orientated phase curves and the multiplication by a negative function changes the orientation only, we can conclude that the vector field $\mu(x)$ is an extension of the germ (1.1).

Example 6.2. Germ (4.5) has an extension

$$\dot{x}_1 = x_2 f_1(x), \quad \dot{x}_2 = f_2(x); \tag{6.1}$$

germ (4.6) has an extension

$$\dot{x}_1 = (x_1 - x_2^2) f_1(x), \quad \dot{x}_2 = f_2(x). \tag{6.2}$$

Given a constrained system $E, E = (A, v)$ denote by $-E$ the constrained system $(A, -v)$.

Proposition 6.2. *Let E_1 and E_2 be germs at $0 \in R^2$ of generic constrained systems and v_1 and v_2 be their extensions, respectively. Then*
a) *if E_1 is phase equivalent to E_2, then v_1 is phase equivalent either to v_2 or to $-v_2$,*
b) *if there exists a local diffeomorphism Φ transforming orientated phase curves of v_1 into those of v_2 and such that $\Phi(Imp(E_1)) = Imp(E_2)$, then E_1 is phase-equivalent either to E_2 or to $-E_2$.*

Proof. We will prove the first statement, the proof of the second being similar. Let Φ be a local smooth diffeomorphism (homeomorphism) transforming orientated phase curves of E_1 into those of E_2 and the set $Imp(E_1)$ into the set $Imp(E_2)$. Then, either

$$i) \quad \Phi(U_{E_1}^+) = U_{E_2}^+, \quad \Phi(U_{E_1}^-) = U_{E_2}^-,$$

or

$$ii) \quad \Phi(U_{E_1}^+) = U_{E_2}^-, \quad \Phi(U_{E_1}^-) = U_{E_2}^+.$$

Consider the case i). Take an orientated phase curve $\gamma \subset U_{E_1}^+$ of the vector field v_1. Then γ is an orientated phase curve of E_1 and $\Phi(\gamma)$

is an orientated phase curve of E_2. Since $\Phi(\gamma) \subset U^+_{E_2}$, then $\Phi(\gamma)$ is an orientated phase curve of v_2. Take now an orientated phase curve $\gamma \subset U^-_{E_1}$ of the vector field v_1. It is an orientated phase curve of $-E_1$. The curve $\Phi(\gamma)$ is an orientated phase curve of $-E_2$. It belongs to $U^-_{E_2}$, therefore $\Phi(\gamma)$ is an orientated phase curve of v_2.

We have shown that Φ transforms any orientated phase curve of v_1 which does not intersect the set of impasse points into an orientated phase curve of E_2. The smooth (topological) phase-equivalence of v_1 and v_2 follows now from condition B.

The arguments in case ii) are similar and we omit them.

7. Singularities of constrained systems on 2-manifolds

In this Section we define singularity classes consisting of germs at impasse point $0 \in \mathbb{R}^2$ of constrained systems which are equivalent to the normal forms given in Section 1.

Let Q be the set of all germs at an impasse point $0 \in \mathbb{R}^2$ of constrained systems on $\mathbb{R}^2$.

Introduce two subclasses of Q:

- a singularity class Q_1, consisting of germs $(A, v) \in Q$ satisfying the following conditions
  a) $dimKerA|_0 = 1$,
  b) $Imp(j^1A, v)$ is a smooth curve (and, therefore $Imp(A, v)$ is a smooth curve),
  c) $v|_0 \notin ImA|_0$,
- a singularity class Q_2, consisting of germs $(A, v) \in Q$ satisfying conditions (a) and (b), but not (c) and such that
  d) the curve $Imp(A, v)$ is transversal to $KerA|_0$ at $0 \in \mathbb{R}^2$.

Lemma 7.1. *We have that*
1) $codimQ_1 = 1, \quad codimQ_2 = 2 \quad (inQ)$,
2) *let E be a generic smooth constrained system on a 2-manifold M, $\alpha \in M$ and $\pi_\alpha E \in Q$. Then, $\pi_\alpha E \in Q_1 \cup Q_2$.*

Proof. Let $\pi_\alpha E = (A, v)$. By Theorem 4.2, conditions (a) and (b) are valid and what we have to prove is that if $v|_0 \in ImA|_0$ then condition (d)

is valid. In Section 4 we proved that any germ of a generic constrained system on a 2-manifold is reducible to the normal form

$$\dot{x}_1 = f_1(x), \quad g(x)\dot{x}_2 = f_2(x) \tag{7.1}$$

A germ belongs to Q_1 if

$$g(0) = 0, \quad dg|_0 \neq 0, \quad f_2(0) \neq 0 \tag{7.2}$$

and it belongs to Q_2 if

$$g(0) = 0, \quad f_2(0) = 0, \quad \frac{\partial g}{\partial x_2}(0) \neq 0 \tag{7.3}$$

Now the lemma follows from the transversality theorem.

Let us divide Q_1 into the following two subclasses

$$Q_{1,1} = \{(A, v) \in Q_1 : Imp(A, v) \ \ is \ \ transversal \ \ to \ \ KerA|_0\},$$

$$Q_{1,2} = \{(A, v) \in Q_1 : Imp(A, v) \ has \ a \ simple \ tangency \ with \ KerA|_0\}.$$

Lemma 7.2. *We have that*
1) *codim* $Q_{1,1} = 1, \quad$ *codim* $Q_{1,2} = 2 \quad$ *(in Q)*,
2) *Let E be a generic smooth constrained system on a 2- manifold M, $\alpha \in M$ and $\pi_\alpha E \in Q_1$. Then $\pi_\alpha E \in Q_{1,1} \cup Q_{1,2}$.*

Proof. The lemma follows from the transversality theorem: in coordinates of the normal form (7.1), a germ belongs to $Q_{1,1}$ if

$$g(0) = 0, \quad \frac{\partial g}{\partial x_2}(0) \neq 0, \quad f_2(0) \neq 0; \tag{7.4}$$

a germ belongs to $Q_{1,2}$ if

$$g(0) = 0, \quad \frac{\partial g}{\partial x_2}(0) = 0, \quad f_2(0) \neq 0, \quad \frac{\partial^2 g}{\partial x_2^2}(0) \neq 0. \tag{7.5}$$

We will also divide Q_2 into subclasses using extensions of constrained systems defined in Section 6.

Lemma 7.3. *Let $(A, v) \in Q_2$ and μ be a smooth extension of (A, v). Then $\mu(0) = 0$.*

Proof. In Section 4 we shown that any germ satisfying conditions (a), (b) and (d) is reducible to the normal form (4.5). For germs of $Q_{1,1}$

the condition $f_2(0) = 0$ holds. On the other hand, the germ (6.1) is an extension of (4.5) and the lemma follows.

Let $(A, v) \in Q_2$ and μ be an extension of (A, v). Let λ_1 and λ_2 be the eigenvalues of μ. Define the singularity subclasses

$$Q_{2,1} = \{(A, v) \in Q_2 : \quad Im\lambda_1 = Im\lambda_2 = 0, \quad \lambda_1\lambda_2 \notin Q,$$

$$Q_{2,2} = \{(A, v) \in Q_2 : \quad Re\lambda_1 = Re\lambda_2 \neq 0, \quad Im\lambda_1 = -Im\lambda_2 \neq 0\}.$$

Lemma 7.4. *Let E be a generic constrained system on a 2- manifold M, $\alpha \in M$ and $\pi_\alpha E \in Q_2$. Then $\pi_\alpha E \in Q_{2,1} \cup Q_{2,2}$.*

Proof. The eigenvalues of the vector field (6.1) are the roots of the equation $\lambda^2 - \lambda c - ab = 0$, where

$$a = f_1(0), \quad b = \frac{\partial f_2}{\partial x_1}(0), \quad c = \frac{\partial f_2}{\partial x_2}(0).$$

Using this and the transversality theorem we obtain the result.

Theorem 7.1. *Let E be a generic constrained system on a 2- manifold M. Then*

1) *for any impasse point $\alpha \in M$, the germ $\pi_\alpha E$ belongs to one and only one of the singularity classes*

$$Q_{1,1}; \; Q_{1,2}; \; Q_{2,1}; \; Q_{2,2}, \tag{7.6}$$

2) *let $M_{1,1}$; $M_{1,2}$; $M_{2,1}$; $M_{2,2}$ be the sets of points at which the germ of E belongs to the singularity classes (7.6), respectively. Assume that none of these sets is empty. Then*

a) *the closure of $M_{1,1}$ is a smooth curve,*

b) *the sets $M_{1,2}$; $M_{2,1}$; $M_{2,2}$ consist of isolated points of the closure of $M_{1,1}$,*

c) *the germs of singularity classes (7.6) are phase-equivalent to the following normal forms given in Section 1:*

germs of the singularity class $Q_{1,1}$ are equivalent either to the normal form 1 or to the normal form 2;

germs of the singularity class $Q_{1,2}$ are equivalent to the normal form 3;

germs of the singularity class $Q_{2,1}$ are equivalent to the normal form 4;

germs of the singularity class $Q_{2,2}$ are equivalent to the normal form 5.

We have already proved the first statement. The second one follows from the results of this section and the transversality theorem. The third statement, from which Theorem 1.1 follows, will be proved in Section 9.

8. Classification of pairs consisting of a vector field and a curve

In Section 6 we shown that the classification of constrained systems can be reduced to a classification of pairs (v, f) consisting of a germ at $0 \in \mathbb{R}^2$ of a vector field and a function-germ $f : \mathbb{R}^2 \to \mathbb{R}$, $f(0) = 0$, $df|_0 \neq 0$. The equivalence relation is as follows: (v, f) is equivalent to $(\widetilde{v}, \widetilde{f})$ if there exists a diffeomorphism Φ transforming phase curves of v into those of $\widetilde{v}$ (the orientation might not be preserved) and transforming the curve $\{f = 0\}$ into the curve $\{\widetilde{f} = 0\}$. In this section we give a classification with respect to this equivalence.

We begin with the simplest case where $v|_0 \neq 0$.

Lemma 8.1. *Let $v|_0 \neq 0$ and $v|_0$ be transversal to the curve $f = 0$ (i.e. $v(f)|_0 \neq 0$). Then the germ (v, f) is equivalent to the germ $(\partial/\partial x_1, x_1)$.*

Proof. We can assume that $v = \partial/\partial x_1$. Then $\partial f/\partial x_1(0) \neq 0$ and there exists a coordinate transformation $\Phi : x_1 \to (x_1, x_2)$, $x_2 \to x_2$ such that $\Phi_*(\partial/\partial x_1) = H\partial/\partial x_1$, $f(\Phi) = x_1$, $H(0) \neq 0$. The lemma follows.

Lemma 8.2. *Let $v|_0 \neq 0$ and $v|_0$ has a simple tangency with the curve $f = 0$ (i.e. $v(f)|_0 = 0$, $v^2(f)|_0 \neq 0$). Then the germ (v, f) is equivalent to the germ $(\partial/\partial x_1, \; x_2 - x_1^2)$.*

Proof. We can assume that $v = \partial/\partial x_1$. Then $\partial f/\partial x_1(0) = 0$, $\partial^2 f/\partial x_1^2(0) \neq 0$, therefore (v, f) is equivalent to $(\partial/\partial x_1, \; x_1^2 + \gamma(x_2))$, where γ is a function-germ. Since $df|_0 \neq 0$, then $\gamma(0) = 0$ and $\gamma(0) \neq 0$. Therefore, we can reduce γ to x_2 and the lemma follows.

Now let us consider the more difficult case $v|_0 = 0$. Normal forms for this case are similar to normal forms obtained in [3].

Lemma 8.3. *Let $v|_0 = 0$, the eigenvalues λ_1 and λ_2 of v are real, $\lambda_1/\lambda_2 \notin \mathbf{Q}$. Assume also that the curve $\{f = 0\}$ is transversal to the eigenvectors of the linearization of v. Then the germ (v, f) is equivalent to the germ $\widetilde{v}_1, x_1 - x_2$, where*

$$\widetilde{v}_1 = x_1\partial/\partial x_1 + \lambda x_2\partial/\partial x_2, \quad |\lambda| > 1. \tag{8.1}$$

Proof. By Chen's linearization theorem we can assume that v has form (8.1) (see [5]). The transversality of the curve $\{f = 0\}$ to the stable and unstable manifold of v means that $\partial f/\partial x_1(0) \neq 0$ and $\partial f/\partial x_2(0) \neq 0$. The transformations $x_1 \to q_1 x_1$, $x_2 \to q_2 x_2$ preserve v and using them we can reduce $f(x_1, x_2)$ to the form $F(x_1, x_2) = x_2 - x_1 + o(||x||)$. Then the curve $\{f = 0\}$ can be given by the equation $x_2 = x_1 + x_1^2\mu(x_1)$, where $\mu(x_1)$ is some germ. The transformations $x_1 \to x_1(1 + h(x_1))$, $x_2 \to x_2(1 + h(x_1))^\lambda$ are locally smooth and they preserve the phase portrait of v (for any germ $h(x_1), h(0) = 0$). To prove the lemma it suffices to show that one of these transformations reduces the curve $\{f = 0\}$ to $\{x_2 = x_1\}$. This condition leads to the following equation for $h = h(x_1)$: $F(x_1, h) = (1 + h)^\lambda - 1 - x_1^2(1 + h)^2\mu(x_1(1 + h)) = 0$. This equation has a solution $h = h(x_1)$, $h(0) = 0$, since near $(0, 0)$ F is a smooth function, $F(0, 0) = 0$ and $\partial F/\partial h(0, 0) = \lambda - 1 \neq 0$.

Lemma 8.4. *Let $v|_0 = 0$, λ_1, λ_2 be the eigenvalues of v and $Re\lambda_1 = Re\lambda_2 \neq 0$, $Im\lambda_1 = -Im\lambda_2 \neq 0$. Then the germ (v, f) is equivalent to the germ $(\widetilde{v}_1, x_1 - x_2)$, where*

$$\widetilde{v}_1 = (-x_1 + \lambda x_2)\partial/\partial x_1 + (-\lambda x_1 - x_2)\partial/\partial x_2, \quad \lambda > 0. \tag{8.2}$$

Proof. We can assume that v has linear normal form (8.2) (see [5]). The transformations

$$x_1 \to \exp(-\gamma(x_1)/\lambda)(x_1 \cos\gamma(x_1) + x_2 \sin\gamma(x_1)),$$

$$x_2 \to \exp(-\gamma(x_1)/\lambda)(-x_1 \sin\gamma(x_1) + x_2 \cos\gamma(x_1)) \tag{8.3}$$

do not change the phase portrait of v for any function $\gamma(x_1)$ (we can check this using polar coordinates). Assume that the curve $\{f = 0\}$ is given by the equation $x_1 = x_G(x_2)$ (the case where the curve is given

by the equation $x_2 = x_1(x_1)$ can be similarly considered). Let us show that one of the transformations (8.3) reduces the curve to $\{x_2 = 0\}$. To prove this we have to show the solvability of the equation

$$F(x_1, \gamma) = \cos\gamma + \sin\gamma G(-e^{-\gamma/\lambda}x_1 \sin\gamma) = 0. \tag{8.4}$$

We can take $\alpha \in R$ such that

$$\cos\alpha + G(0)\sin\alpha = 0. \tag{8.5}$$

Then $F(0, \alpha) = 0$, and to prove the solvability of (8.4) with respect to $\gamma = \gamma(x_1)$, $\gamma(0) = \alpha$ we have to show that $\partial F/\partial\gamma(0, \alpha) \neq 0$, that is

$$-\sin\alpha + G(0)\cos\alpha \neq 0. \tag{8.6}$$

But condition (8.6) follows from (8.5). The lemma follows.

9. Proof of Theorem 1.1

In this section we prove the third statement of Theorem 7.1 and Theorem 1.1.

End of proof of Theorem 7.1. Let e be a germ of one of the singularity classes (7.6). Then the germ e is is phase-equivalent to the normal form (7.1). This normal form has an extension

$$g(x)f_1(x)\partial/\partial x_1 + f_2(x)\partial/\partial x_2. \tag{9.1}$$

In the case $e \in Q_{1,1}$ conditions (7.4) hold. These conditions imply that the extension (9.1) is transversal at $0 \in \mathbb{R}^2$ to the impasse curve $\{g(x) = 0\}$. By Lemma 8.1 and Proposition 6.2, the germ e is phase-equivalent either to the normal form 1 or to the normal form 2.

In the case $e \in Q_{1,2}$ conditions (7.5) hold. They imply that the extension (9.1) has a simple tangency with the impasse curve $\{g(x) = 0\}$. Denote by e_3 the normal form 3. By Lemma 8.2 and Proposition 6.2, the germ e is phase-equivalent either to the germ e_3 or to $-e_3$. The change of the coordinate x_2 by $-x_2$ transforms the germ $-e_3$ to e_3, therefore e is phase-equivalent to e_3.

Consider now the case $e \in Q_{2,1} \cup Q_{2,2}$. By Theorem 4.2, e is reducible to the normal form (4.5). The vector field (6.1) is an extension of the normal form (4.5). Using the condition that 0 is not an eigenvalue of the vector field (6.1), we can conclude that the vector $\partial/\partial x_1$ is not an eigenvector of the linearization of (6.1). Therefore the eigenvectors of the linearization of (6.1) are transversal to the impasse curve $\{x_2 = 0\}$. By Lemma 8.3 and Lemma 8.4, there exists a diffeomorphism transforming the curve $\{x_2 = 0\}$ into the curve $\{x_1 = x_2\}$ and transforming the phase curves of the vector field (6.1) into those of one of the fields (8.1) or (8.2). Denote by e_4 and e_5 the normal forms 4 and 5, respectively. Note that $Imp(e_4) = Imp(e_5) = x_1 = x_2$, the vector field (8.1) is an extension of e_4 and the vector field (8.2) is an extension of e_5. By Proposition 6.2, the germ e is phase-equivalent to one of the germs $e_4, e_5, -e_4, -e_5$. The change of coordinates $x_1 \to -x_1$, $x_2 \to -x_2$ transforms the germ $-e_4$ into the germ e_4 and the germ $-e_5$ to e_5. Therefore, the germ e is phase-equivalent either to the normal form 4 or to the normal form 5. The proof is now complete.

Proof of Theorem 1.1. Let E be a generic constrained system on a 2-manifold M, and α a point in M. By Theorem 7.1, the germ of E at α is phase-equivalent to one of the normal forms 1-5. It remains to show that the normal form 1 is not phase-equivalent to the normal form 2 and that none of the different germs of the forms (1.11) or (1.12) are phase-equivalent. The first statement follows from the stability of the manifold of impasse points for normal form 1 and its instability for normal form 2 (see Fig.1,a,b). The second statement follows from Proposition 6.2: the vector field (8.1) is an extension of the normal form 4, the vector field (8.2) is an extension of the normal form 5 and none of the different vector fields v_1 and v_2 of the forms (8.1) or (8.2) are orbitally equivalent (i.e. v_1 is phase-equivalent neither to v_2 nor to $-v_2$, see [5]).

10. Discussion on the classification of constrained systems on $\mathbb{R}^n$, $n \geq 3$

Trying to extend the obtained results to constrained systems $A(x)\dot{x} = F(x)$, $x \in \mathbb{R}^n$, $n \geq 3$ one meets the following difficulties:

1. If $n \geq 4$, then the degeneracy $rankA(0) = n - 2$ becomes typical. It

is not even clear how to obtain a normal form for the matrix $A(x)$ and a scheme of adjacencies in this case.

2. Let $rankA(0) = n-1$. The hypersurface of impasse points $\{detA(x) = 0\}$ might be tangent to $KerA(0)$ and, on the top of this, $F(0) \in ImA(0)$. This codimension 3 degeneracy is not typical in the 2-dimensional case.

3. Let $rankA(0) = n - 1$ and the hypersurface of impasse points be transversal to $KerA(0)$, but $F(0) \in ImA(0)$. Arguments similar to those in the 2- dimensional case show that the system can be reduced to

$$\dot{x}_1 = f_1(x), \ldots, \dot{x}_{n-1} = f_{n-1}(x), x_n\dot{x}_n = f_n(x)$$

where $f_n(0) = 0$. To investigate the phase portrait of this system one has to deal with a pair consisting of the vector field μ

$$\dot{x}_1 = x_n f_1(x), \ldots, \dot{x}_{n-1} = x_n f_{n-1}(x), \dot{x}_n = f_n(x)$$

(the extension of the constrained system, see Section 6) and the hypersurface of impasse points $\{x_n = 0\}$. The vector field μ has a codimension 2 submanifold L of singular points, $L = \{x_n = f_n(x) = 0\}$, while in the 2- dimensional case the singular points of the extension of a constrained system are typically isolated.

4. Furthemore, at any point of the submanifold L the vector field μ has $n - 2$ zero eigenvalues and two more eigenvalues λ_1, λ_2. There are points of L (forming a codimension one subset) at which the tuple λ_1, λ_2 is resonant. In particular, there are points at which it is not hyperbolic ($Re\lambda_1 Re\lambda_2 = 0$). This is a typical degeneracy (of codimension three), and there is no analogous degeneracy in typical 2-dimensional case.

Note, that the degeneracies 1) and 4) seem to be most difficult. The work on typical singularities in n-dimensional case is in development.

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