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A Constrained Minimization Problem With Integrals on the Entire Space

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—Dedicated to the memory of Antonio Gilioli (1945-1989)

Abstract. In this paper we consider the question of minimizing functionals defined by improper integrals. Our approach is alternative to the method of concentration-compactness and it does not require the verification of strict subaddivity.

I. Introduction

In this paper we study the problem of minimizing

$$V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\mathrm{grad}\, u(x)|^2 dx + \int_{\mathbb{R}^N} F(u(x))dx$$

subject to

$$I(u) = \int_{\mathbb{R}^N} G(u(x))dx = \lambda \neq 0 \ .$$

This minimization problem is considered in the space $H^1(\mathbb{R}^N)$; under certain growth assumptions $V(u)$ and $I(u)$ are well defined smooth functionals on $H^1(\mathbb{R}^N)$.

This problem has been studied by many authors in connection with the existence of solution of a semilinear elliptic equation (or system) and/or the existence and stability of special solutions of some evolution equation. References [**1**] to [**8**] are a partial list of papers about this topic.

As far as the convergence of minimizing sequences is concerned, our approach is based on Theorem I which states, using the terminology adopted in [**3**], that dichotomy never occurs in the problem above; so,

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all we have to worry about is to avoid vanishing minimizing sequences.

Our growth assumptions are much more restrictive than in [**3**], for instance (because we assume two sided growth conditions on $F(u)$, $G(u)$ and their first and second derivatives), but we allow $G(u)$ to change sign and not to be even.

II. Statement of the Results

Let $V(u)$ and $I(u)$ be as above. For $N \geq 2$ we set $l(N) = \frac{2N}{N-2}$ and denote by $M = \{u \in H^1(\mathbb{R}^N) : I(u) = \lambda\}$ the admissible set (which is supposed to be non empty) and by $f(u)$ and $g(u)$ the derivatives of $F(u)$ and $G(u)$. We rewrite $F(u)$ and $G(u)$ in the form $F(u) = mu^2 + F_1(u)$ and $G(u) = m_0u^2 + G_1(u)$ and we make the following assumptions:

H1. $F_1(u)$ and $G_1(u)$ are C^2 functions with $F_1(0) = G_1(0) = 0 = F_1'(0) = G_1'(0)$ and for some constant k and $2 < q \leq p < l(N)$ we have

$$|F_1''(u)|,\ \ |G_1''(u)| \leq k(|u|^{q-2} + |u|^{p-2});$$

H2. V is bounded below on M and any minimizing sequence is bounded in $H^1(\mathbb{R}^N)$;

H3. if $u \in H^1(\mathbb{R}^N)$ and $u \not\equiv 0$, then $g(u(\cdot)) \not\equiv 0$.

Remarks.

1. If $N = 1$ we assume $F_1(u)$ and $G_1(u)$ are C^2 functions satisfying $F_1(0) = F_1'(0) = F_1''(0) = 0 = G_1(0) = G_1'(0) = G_1''(0)$.

2. Assumption H_3 is satisfied if $g(u) \neq 0$ for $u \neq 0$ and small.

3. for $N = 3$ we give two examples verifying assumption H_2:

a) $G(u) = u^2$ and $\lim_{u\to+\infty} F_-(u)/|u|^{\frac{10}{3}} = 0$, where $F_-(u)$ is the negative part of $F(u)$; this type of growth condition has also appeared in [**3**], part II, page 240; the fact that H_2 is satisfied is a consequence of the interpolation inequality $|u|_{Lp} \leq C|\text{grad } u|^a_{L^2}|u|^{1-a}_{L_2}$ with $a = \frac{3}{2} - \frac{3}{p}$. Since we need $ap < 2$ we should ask for $p < \frac{10}{3}$ but the fact the limit above is zero is sufficient for

$$\int_{\mathbb{R}^3} |\text{grad } u(x)|^2 dx \text{ to dominate } \int_{\mathbb{R}^3} F(u(x))dx.$$

b) $G(u) = u^3 + u^5$ and $F(u) = u^2 + u^4$.

Under assumptions H1, H2 and H3 our results are the following:

Theorem I. *If u_n is a minimizing sequence and u_n converges weakly in $H^1(\mathbb{R}^N)$ to $u \not\equiv 0$, then u_n converges to u strongly in $L_r(\mathbb{R}^N)$, $2 < r < \ell(N)$ (for $N = 1$ this interval becomes $2 < r \leq \infty$).*

In order to analyze the precompactness of minimizing sequences we have to consider several cases.

First case. $m_0 > 0$ and $\lambda > 0$. In this case the constraint gives

$$\int_{\mathbb{R}^N} u^2(x)dx = -\frac{1}{m_0}\int_{\mathbb{R}^N} G_1(u(x))dx + \frac{\lambda}{m_0}$$

and so, replacing this expression in $V(u)$ we get

$$V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\text{grad } u(x)|^2 dx + \int_{\mathbb{R}^N} \overline{F}(u(x))dx + \frac{m\lambda}{m_0}$$

where $\overline{F}(u) = F_1(u) - \frac{m}{m_0}G_1(u)$. If we drop the constant $\frac{m\lambda}{m_0}$ and we keep the notation $F(u)$ for $\overline{F}(u)$ we get $V(u)$ of the same form and $m = 0$.

Theorem II. *Assume $m_0 > 0$, $\lambda > 0$ and $m = 0$. Then $\inf V(u) \leq 0$; moreover, any minimizing sequence is precompact in $H^1(\mathbb{R}^N)$ modulo translation in the x variable if and only if $\inf V(u) < 0$; in this case, the Lagrange multiplier is different from zero.*

Second case. $m_0 > 0$ and $\lambda < 0$. Arguing as in the previous case, we may assume $m = 0$.

Theorem III. *Assume $N \geq 2$, $m_0 > 0$, $\lambda < 0$ and $m = 0$. Then modulo translation in the x variable any minimizing sequence is precompact in $H^1(\mathbb{R}^N)$.*

Remark. For the case $N = 1$ see the remark following the proof of Theorem IV.

Third case. $m_0 = 0$ and $m > 0$.

Theorem IV. *Assume $m_0 = 0$ and $m > 0$. Then modulo translation in the x variable any minimizing sequence is precompact in $H^1(\mathbb{R}^N)$.*

Remark. In the case $m_0 = 0$, the condition $m \geq 0$ is necessary for the existence of a minimizer. If $m = 0$ (the zero mass case) the proof of theorem V shows that, modulo translation in the x variable, any bounded minimizing sequence is precompact with respect to the norm $|\text{grad}\, u|_{L^2} + |u|_{L^r}$, for some r, $2 < r < \ell(N)$; however, since the L_2 norm of u is absent in $V(u)$ and $I(u)$, we cannot expect to have boundedness of a minimizing sequence in the $H^1(\mathbb{R}^N)$ norm. This means that we have to change the space where we want to solve our minimization problem in the case $m = m_0 = 0$ as in [**4**] for instance.

Before passing to the proof of theorems I to IV, we state a few propositions which will be very useful.

The following statement known as Lieb's lemma [**10**] will play a crucial role in the proof.

Lemma 1. *Let u_n be a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying the following condition: there are $\varepsilon > 0$, $\delta > 0$ and n_0 such that meas $(\{x : |u_n(x)| \geq \delta\}) \geq \varepsilon$ for $n \geq n_0$. Then there is a sequence $d_n \in \mathbb{R}^N$ such that if we let $v_n(x) = u_n(x + d_n)$ then $v_{nj} \rightharpoonup v \not\equiv 0$ in $H^1(\mathbb{R}^N)$, for some subsequence v_{nj}.*

We need also the following version of Lieb's lemma.

Lemma 2. *Let u_n be a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying the following condition: there are $\varepsilon > 0$, $\delta > 0$ and n_0 and a sequence R_n converging to $+\infty$ such that meas $(\{x : |x| \geq R_n,\ |u_n(x)| \geq \delta\}) \geq \varepsilon$ for $n \geq n_0$. Then there is a sequence $d_n \in \mathbb{R}^N$ with $|d_n| \to +\infty$ such that if we let $v_n(x) = u_n(x + d_n)$ then $v_{nj} \rightharpoonup v \not\equiv 0$ in $H^1(\mathbb{R}^N)$, for some subsequence v_{nj}.*

The growth assumption $H1$ implies that the functionals V, I : $H^1(\mathbb{R}^N) \to \mathbb{R}$ are of class C^2 with first and second derivatives uniformly bounded on bounded sets and uniformly continuous on bounded sets.

Now if $u(t, x)$ is a C^2 curve satisfying

$$\int_{\mathbb{R}^N} G(u(t, x)) = \lambda$$

then

$$\int_{\mathbb{R}^N} g(u(t, x))\dot{u}(t, x)dx = 0$$

and

$$\int_{\mathbb{R}^N} g'(u(t, x))\dot{u}^2(t, x)dx + \int_{\mathbb{R}^N} g(u(t, x))\ddot{u}(t, x)dx = 0.$$

So, if $u(0, x) = u(x)$, the admissible $\dot{h}$ and $\ddot{h}$ are those satisfying

$$\int_{\mathbb{R}^N} g(u)\dot{h}dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} g'(u)\dot{h}^2 dx + \int_{\mathbb{R}^N} g(u)\ddot{h}dx = 0.$$

We need the converse (with some uniformity) for the whole sequence u_n.

Lemma 3. *Let u_n be a minimizing sequence converging weakly in $H^1(\mathbb{R}^N)$ to $u \not\equiv 0$ and $\dot{h}_n$ and $\ddot{h}_n$ be bounded sequences in $H^1(\mathbb{R}^N)$ satisfying*

$$\int_{\mathbb{R}^N} g(u_n(x))\dot{h}_n(x)dx = 0$$

and

$$\int_{\mathbb{R}^N} [g'(u_n(x))\dot{h}_n^2(x) + g(u_n(x))\ddot{h}_n(x)]dx = 0.$$

Then there is a $\delta_0 > 0$ such that for n large there is a sequence of C^2 functions $h_n : (-\delta_0,\ \delta_0) \to H^1(\mathbb{R}^N)$ satisfying: i) $u_n + h_n(t)$ is admissible; ii) $h_n(0) = 0$; $\dot{h}_n(0) = \dot{h}_n$, $\ddot{h}_n(0) = \ddot{h}_n$; iii) $h_n(t) - h_n(0)$, $\dot{h}_n(t) - \dot{h}_n(0)$, $\ddot{h}_n(t) - \ddot{h}_n(0)$ go to zero as t goes to zero, uniformly on n.

Proof. From assumption H3 we know $g(u(x)) \neq 0$. If $\psi(x)$ is a, say, smooth function with compact support such that $\int_{\mathbb{R}^N} g(u(x))\psi(x)dx \neq 0$, we see that lemma 4 is a consequence of the application of the implicit function theorem to the function

$$H_n(\sigma,\ t) = \int_{\mathbb{R}^N} G(u_n + \sigma\psi + t\dot{h}_n + \frac{t^2}{2}\ddot{h}_n)dx \quad - \lambda$$

at $(0,\ 0)$ provided we define $h_n(t) = u_n + \sigma(t)\psi + t\dot{h}_n + \frac{t^2}{2}\ddot{h}_n$.

Lemma 4. *Let u_n be a minimizing sequence converging weakly in $H^1(\mathbb{R}^N)$ to $u \not\equiv 0$. Then*

1) $|V'(u_n)| \to 0$ as $n \to +\infty$, the norm of derivative being calculated on the admissible elements;

2) *if for some* $\delta_0 > 0$, $h_n : (-\delta_0, \delta_0) \to H^1(\mathbb{R}^N)$ *is a sequence of* C^2 *admissible curves such that* $h_n(0) = u_n$, $\dot{h}_n(0)$ *and* $\ddot{h}(0)$ *are uniformly bounded (the dots mean derivative) and* $h_n(t) - h_n(0)$, $\dot{h}_n(t) - \dot{h}_n(0)$, $\ddot{h}_n(t) - \ddot{h}_n(0)$, *go to zero as* $t \to 0$, *uniformly on* n, *then* $\liminf \frac{d^2}{dt^2} V(h_n(t))|_{t=0} \geq 0$.

Lemma 4 has been used in **[11]** and it has a sort of "calculus" proof. We show the part concerning $|V'(u_n)| \to 0$ because the other is similar.

Proof of part 1 of lemma 3. By contradiction, if it was false then, passing to a subsequence if necessary, there would exist $\dot{h}_n$ and $\eta > 0$ with $|\dot{h}_n|_{H^1(\mathbb{R}^N)} = 1$, $\dot{h}_n$ admissible for u_n, such that $V'(u_n)\dot{h}_n \leq -\eta$. We define $\ddot{h}_n = c_n \psi$ where ψ is a smooth function with compact support satisfying

$$\int_{\mathbb{R}^N} g(u(x))\psi(x)dx \neq 0$$

and c_n is chosen in such way that the compatibility condition for $\ddot{h}_n$ in the previous lemma is satisfied. Let $h_n : (-\delta_0, \delta_0) \to H^1(\mathbb{R}^N)$ be the curve whose existence is guaranteed by that lemma. Then

$$\begin{aligned} V(u_n + h_n(t)) - V(u_n) = \int_0^t V'(u_n + h_n(s))\dot{h}_n(s)ds = tV'(u_n)\dot{h}_n + \\ + \int_0^t (V'(u_n + h_n(s)) - V'(u_n))\dot{h}_n(s)ds + \\ + \int_0^t V'(u_n)(\dot{h}_n(s) - \dot{h}_n)ds. \end{aligned}$$

Let $t_0 > 0$ be a fixed (independent of n) number such that for $t = t_0$ the absolute value of the last two integrals are less than $\frac{\eta t_0}{4}$; then we would have

$$V(u_n + h_n(t_0)) - V(u_n) \leq -\frac{\eta t_0}{2},$$

a contradiction, and so part 1 of lemma 4 is proved.

Let u be a generic admissible element in $H^1(\mathbb{R}^N)$. In order to compute $|V'(u)|$ we have to maximize

$$V'(u)\varphi = \int_{\mathbb{R}^N} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle dx + \int_{\mathbb{R}^N} f(u)\varphi dx$$

subject to

$$\int_{\mathbb{R}^N} g(u)\varphi dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} (|\operatorname{grad}\varphi|^2 + \varphi^2)dx = 1.$$

If $\overline{\varphi}$ is the place where the maximum is achieved, there are numbers α and γ such that

$$\begin{aligned} \int_{\mathbb{R}^N} (\langle \operatorname{grad} u, \operatorname{grad}\varphi \rangle + f(u)\varphi + \alpha g(u)\varphi + \\ + \gamma \langle \operatorname{grad}\overline{\varphi}, \operatorname{grad}\varphi \rangle + \gamma\overline{\varphi}\varphi)dx = 0 , \ \forall \varphi \in H^1(\mathbb{R}^N). \end{aligned} \tag{1}$$

In particular

$$-\Delta(u + \gamma\overline{\varphi}) + f(u) + \alpha g(u) + \gamma\overline{\varphi} = 0. \tag{2}$$

If we set $\varphi = \overline{\varphi}$ in (1) we get $V'(u)\overline{\varphi} = -\gamma$ and this shows that $|V'(u)| = -\gamma$.

Moreover, if $h(0) = 0$ and $\dot{h}$ and $\ddot{h}$ are admissible, we have

$$\begin{aligned} \frac{d^2}{dt^2} V(u + h(t)|_{t=0} = \\ = \int_{\mathbb{R}^N} (|\operatorname{grad}\dot{h}|^2 + (f'(u) + \alpha g'(u))\dot{h}^2 - \gamma \langle \operatorname{grad}\overline{\varphi}, \operatorname{grad}\ddot{h} \rangle - \gamma\overline{\varphi}\ddot{h})dx. \end{aligned} \tag{3}$$

Proof of theorem I. We give the proof for $N \geq 2$. The case $N = 1$ requires minor modifications.

Let u_n be a minimizing sequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u \not\equiv 0$. From lemma 3 and equations (1) and (2) we know that there are sequences $\alpha_n, \gamma_n, \overline{\varphi}_n$ with $|\overline{\varphi}_n|_{H^1(\mathbb{R}^N)} = 1$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} (\langle \operatorname{grad} u_n, \operatorname{grad}\varphi \rangle + f(u_n)\varphi + \alpha_n g(u_n)\varphi + \gamma_n \langle \operatorname{grad}\overline{\varphi}_n, \operatorname{grad}\varphi \rangle + \\ + \gamma_n \overline{\varphi}_n \varphi)dx = 0 \quad \text{for any} \varphi \in H^1(\mathbb{R}^N) \end{aligned} \tag{1'}$$

and

$$-\Delta(u_n + \gamma_n\overline{\varphi}_n) + f(u_n) + \alpha_n g(u_n) + \gamma_n\overline{\varphi}_n = 0 \tag{2'}$$

Step 1. α_n is bounded. In fact, we know that $\gamma_n \to 0$. Suppose that for some subsequence, for which we keep the same notation, we have $|\alpha_n| \to \infty$. If we divide (1′) by α_n and let $n \to +\infty$ keeping φ fixed, we get

$$\int_{\mathbb{R}^N} g(u(x))\varphi(x)dx = 0$$

for any φ in $H^1(\mathbb{R}^N)$ and this contradicts the assumption H3 and this proves step 1.

Passing to a subsequence if necessary we can assume that $\alpha_n \to \alpha$.

Step 2. For any $\varepsilon > 0$ and $\delta > 0$ there are $\mathbb{R}$ and n_0 such that meas $\{x : |x| \geq R,\ |u_n(x)| \geq \delta\} < \varepsilon$ for $n \geq n_0$.

If not, there are $\varepsilon > 0$ and $\delta > 0$ and a sequence $R_n \to +\infty$ such that meas $\{x : |x| \geq R_n,\ |u_n(x)| \geq \delta\} \geq \varepsilon$ for infinitely many n. By lemma 2 and passing to a subsequence if necessary, we know that there is a sequence $d_n \in \mathbb{R}^N$, $|d_n| \to +\infty$, such that $v_n(x) = u_n(x + d_n) \rightharpoonup v \not\equiv 0$ and so, from $(1')$, we conclude that u and v satisfy

$$- \Delta u + f(u) + \alpha g(u) = 0$$

$$- \Delta v + f(v) + \alpha g(v) = 0$$

Due to the growth assumptions on f and g, u and v are continuous and tend to zero at infinity.

Since not all derivatives $\frac{\partial u}{\partial x_i}$ are identically zero, if we let

$$\mu = f'(0) + \alpha g'(0),\ p(x) = f'(u(x)) + \alpha g'(u(x)) - \mu,$$

we see that the equation

$$-\Delta w + (p(x) + \mu)w = 0$$

has $\frac{\partial u}{\partial x_i}$ as a nontrivial solution.

Before continuing, we show that $\mu \geq 0$; in fact, let ψ a smooth function with compact support such that

$$\int_{\mathbb{R}^N} g(u(x))\psi(x)dx \neq 0$$

and let $\dot{h}$ be any function in $H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} g(u(x))\dot{h}(x)dx = 0;$$

we define $\dot{h}_n = \dot{h} + \varepsilon_n\psi$ and we choose ε_n such that

$$\int_{\mathbb{R}^N} g(u_n(x))\dot{h}_n(x)dx = 0$$

and $\dot{h}_n$ admissible as in the proof of lemma 4. Using that lemma, equation (3), we conclude that

$$\frac{d^2}{dt^2}V(u + h(t))_{t=0} = \int_{\mathbb{R}^N} [|\text{grad}\ \dot{h}(x)|^2 + (f'(u)(x)) + \alpha g'(u(x))\dot{h}^2]dx \geq 0,$$

for any $\dot{h}$ such that

$$\int_{\mathbb{R}^N} g(u(x))\dot{h}(x)dx = 0.$$

As a consequence the spectrum of the linear operator

$$Lh = -\Delta h + (f'(u(x)) + \alpha g'(u(x))h$$

cannot have two distinct points μ_1, μ_2 in the negative half-line because, otherwise, there would be two sequences φ_n, ψ_n, $|\varphi_n|_{L_2} = 1$, $|\psi_n|_{L_2} = 1$, such that $v_n = L\varphi_n - \mu_1\varphi_n$ and $w_n = L\psi_n - \mu_2\psi_n$ tend to zero in $L_2(\mathbb{R}^N)$, and choosing a_n and b_n in such way that $a_n^2 + b_n^2 = 1$ and $k_n = a_n\varphi_n + b_n\psi_n$ is admissible, we would have $\langle Lk_n, k_n\rangle < 0$ for n large, and this is a contradiction. Noticing that $p(x) \to 0$ as $|x| \to +\infty$, we conclude from theorem 5.7 page 304 in **[12]** that the half line $[\mu, +\infty]$ is contained in the spectrum of L; since we have showed that the spectrum of L cannot have two distinct points on the half-line $(-\infty, 0)$ we must have $\mu \geq 0$.

Next we claim that there are ψ_1 and $\beta_1 > \mu$ such that

$$\int_{\mathbb{R}^N} \psi_1^2 dx = 1 \quad \text{and} \quad - \Delta\psi_1 + (p(x) + \beta_1)\psi_1 = 0.$$

Similarly, there are ψ_2 and $\beta_2 > \mu$ such that

$$\int_{\mathbb{R}^N} \psi_2^2 dx = 1 \quad \text{and} \quad - \Delta\psi_2 + (q(x) + \beta_2)\psi_2 = 0,$$

where

$$q(x) = f'(v(x)) + \alpha g'(v(x)) - \mu.$$

The existence of ψ_1 and ψ_2, of ψ_1 for instance, is obtained by minimizing

$$W(\psi) = \int_{\mathbb{R}^N} (|\text{grad}\,\psi(x)|^2 + p(x)\psi^2(x))dx \quad \text{under} \quad \int_{\mathbb{R}^N} \psi^2(x)dx = 1\ .$$

In order to see that this minimum is attained we have to notice that

$$W\left(\frac{\partial u}{\partial x_1}\right) = -\mu\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial x_1}\right)^2 dx \leq 0$$

and then W assumes negative values because, otherwise, $W(\psi)$ would have a minimum at both $\frac{\partial u}{\partial x_1}$ and $\left|\frac{\partial u}{\partial x_1}\right|$ and this is a contradiction (by the unique continuation principle). So, the infimum ℓ of $W(\psi)$ on the admissible set is strictly negative. Let (ψ_n) be a minimizing sequence converging weakly in $H^1(\mathbb{R}^N)$ to ψ. Since

$$\int_{\mathbb{R}^N} p(x)\psi_n^2(x) \text{ converges to } \int_{\mathbb{R}^N} p(x)\psi^2(x)dx$$

(because $p(x)$ is continuous, tends to zero at infinite and $\psi_n \longrightarrow \psi$ strongly in $L_2^{loc}(\mathbb{R}^n)$) we see that $W(\psi) \leq \ell < 0$. But if we had

$$\int_{\mathbb{R}^N}\psi^2(x) < 1,$$

defining $\widetilde{\psi} = c\psi, c > 0$, such that

$$\int_{\mathbb{R}^N}\widetilde{\psi}^2(x)dx = 1$$

we would have $c > 1$ and $W(\widetilde{\psi}) = c^2 W(\psi) < \ell$, a contradiction. So, we conclude that W has a minimum at ψ and the rest is trivial because $\frac{\partial u}{\partial x_1}$ changes sign.

Notice that $\psi_1(x)$ and $\psi_2(x)$ are continuous and tend to zero as $|x| \to +\infty$; in particular, $\psi_1(x)\ \psi_2(x - d_n)$ tends to zero in $L_s(\mathbb{R}^N)$ as n tends to $+\infty$, for $1 \leq s \leq \infty$.

Next define $\dot{h}_n(x) = a_{1,n}\psi_1(x) + a_{2,n}\psi_2(x - d_n)$ imposing that $a_{1,n}^2 + a_{2,n}^2 = 1$ and

$$\int_{\mathbb{R}^N} g(u_n(x))\dot{h}_n(x)dx = 0;$$

notice that

$$\int_{\mathbb{R}^N}\dot{h}_n^2(x)dx \to 1$$

because $\int_{\mathbb{R}^N}\psi_1(y)\psi_2(y - d_n)dy \to 0$.

For $\ddot{h}_n$ we make the choice $\ddot{h}_n = c_n\psi$ where ψ is as in lemma 3 and c_n is chosen to satisfy

$$\int_{\mathbb{R}^N}[g'(u_n(x))\dot{h}_n^2(x) + c_n g(u_n(x))\psi(x)]dx = 0$$

Let $h_n(t)$, $|t| < \delta_0$, be the sequence whose existence is guaranteed by lemma 3; then

$$\frac{d^2}{dt^2}V(u_n + h_n(t)|_{t=0} = \int_{R^N}(|\text{grad}\,\dot{h}_n|^2 + (f'(u_n(x)) + \\ +\alpha g'(u_n(x))\dot{h}_n^2(x))dx - \gamma_n\int_{\mathbb{R}^N}(\langle\text{grad}\,\overline{\varphi}_n, \text{grad}\,\ddot{h}_n\rangle + \overline{\varphi}_n\ddot{h}_n)dx.$$

This last term tends to zero and the first is equal to

$$\begin{aligned}&\int_{\mathbb{R}^N}(a_{1,n}^2|\text{grad}\,\psi_1(x)|^2 + 2a_{1,n}a_{2,n}\langle\,\text{grad}\,\psi_1(x), \text{grad}\,\psi_2(x - d_n)\rangle + \\ &\qquad + a_{2,n}^2|\text{grad}\,\psi_2(x - d_n)|^2)dx + \\ &\qquad + \int_{\mathbb{R}^N}(f'(u_n(x)) + \alpha g'(u_n(x))(a_{1,n}^2\psi_1^2(x) + \\ &\qquad + 2a_{1,n}a_{2,n}\psi_1(x)\psi_2(x - d_n)a_{2,n}^2\psi_2^2(x - d_n))dx\ .\end{aligned} \tag{4}$$

The mixed terms in (4) go to zero and the rest is equal to

$$\int_{\mathbb{R}^N} a_{1,n}^2(-p(x) - \beta_1 + f'(u_n(x)) + \alpha g'(u_n(x))\psi_1^2(x)dx +$$
$$+ \int_{\mathbb{R}^N} a_{2,n}^2(-q(x) - \beta_2 + f'(v_n(x)) + \alpha g'(v_n(x))\psi_2^2(x)dx =$$
$$= a_{1,n}^2\int_{\mathbb{R}^N}(\mu - \beta_1 + f'(u_n(x)) + \alpha g'(u_n(x)) - f'(u(x)) - \alpha g'(u(x))\psi_1^2(x)dx +$$
$$+ a_{2,n}^2\int_{\mathbb{R}^N}(\mu - \beta_2 + f'(v_n(x)) + \alpha g'(v_n(x)) - f'(v(x)) - \alpha g'(v(x))\psi_2^2(x)dx.$$

We claim that the first integral tends to $\mu - \beta_1$ and the second to $\mu - \beta_2$. Let us look, for instance, at the term $\int_{\mathbb{R}^N}(f'(u_n(x)) - f'(u(x))\psi_1^2(x)dx$. Define $h(u) = f'(u)$, $h_1(u) = h(u)$ for $|u| \leq 1$ and zero otherwise, $h_2(u) = h(u)$ for $|u| > 1$ and zero otherwise. From growth assumption, we have $|h_1(u)| \leq$ const. $|u|$ and $|h_2(u)| \leq$ const. $|u|^{p-1}$; if r is large, the term $\int_{|x|\geq r}(h_1(u_n(x)) - h_1(u(x)))\psi_1^2(x)dx$ is small by Holder's inequality because $h_1(u_n) - h_1(u)$ is bounded in $L_2(\mathbb{R}^N)$ and ψ_1^2 belongs to $L_2(\mathbb{R}^N)$. On the other hand, for r fixed, the term

$\int_{|x|\leq r}(h_1(u_n(x)) - h_1(u(x)))\psi_1^2(x)dx$ tends to zero because $h_1(u_n) \to h_1(u)$ in $L_1^{loc}(\mathbb{R}^N)$ and ψ_1^2 belongs to $L_\infty(\mathbb{R}^N)$. This shows

$$\int_{\mathbb{R}^N}(h_1(u_n(x)) - h_1(u(x)))\psi_1^2(x)dx$$

tends to zero. The term

$$\int_{\mathbb{R}^N}(h_2(u_n(x)) - h_2(u(x))\psi_1^2(x)dx$$

is treated in a similar way.

We conclude that $\liminf \frac{d^2}{dt^2}V(u_n + h_n(t))_{t=0} < 0$; this contradiction with lemma 4 proves step 2.

Final Step. u_n is precompact in L_r, $2 < r < \ell(N)$.

Consider first the case $N \geq 3$. In this case u_n is bounded in $L_{l(N)}(\mathbb{R}^N)$. For $\varepsilon > 0$ and $\delta > 0$ given, say $\delta = \varepsilon$, let R and n_0 be as in step 2. If we let $A = \{x : |x| \geq R\}$, $A_n = \{x \in A : |u_n(x)| \geq \delta\}$ and $s = \frac{\ell(N)}{r} > 1$, then for $n \geq n_0$ we have

$$\int_{A_n} 1 \cdot |u_n(x)|^r dx \leq (\text{meas}(A_n))^{\frac{1}{s'}} \left(\int_{A_n} |u_n(x)|^{\ell(N)}\right)^{\frac{1}{s}}$$

and

$$\int_{A_n^c\cap A} |u_n(x)|^r dx \leq \delta^{r-2} \int_{A_n^c\cap A} |u_n(x)|^2 dx$$

and this proves the final step in the case $N \geq 3$. If $N = 2$ we notice that $H^1(\mathbb{R}^2) \subset L_r(\mathbb{R}^2)$ for $2 \leq r < \infty$ and the rest goes as in the case $N = 3$ and theorem I is proved.

Proof of theorem II. We may assume $m_0 = 1$. Let v_n be a sequence of functions satisfying the following conditions:

$$\int_{\mathbb{R}^N} v_n^2(x)dx = \lambda, \int_{\mathbb{R}^N} |\text{grad}\, v_n(x)|^2 dx \to 0 \quad \text{and} \quad |v_n|_{L_\infty} \to 0.$$

For instance, take v_n to be radial and defined by $v_n(r) = \varepsilon_n$ for $0 \leq r \leq n-1$, $v_n(r) = 0$ for $r \geq n$ and linear in the rest and choose ε_n properly. Define $u_n(x) = v_n(\tau_n x)$ where

$$\tau_n^N = \frac{1}{\lambda}\int_{\mathbb{R}^N} G(v_n(x))dx;$$

with this definition we see that u_n is admissible and $V(u_n) \to 0$ because $\tau_n \to 1$ and this shows that $\inf V(u) \leq 0$ and that the condition $\inf V(u) < 0$ is necessary for precompactness of minimizing sequences, modulo translation in the x variable.

Next we show it is sufficient. Let u_n be a minimizing sequence and α_n, α, γ_n and $\overline{\varphi}_n$ as in the proof of step 1 in theorem II; the first thing to be noticed is that u_n satisfies the assumptions of lemma 1 because, if not, u_n would converge to zero in $L_r(\mathbb{R}^N)$, $2 < r < \ell(N)$ and this would imply $\liminf V(u_n) \geq 0$ and this is a contradiction. So, passing to a subsequence if necessary and making a translation in the x variable, we may assume that $u_n \rightharpoonup u \not\equiv 0$ in $H^1(\mathbb{R}^N)$ and then, by theorem II, $u_n \to u$ in L_r, $2 < r < \ell(N)$. Next we notice that $V(u) < 0$ because

$$\int_{\mathbb{R}^N} |\text{grad}\, u(x)|^2 dx \leq \liminf \int_{\mathbb{R}^N} |\text{grad}\, u_n(x)|^2 dx$$

and

$$\int_{\mathbb{R}^N} F(u_n(x))dx \to \int_{\mathbb{R}^N} F(u(x))dx.$$

Moreover, since $-\Delta u + f(u) + \alpha g(u) = 0$ we conclude that $\alpha > 0$ because, otherwise, Pohozaev's identity would imply $V(u) \geq 0$. We make the decomposition $g(u) = 2u + g_1(u)$. If we multiply both sides of the equality

$$-\Delta(u_n + \gamma_n\overline{\varphi}_n - u) = f(u) - f(u_n) + \alpha(g(u) - g(u_n)) + (\alpha - \alpha_n)g(u_n) - \gamma_n\overline{\varphi}_n$$

by $(u_n - u + \gamma_n\overline{\varphi}_n)$ and integrate we get

$$0 \leq \int_{\mathbb{R}^N} |\text{grad}(u_n(x) - u(x) + \gamma_n\overline{\varphi}_n(x))|^2 dx + 2\alpha \int_{\mathbb{R}^N} (u(x) - u_n(x))^2 dx \leq$$
$$\leq \int_{\mathbb{R}^N} [(f(u) - f(u_n))(u_n - u + \gamma_n\overline{\varphi}_n) +$$
$$+ \alpha\gamma_n(g_1(u) - g_1(u_n))\overline{\varphi}_n + (\alpha - \alpha_n)g(u_n)(u_n - u + \gamma_n\overline{\varphi}_n) -$$
$$- \gamma_n(u_n - u + \gamma_n\overline{\varphi}_n)\overline{\varphi}_n]dx\ .$$

Taking in account the growth assumptions and that $u_n \to u$ in $L_r(\mathbb{R}^N)$, $2 < r < \ell(N)$, we can show that the right hand side of this inequality tends to zero. Let us consider, for example, the term $\int_{\mathbb{R}^N}(f(u) - f(u_n))(u_n - u)dx$; from the growth condition H_2 we get

$$|(f(u) - f(u_n))(u_n - u)| \leq k_1(|u|^{q-2} + |u_n|^{q-2} + |u|^{p-2} + |u_n|^{p-2})(u_n - u)^2$$

since $|u(x)|^{q-2}$ is bounded in $L_s(\mathbb{R}^N)$, $s = \frac{l(N)}{q-2}$, and $|u_n(x)-u(x)|^2$ tends to zero in $L_{s'}(\mathbb{R}^N)$,

$$s' = \frac{l(N)}{l(N)-q+2}, \quad 1 < s' < \frac{l(N)}{2},$$

we see that the integral

$$\int_{\mathbb{R}^N} |u(x)|^{q-2}|u_n(x)-u(x)|^2 dx$$

tends to zero. The other terms can be treated by similar arguments we conclude that $u_n \to u$ in $H^1(\mathbb{R}^N)$ and this proves theorem II.

Proof of theorem III. Let u_n be a minimizing sequence; such a sequence satisfies the assumption of lemma 1 because, if not, u_n would converge to zero in $L_r(\mathbb{R}^N)$, $2 < r < \ell(N)$, and the constraint would be violated. Let α_n, α, γ_n and $\overline{\varphi}$ as in the proof of step 1 in theorem II. By theorem I and passing to a subsequence if necessary, we can assume $\alpha_n \to \alpha \geq 0$, $u_n \rightharpoonup u \not\equiv 0$ in $H^1(\mathbb{R}^N)$ and $u_n \to u$ in L_r, $2 < r < \ell(N)$. If $\alpha > 0$ the argument given in Theorem II shows that $u_n \to u$ in $H^1(\mathbb{R}^N)$; however, if $\alpha = 0$ the same argument gives only that $\operatorname{grad} u_n \to \operatorname{grad} u$ in $L^2(\mathbb{R}^N)$ and some extra work is needed to get convergence in $H^1(\mathbb{R}^N)$. In order to get that all we have to show is $|u|^2_{L_2} = \lim |u_n|^2_{L_2}$. By contradiction, suppose $|u|^2_{L_2} < \lim |u_n|^2_{L_2}$ (we can pass to a subsequence if necessary); then $\int_{\mathbb{R}^N} G(u(x))dx < \lambda < 0$. Suppose first $N \geq 3$.

Defining $\sigma^N = \lambda / \int_{\mathbb{R}^N} G(u(x))dx$, and $v(x) = u(\frac{x}{\sigma})$ we see that $0 < \sigma < 1$, v is admissible and then $V(u) \leq V(v)$, that is,

$$\frac{1}{2}\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2 + \int_{\mathbb{R}^N} F(u(x))dx \leq$$
$$\leq \frac{\sigma^{N-2}}{2}\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2 dx + \sigma^N \int_{\mathbb{R}^N} F(u(x))dx$$

Moreover, since $-\Delta u = -f(u)$, we have

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2 dx = -N\int_{\mathbb{R}^N} F(u(x))dx$$

and then

$$\left(\frac{1}{2} + \frac{2-N}{2N}\right)\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2 dx \leq$$
$$\left(\frac{\sigma^{N-2}}{2} + \frac{(2-N)}{2N}\sigma^N\right)\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2 dx .$$

This implies $\operatorname{grad} u(x) \equiv 0$ which is a contradiction. If $N = 2$ we have

$$\int_{\mathbb{R}^N} F(u(x))dx = 0$$

and this shows $V(u) = V(v)$, hence V has a minimum at v and then $-\Delta v + f(v) = \beta g(v)$. Using Pohozaev's identity again we get $\beta = 0$ and then $-\frac{1}{\sigma^2}\Delta u + f(u) = 0$; this implies $\Delta u \equiv 0$ and this is a contradiction. So theorem IV is proved.

Remark. If $\alpha = 0$ and $N = 1$ the argument above fails. So, in the case $N = 1$, we are able to prove theorem IV provided $\alpha > 0$. This condition is verified if either the only solution of $-u_{xx} + f(u) = 0$, $u \in H^1(\mathbb{R})$, is $u \equiv 0$ or V assumes negative values on the admissible set.

Proof of theorem IV. Let u_n be a minimizing sequence. As before, u_n satisfies the assumptions of lemma 1 and then, passing to a subsequence and making translation in the x variable, we can assume $u_n \rightharpoonup u \not\equiv 0$ in $H^1(\mathbb{R}^N)$. From theorem II we conclude u satisfies the constraint and this together with the inequality $V(u) \leq \liminf V(u_n)$ gives $u_n \to u$ in $H^1(\mathbb{R}^N)$ and theorem V is proved.

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