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Expansive Dynamics and Hyperbolic Geometry

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Abstract. Let M be a compact Riemannian manifold with no conjugate points such that its geodesic flow is expansive. Then we show that the universal Riemannian covering of M is a hyperbolic geodesic space according to the definition of M. Gromov. This allows us to extend a series of relevant geometric and topological properties of negatively curved manifolds to M and in particular, geometric group theory applies to the fundamental group of M.

0. Introduction

The divergence of geodesics in the universal covering of a Riemannian manifold with no conjugate points is one of the indicators of the presence of hyperbolic geometric and topological fenomena in the manifold. The classical theory of manifolds with no conjugate points provides a lot of information concerning the behavior of geodesics once we impose some restrictions in the metric. The case of strictly negatively curved manifolds, in which every pair of different geodesics diverges at a exponential rate, contrasts with the zero curvature case in which there exist a lot of 'parallel' geodesics. Busemann [**3**] shows that a torus without conjugate points has the following property: given any homotopy class there exists a foliation of the torus by closed geodesics belonging to that homotopy class and all having the same period. In this example the topological nature of the manifold determines the lack of expansivity of the geodesic flow, and no convexity assumptions are needed to deduce this fact. Recall that the geodesic flow $\varphi_t: T_1M \to T_1M$ is the one parameter family of diffeomorphisms acting on the unit tangent bundle T_1M of a com-

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plete, C^∞ Riemannian manifold M as follows: $\varphi_t(p,v) = (\gamma(t), \gamma'(t))$ where $\gamma(t)$ is the geodesic of M such that $\gamma(0) = p$, $\gamma'(0) = v$. One defines the expansivity as follows:

Definition. Let $\psi_t: N \to N$ be a continuous flow acting on a metric space N. The flow ψ_t is said to be *expansive of constant* $\epsilon > 0$ if for every $x \in N$ we have the following property: given $y \in N$ if there exists a non-decreasing surjective map $h: \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ and $d(\psi_t(x), \psi_{h(t)}(y)) \leq \epsilon$ then there exists $t(\epsilon, y) \in \mathbb{R}$ such that $\psi_{t(\epsilon,y)}(x) = y$.

In other words, two different orbits eventually become separated by a distance of at least ϵ. It is clearly one of the properties of geodesic flows of compact manifolds with negative curvature. In this work we are interested in obtaining geometric information of the manifold from dynamical data of the geodesic flow. Busemann's result suggests that if the geodesic flow of a manifold is expansive then the manifold should be 'closer' to a negatively curved manifold than to a torus. In fact, we are able to show that the expansivity condition implies that the universal covering of the manifold 'looks' like the hyperbolic space. Let us define the so-called Axiom of Visibility:

Definition. A complete, simply connected Riemannian manifold N with no conjugate points (i.e., the exponential map is non-singular) satisfies the *Axiom of Visibility* if for each $p \in N$, $\alpha > 0$ there exists $R = R(p, \alpha) > 0$ such that if a geodesic segment $\gamma: [a,b] \to N$ satisfies $\inf_{t \in [a,b]} d(p, \gamma(t)) \geq R$ we have that the angle at p formed by the geodesic segments joining p to $\gamma(a)$ and p to $\gamma(b)$ is less than α.

Such manifolds are called *visibility manifolds*. If the number R does not depend on p then is said to be a *uniform* visibility manifold. It is clear that flat manifolds are not visibility manifolds. On the other hand, negatively curved manifolds are visibiliby manifolds and more generally, if a compact manifold with no conjugate points is such that its universal covering is a visibility manifold then every metric with no conjugate points in the same manifold satisfies the same property [**5**]. In particular every metric with no conjugate points of a compact surface

of genus greater than one induces a visibility structure for its universal covering. The theory of visibility manifolds is well developed in many aspects, specially those concerning automorphic function theory [**2**], [**5**]. There exists also very interesting geometric characterizations of these manifolds in terms of the asymptotic behavior of geodesics [**5**]. In the non-positive curvature case they have a lot of rigidity and the topological dynamics of the geodesic flow is – modulo the existence of flat strips – very similar to hyperbolic dynamics [**2**]. The purpose of this work is to show that:

Theorem A. *Let (M,g) be a compact Riemannian manifold with no conjugate points. If the geodesic flow of M is expansive then $\widehat{M}$ is a visibility manifold.*

Note that if the curvature of M in Theorem A is non-positive the theorem holds since, on one hand, it is clear that if the geodesic flow is expansive there cannot exists flat strips in the universal covering of M and on the other hand Eberlein shows that [**5**]:

Proposition. *A simply connected complete Riemannian manifold with non-positive curvature is a visibility manifold if and only if there is no complete isometric embedding of a flat half plane into the manifold.*

In general, manifolds with no conjugate points have neither convexity of the metric, nor relations between the angles of geodesic triangles as one has in the non-positive curvature case. These last two properties of non-positive curvature metrics are crucial to the proof of the above proposition. Nevertheless, we can show that the expansivity of the geodesic flow implies that the universal covering of M endowed with the induced metric is a quasi-convex metric space.

Definition. A complete metric space (X,d) is K,C-*quasi-convex* for two given positive constants K and C if every two geodesic segments $[a,b]$, $[c,d]$ in X satisfy the following property:

$$d([a,b],[c,d]) \leq K \sup\{d(a,c), d(b,d)\} + C$$

where $d([a,b],[c,d])$ is the Hausdorff distance between $[a,b]$ and $[c,d]$.

The metric d is also called quasi-convex. This is shown in section 1. Then we prove the main result of the paper:

Theorem B. *Let M be a compact manifold with no conjugate poits and suppose that the universal covering $\widehat{M}$ is a quasi-convex metric space. Then $\widehat{M}$ is not a visibility manifold if and only if for every $L \in \mathbb{N}$ there exist $E > 0$ and a pair of geodesics $\gamma, \beta \colon (-\infty, \infty) \to \widehat{M}$ such that*
i) $d(\gamma(t), \beta(t)) \leq E$, $\forall t \in \mathbb{R}$
ii) $\inf_{t \in \mathbb{R}}\{d(\gamma(t), \beta)\} \geq L$, $\forall t \in \mathbb{R}$
where $d(\gamma(t), \beta)$ is the distance from $\gamma(t)$ to the geodesic β.

So theorem A is straightforward from theorem B and the following result of [**11**]:

Let M be a compact manifold with no conjugate points such that its geodesic flow is expansive. Then if γ, $\beta \colon (-\infty, \infty) \to \widehat{M}$ have the property that

$$d(\gamma(t), \beta(t)) \leq D, \ \forall t \in \mathbb{R}$$

then there exists $t_0 \in \mathbb{R}$ such that $\gamma(t) = \beta(t + t_0)$, $\forall t \in \mathbb{R}$.

Note that theorem B implies that the theorem of Eberlein mentioned above is true for the universal covering of a compact manifold with no focal points. This is a consequence of the fact that in the absence of focal points bi-asymptotic geodesics at Hausdorff distance more than $A > 0$ bound a flat strip of width more than A as well as in non-positively curved spaces [**10**]. The fact that the visibility axiom implies the non-existence of pairs of geodesics satisfying the statement of theorem B is not hard to deduce from the definition of the visibility property. The proof of the reciprocal assertion is essencially the content of this paper. The main point of the proof is that the lack of visibility implies the existence of geodesic triangles in $\widehat{M}$ of arbitrarilly large size satisfying certain remarkable properties of triangles of $\mathbb{R}^n$. Roughly speaking, we can find large geodesic triangles 'bounding regions or arbitrarilly large area' and whose perimeters increase in a certain controlled way. This statement, that will be made more precise in section 2, can be viewed as a reffinement of the non-existence of an isoperimetric inequality for triangles. The proof of Theorem B will be performed in the first three

sections. Finally, in section 5 we show that visibility manifolds are hyperbolic geodesic spaces according to the definition of M. Gromov [**8**]. So the results of the geometric theory of groups – developed recently by many authors and specially by Gromov [**9**] (see also [**3**], [**8**]) – can be extended as well to manifolds with expansive geodesic flows. In particular, the divergence of geodesics in the universal covering is exponential, and every two non-commuting elements of the fundamental group, up to a finite power, generate a free group.

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1. Expansivity and quasi-convexity

We begin by fixing some notations. From now on, M will be a compact, boundaryless C^∞ Riemannian manifold, $\widehat{M}$ will be its universal Riemannian covering and $\pi_1(M)$ its fundamental group. If p, $q \in \widehat{M}$ we shall denote by $[p, q]$ the geodesic segment joining p to q. Given a point $z \in \widehat{M}$ the angle at z subtended by two geodesic segments $[z, p]$ and $[z, q]$ will be denoted by $\measuredangle_z(p, q)$.

Definition 1.1. Given a complete, simply connected Riemmanian manifold N with no conjugate points, and a unit geodesic $\gamma(t) \subset N$ (i.e., parametrized by arclength), a unit geodesic $\beta(t)$ is called an *asymptote* of γ or *asymptotic* to γ if there exists $C > 0$ such that

$$d(\beta(t), \gamma(t)) \leq C$$

for every $t \geq 0$. The geodesics γ, β are said to be asymptotic. The manifold N satisfies the *Axiom of Asymptoticity* if for every unit geodesic $\gamma(t)$ with $\gamma(0) = p$, $\gamma'(0) = v$ it holds the following property:

Let $q \in N$, let $q_n \to q$ be a sequence of points in N, let $(p_n, v_n) \in T_{p_n}N$ be a sequence of unit vectors such that $(p_n, v_n) \to (p, v)$ and let $\gamma_n(t)$ be the unit geodesic of N such that $\gamma_n(0) = p_n$, $\gamma_n'(0) = v_n$. Then, for every sequence of numbers $t_n \to +\infty$ the sequence β_n of geodesics joining q_n with $\gamma_n(t_n)$ converges to an asymptote β of γ.

If $\beta(t)$ is asymptotic to $\gamma(t)$ and $\beta(-t) = \overline{\beta}(t)$ is asymptotic to

$\gamma(-t) = \overline{\gamma}(t)$ then γ, β are called *bi-asymptotic*. Simply connected manifolds without focal points, and in particular those having non-positive curvature, satisfy the Axiom of Asymptoticity. More generally we can show (see **[11]**, proof of Prop. 1.1, for instance):

Lemma 1.1. *If N is a complete, simply connected, quasi-convex manifold with no conjugate points then it satisfies the Axiom of Asymptoticity.*

Definition 1.2. Let N be a complete simply connected manifold with no conjugate points. We say that N satisfies the *uniqueness condition* if the only bi-asymptote of each geodesic $\gamma \subset N$ is γ itself.

Examples of such manifolds are the universal coverings of compact manifolds of negative curvature. In **[11]** it is proved that:

Theorem 1.1. *Let M be a compact manifold with no conjugate points. The geodesic flow is expansive if and only if $\widehat{M}$ satisfies the uniqueness condition.*

We shall write down the proof of this theorem for the sake of completeness. To begin with, we show

Lemma 1.2. *Let $\varphi_t\colon T_1M \to T_1M$ be expansive. Then for every $n > 0$ there exists $\delta = \delta(n) > 0$ with $\lim_{n\to+\infty}\delta(n) = 0$ such that for every pair of geodesics γ, β in $\widehat{M}$ satisfying*

$$d(\gamma(t), \beta(t)) < \frac{1}{n}$$

$$d(\gamma(s), \beta(s)) < \frac{1}{n}$$

for some $t < s$ then we have that

$$d(\gamma(r), \beta(r)) < \delta$$

for every $r \in [t, s]$.

Proof. Suppose that the statement does not hold. Then we get sequences $n_k \to +\infty$ of integers, t_k, s_k, a_k of real numbers with $t_k < a_k < s_k$, γ_k, β_k of geodesics in $\widehat{M}$ and a number $a > 0$ such that:

i) $d(\gamma_k(t_k), \beta_k(t_k)) < \frac{1}{n_k}$

ii) $d(\gamma_k(s_k), \beta_k(s_k)) < \frac{1}{n_k}$

iii) $d(\gamma_k(a_k), \beta_k(a_k)) \geq a$

for every $k \in N$. Let $\epsilon > 0$ be the expansivity constant of ε_t. It is easy to see that there exists $\rho > 0$, $C > 0$ such that in every open ball $B_\rho(\theta)$ of radius ρ in T_1M we have that for every $\omega = (p_1, v_1)$, $\tau = (p_2, v_2)$ belonging to $B_\rho(\theta)$ the following holds:

$$\overline{d}(\omega, \tau) \leq C \sup\{d(p_1, p_2), \|P_{p_1,p_2}(v_1) - v_2\|, \|P_{p_2,p_1}(v_2) - v_1\|\} \qquad (*)$$

where $\overline{d}$ is the distance function associated to the canonical metric of T_1M, and $P_{q,z}$ is the parallel transport along the shortest geodesic of M between q and z. We can suppose without loss of generality that $\rho \leq \epsilon$. Since the geodesic segment between two points in $\widehat{M}$ is unique and it depends continuously on its endpoints we can deduce from (i), (ii) and (iii) above that for every $D > 1$ there is a sequence of geodesics η_k in $\widehat{M}$ and numbers $b_k \in (t_k, s_k)$ satisfying

i) $d(\gamma_k(t_k), \eta_k(t_k)) < \frac{1}{n_k}$

ii) $d(\gamma_k(s_k), \eta_k(s_k)) < \frac{1}{n_k}$

iii) $d(\gamma_k(b_k), \eta_k(b_k)) = \inf\,(a, \frac{\rho}{D}) = \sup_{r\in(t_k,s_k)}\{d(\gamma_k(r), \eta_k(r))\}$.

Clearly, $|t_k - s_k| \to +\infty$, $|t_k - b_k| \to +\infty$, and $|s_k - b_k| \to +\infty$. Let $M_0 \subset \widehat{M}$ be a fundamental domain of M and let $g_k \in \pi_1(M)$ be such that $g_k(\gamma_k(b_k)) \in M_0$, $\forall k$. There exists convergent subsequences of pairs of points $(g_k(\gamma_k(b_k)), dg_k(\gamma_k'(b_k))$ and $(g_k(\eta_k(b_k)), dg_k(\eta_k'(b_k))$ to points (p, v) and (q, w) respectively. Now it is straightforward to check that the geodesics $\gamma_v(t)$, $\gamma_v(0) = p$ and $\gamma_w(t)$, $\gamma_w(0) = q$ verify the following properties:

i) $d(\gamma_v(0), \gamma_w(0)) = \inf\left(a, \frac{\rho}{D}\right) > 0$

ii) $d(\gamma_v(t), \gamma_w(t)) \leq \frac{\rho}{D}$, $\forall t \in \mathbb{R}$.

Then it is clear from (*) that if D is big enough we shall get that

$$\overline{d}(\varphi_t(p, v), \varphi_t(q, w)) \leq \varepsilon$$

$\forall t \in \mathbb{R}$ which contradicts the expansivity of the geodesic flow. □

Proof of Theorem 1.1. It is clear that if $\widehat{M}$ satisfies the uniqueness condition then the geodesic flow of M must be expansive for some constant $\epsilon > 0$. So it remains to show the reciprocal statement. Suppose that the geodesic flow of M is expansive and let $\epsilon > 0$ be the expansivity constant. Let $\gamma(t)$ be a geodesic of $\widehat{M}$ and suppose that $\beta(t)$ is another geodesic in $\widehat{M}$ such that

$$d(\gamma(t), \beta(t)) \leq C$$

for every $t \in \mathbb{R}$. For every $t \geq 0$ let $\sigma_t(s)\colon [0,1] \to \widehat{M}, \sigma_{-t}(s)\colon [0,1] \to \widehat{M}$ be the geodesic segments defined by

$$\sigma_t(0) = \gamma(t), \sigma_t(1) = \beta(t)$$
$$\sigma_{-t}(0) = \gamma(-t), \sigma_t(1) = \beta(-t).$$

From the hypotheses we have that the length of both $\sigma_t(s)$ and $\sigma_{-t}(s)$ is less than C for every $t \geq 0$. Let $\gamma_{t,s}(\rho)\colon [0,1] \to \widehat{M}$ be the geodesic segment joining $\sigma_t(s)$ with $\sigma_{-t}(s)$. From lemma 1.1 we get that there exists $\delta > 0$ such that if $d(\gamma_{t,s}(0), \gamma_{t,a}(0)) \leq \delta$ and $d(\gamma_{t,s}(1), \gamma_{t,a}(1)) \leq \delta$ then

$$d(\gamma_{t,s}(\rho), \gamma_{t,a}(\rho)) \leq \epsilon$$

for every $\rho \in [0,1]$. Since the length of the segments $\sigma_t(s)$ is uniformly bounded above by C there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there are points $s_{n,0} = 0, s_{n,1}, s_{n,2}, \ldots, s_{n,m} = 1$ in $[0,1]$ such that

$$d(\sigma_n(s_{n,1}), \sigma_n(s_{n,i+1})) \leq \delta$$
$$d(\sigma_{-n}(s_{n,1}), \sigma_{-n}(s_{n,i+1})) \leq \delta$$

for every $1 \leq i \leq m-1$. This means that

$$d(\gamma_{n,s_{n,i}}(\rho), \gamma_{n,s_{n,i+1}}(\rho)) \leq \epsilon$$

for every $\rho \in [0,1]$, and in particular, every geodesic $\gamma_{n,s_{n,i}}(\rho)$ intersects the ball of radius $C + m\epsilon$ with center at the point $\gamma(0)$. Letting n go to infinity we subsequences of the sequences $\gamma_{n,s_{n,i}}$ converging to geodesics $\gamma_k, k = 0, 1, 2, \ldots, m$ satisfying

$$d(\gamma_k(t), \gamma_{k+1}(t)) \leq \epsilon$$

for every $t \in \mathbb{R}$ which implies that $\gamma_k = \gamma_{k+1}$, $\forall k$ from the expansivity of the geodesic flow. But since $\gamma_0 = \gamma$ and $\gamma_m = \beta$, modulo reparametrization, we get that $\gamma = \beta$. This finishes the proof of the theorem. □

Corollary 1.1. *Let M be a compact manifold with no conjugate points. If $\widehat{M}$ satisfies the uniqueness condition then it is a quasi-convex metric space.*

Proof. The geodesic flow of M is expansive with constant $\epsilon > 0$. Let $m \in \mathbb{N}$ be sufficiently large and let $\delta = \delta(m)$ the number defined in lemma 1.2. Let $[a,b]$, $[c,d]$ be two geodesic segments in $\widehat{M}$. Assume that $\sup\{d(a,c), d(b,d)\} = d(a,c)$ and subdivide $[a,c]$ in segments $[x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$, where $x_0 = a, x_n = c, n = [[d(a,c)/\frac{1}{m}]] + 1$ – where $[[z]]$ is the integer part of $z \in \mathbb{R}$ – and $d(x_i, x_{i+1}) = d(x_{i-1}, x_i)$, $\forall i \leq n-1$. Obviously $d(x_{i-1}, x_i) \leq \frac{1}{m}$, $\forall i$. Take a partition of $[b,d]$ by segments $[z_i, z_{i+1}]$, $i = 0, 1, \ldots n-1$ such that $z_0 = b$, $z_n = d$ and $d(z_i, z_{i+1}) = d(b,d)/n$. Then we have that $d(z_i, z_{i+1}) \leq d(a,c)/n \leq d(x_i, x_{i+1}) \leq \frac{1}{m}$, $\forall i$. Consider the geodesic segments $[x_i, z_i]$, $i = 0, 1, \ldots, n-1$ which according to lemma 1.2 satisfy

$$d([x_i, z_i], [x_{i+1}, z_{i+1}]) \leq \delta$$

for every $0 \leq i \leq n-1$ which implies

$$d([a,b],[c,d]) \leq n\delta = ([[d(a,c)/\frac{1}{m}]] + 1)\delta \leq (md(a,c) + 1)\delta$$
$$\leq m\delta \sup\{d(a,c), d(b,d)\} + \delta\,.$$

So taking $K = m\delta$ and $C = \delta$ we get corollary 1.1. □

2. On non-hyperbolic triangles

The purpose of this section is to show that the lack of visibility determines some flat-like behavior in the geodesic triangles of the universal covering. Remark that the hypotheses of the main theorem do not grant the existence of any relationships between the angles in a geodesic triangle of the type occurring in the non-positive curvature case. The main result of this section is the following:

Theorem 2.1. *Let M be a compact manifold with no conjugate points such that $\widehat{M}$ is a quasi-convex metric space. If $\widehat{M}$ is not a visibility manifold then the following is true:*

There exist constants $B > 0$, $D > 0$ such that for a given $m \in \mathbb{N}$

there are sequences a_n, b_n, c_n, d_n *of points in* $\widehat{M}$ *such that*

i) $d(a_n, b_n) \to +\infty, d(b_n, d_n) \to +\infty, d(c_n, d_n) \to +\infty, d(a_n, c_n) = m$ *and* $\inf_{s,t\in[0,1]}\{d([a_n, b_n](t), [c_n, d_n](s))\} \geq m, \ \forall n \geq n_0$, *where* $[a_n, b_n]\colon [0,1] \to \widehat{M} [c_n, d_n]\colon [0,1] \to \widehat{M}$ *are parametrizations of the segment* $[a_n, b_n]$ *and* $[c_n, d_n]$.

ii) $d(b_n, d_n) \leq Bd(a_n, b_n)$ *and* $d(c_n, d_n) \leq Bd(a_n, b_n)$

iii) $\underline{d}([a_n, c_n], [b_n, d_n]) = \inf_{t,s\in[0,1]}\{d([a_n, c_n](t), [b_n, d_n](s))\} \geq Dd(a_n, b_n)$.

Let us comment briefly the statement of theorem 2.1. For every $m \in \mathbb{N}$ there exists a sequence of geodesic quadrilaterals having sides of increasing lengths. The lengths of the sides depend at most linearly on the length of one of them, on which we could think as the base of the quadrilateral. This condition ressembles the law of cosines of triangles in $\mathbb{R}^n$. Moreover, condition (i) says that the distances between points in $[a_n, b_n]$ and points in $[c_n, d_n]$ is greater than m, and condition (iii) tells us that the distances between the points of $[a_n, c_n]$ and $[b_n, d_n]$ is proportional to the distance between a_n and b_n. This means that the 'area' bounded by the quadrilateral $\square(a_n, b_n, c_n, d_n)$ with sides $[a_n, b_n], [b_n, d_n], [d_n, c_n], [c_n, a_n]$ goes to $+\infty$ if $n \to +\infty$ in a controlled way.

We shall show some lemmas before proving the theorem. First recall the following basic results concerning triangles in manifolds with no conjugate points [**5**]:

Lemma 2.1. *Given* $\alpha > 0$ *and* $m \in \mathbb{N}$ *there exists* $R > 0$ *such that if* $d(a_n, b_n) > R$, $d(a_n, c_n) > R$ *and* $\measuredangle_a([a, b], [a, c]) \geq \alpha$ *then we have*

$$\inf_{s,t\geq R} d([a, b](t), [a, c](s)) \geq m$$

where t and s are the arclength parameters of $[a, b]$ *and* $[a, c]$ *with* $[a, b](0) = [a, c](0) = a$.

We shall need also the following definitions.

Definition 2.1. Let $m > 0$. A sequence of geodesic triangles $\triangle(a_n, b_n, c_n)$ is **m-fat** with respect to $[a_n, b_n]$ if there exists $t_m(n) \in [0, \ell[a_n, b_n]]$ such that:

i) $\ell[a_n, b_n] - t_m(n) \to +\infty$ if $n \to +\infty$

ii) $d([a_n, b_n](t), [a_n, c_n]) \geq m, \ \forall t \geq t_m(n)$.

Definition 2.2. Let $D > 0$. Consider a geodesic triangle $\triangle(a_1, a_2, a_3)$. We say that $[a_i, a_j], i \neq j$ is **D-bounded** with respect to $[a_k, a_m]$ if $\ell[a_i, a_j] \leq D\ell[a_k, a_m]$.

Let us begin with the proof of the theorem. If $\widehat{M}$ is not a visibility manifold then there exist $\alpha > 0$ and sequences p_n, q_n, r_n of points in $\widehat{M}$ such that

a) $d(p_n, [q_n, r_n]) \to +\infty$ if $n \to +\infty$

b) $\measuredangle_{p_n}([p_n, q_n], [p_n, r_n]) \geq \alpha > 0$

In other words, there are geodesic segments which are far from p_n and subtend at this point an angle greather than some strictly positive constant. From lemma 2.1 it is clear that $d(q_n, r_n) \to +\infty$ with $n \to +\infty$. We can assume without loss of generality that $\alpha \leq \frac{\pi}{2}$.

Claim. We can assume that either

$$\measuredangle_{q_n}([q_n, p_n], [q_n, r_n]) \geq \frac{\pi}{2}$$

and then $d(p_n, [q_n, r_n]) = d(p_n, q_n)$ or

$$\measuredangle_{r_n}([r_n, p_n], [q_n, r_n]) \geq \frac{\pi}{2}$$

and $d(p_n, [q_n, r_n]) = d(p_n, r_n)$.

This is because either there exists $z_n \in (q_n, r_n)$ – the interior of $[q_n, r_n]$ – such that

$$d(p_n, [q_n, r_n]) = d(p_n, z_n)$$

and then from the first variation formula we get that

$$\measuredangle_{z_n}([p_n, z_n], [q_n, z_n]) = \frac{\pi}{2}, \measuredangle_{z_n}([z_n, p_n], [z_n, r_n]) = \frac{\pi}{2},$$

(and either $\measuredangle_{p_n}([p_n, q_n], [p_n, z_n]) \geq \frac{\alpha}{2}$ or $\measuredangle_{p_n}([p_n, r_n], [p_n, z_n]) \geq \frac{\alpha}{2}$), or the minimum distance from p_n to $[q_n, r_n]$ is one of the endpoints of this geodesic segment, q_n for instance. In this case, again from the first variation formula we deduce that

$$\measuredangle_{q_n}([q_n, p_n], [q_n, r_n]) \geq \frac{\pi}{2}.$$

So let us start with sequences $p_n, q_n, r_n \in \widehat{M}$ such that

a) $d(p_n, [q_n, r_n]) = d(p_n, q_n) \geq n$
b) $\measuredangle_{p_n}([p_n, q_n], [p_n, r_n]) \geq \alpha$
for some constant $\alpha > 0$, for every $n \in \mathbb{N}$. Let $h_n \in [p_n, r_n]$ be defined by $d(p_n, h_n) = d(p_n, q_n)$. Observe that lemma 2.1 implies that $d(h_n, q_n) \to +\infty$ if $n \to +\infty$.

Lemma 2.2. *For the sequence of triangles $\Delta(p_n, q_n, r_n)$ we have the following possibilities:*

1. *Either there exist $M > 0$, $P > 0$ and a subsequence $n_k \to +\infty$ such that for every $k \in \mathbb{N}$ there exists $x_k \in [h_{n_k}, r_{n_k}]$ satisfying*
i) $d(x_k, h_{n_k}) \leq P d(h_{n_k}, q_{n_k})$
ii) $d(x_k, [r_{n_k}, q_{n_k}]) \leq M$

In this case there exists $B > 0$ such that for every $L > 0$ we get $k(L) > 0$ such that the sequence $\Delta(p_{n_k}, q_{n_k}, x_k)$ is L-fat with respect to $[p_{n_k}, q_{n_k}]$, $\forall k \geq k(L)$, and $[q_{n_k}, x_k]$ is B-bounded by $[p_{n_k}, q_{n_k}]$.

2. *Or, for every $M \in \mathbb{N}$ there exists $n(M) \in \mathbb{N}$ such that for every $n \geq n(M)$ the sequence of triangles $\Delta(r_n, h_n, q_n)$ is M-fat with respect to $[r_n, h_n]$ and $[h_n, q_n]$ is $\frac{1}{M-1}$-bounded by $[r_n, h_n]$.*

Proof. Parametrize $[r_n, h_n]\colon [0, \ell[r_n, h_n]] \to \widehat{M}$ by arclength, with $[r_n, h_n](0) = r_n$ and decompose

$$[r_n, h_n] = \bigcup_{j=0}^{k_n} [[r_n, h_n](t_j), [r_n, h_n](t_{j+1})]$$

in k_n intervals of length $\leq \ell[h_n, q_n]$, i.e., $t_0 = 0$, $t_{k_n} = \ell[h_n, q_n]$ and
i) $d([h_n, r_n](t_j), [h_n, r_n](t_{j+1})) = \ell[h_n, q_n]$, $\forall j = 0, 1, \ldots, k_{n-2}$
ii) $d([h_n, r_n](t_{k_{n-1}}), [h_n, r_n](t_{k_n})) \leq \ell[h_n, q_n]$

For $m \in \mathbb{N}$ define $t_n(m) = \sup\{t \in [0, \ell[r_n, h_n]], d([r_n, h_n](s), [r_n, q_n] \leq Km + C, \forall s \leq t\}$, where $K > 0$ is the quasi-convexity constant. Remark that

$$d([r_n, h_n](t), [r_n, q_n]) \geq m$$

$\forall t > t_n(m)$, for otherwise, if there exists $s > t_n(m)$ with

$$d([h_n, r_n](t), [q_n, r_n]) < m$$

we would get by the quasi-convexity

$$d([r_n, h_n](t), [r_n, q_n]) \leq Km + C$$

for every $t < s$ contradicting the choice of $t_n(m)$. Let

$$t_n(m) \in [[h_n, r_n](t_{i_n}), [h_n, r_n](t_{i_n+1})] = [i_n \ell[h_n, q_n], (i_n + 1)\ell[h_n, q_n]].$$

Then either
a) There exist a constant $E > 0$ and a subsequence $n_j \to +\infty$ such that $k_{n_j} - i_{n_j} \leq E$
b) For every $E \in \mathbb{N}$ there exists $n(E) \in \mathbb{N}$ such that $k_n - i_n \geq E$, $\forall n \geq n(E)$. In particular,

$$\frac{\ell[h_n, q_n]}{\ell[h_n, r_n] - t_n(m)} \leq \frac{\ell[h_n, q_n]}{(k_n - (i_n + 1))\ell[h_n, q_n]} \leq \frac{1}{E - 1}.$$

Case (b) is just statement (2) of lemma 2.1, i.e., the sequence of triangles $\Delta(r_n, h_n, q_n)$ is m-fat and $[h_n, q_n]$ is $\frac{1}{E-1}$ – bounded with respect to $[r_n, h_n]$, $\forall n \geq n(m)$. If (a) holds we deduce

Claim. Given $L > 0$ there exist $j(L) > 0$ and $B > 0$ such that the sequence of triangles $\Delta(p_{n_j}, q_{n_j}, c_{n_j}) = [r_{n_j}, h_{n_j}](t_{i_{n_j}}))$ is L-fat with respect to $[p_{n_j}, q_{n_j}]$ for every $j \geq j(L)$ and $[q_{n_j}, c_{n_j}]$ is B-bounded by $[p_{n_j}, q_{n_j}]$, $\forall n \geq 0$.

Let us first, for simplicity, denote $r_j = r_{n_j}, h_j = h_{n_j}, c_j = c_{n_j}$. To show the claim let $\overline{q}_j \in [r_j, q_j]$ be such that $d([r_j, h_j](t_{i_j}), \overline{q}_j) \leq Km$. By the quasi-convexity we get

$$d([q_j, \overline{q}_j], [q_j, [r_j, h_j](t_{i_j})]) \leq K^2 m$$
$$\Downarrow$$
$$d(p_j, [q_j, [r_j, h_j](t_{i_j})]) \geq d(p_j, [r_j, q_j]) - K^2 m \qquad (*)$$
$$\geq (1 - \delta) d(p_j, [r_j, q_j])$$

for a certain $\delta > 0$ small and j suitably big. We have also that

$$\measuredangle_{q_j}([p_j, q_j], [q_j, [r_j, h_j](t_{i_j})]) \geq \frac{\pi}{2} - \nu_j$$

where $\nu_j \to 0$, which follows from (*) and lemma 2.1. On the other hand

$$\begin{aligned} d(q_j, [r_j, h_j](t_{i_j})) &\leq d(q_j, p_j) + d(p_j, [r_j, h_j](t_{i_j})) \\ &\leq d(p_j, q_j) + E\ell(h_j, q_j) \\ &\leq d(p_j, q_j) + 2Ed(p_j, q_j) \\ &\leq (1 + 2E)d(p_j, q_j) \end{aligned}$$

which shows that $[q_j, c_j = [r_j, h_j](t_{i_j})]$ is $(1 + 2E)$ – bounded by $[p_j, q_j]$. Finally, since

$$\measuredangle_{pn}([p_n, q_n], [p_n, [r_n, h_n](t_{in})]) \geq \alpha$$

we can apply lemma 2.1 in this case, from which we deduce that for every $L > 0$ there exist $R(L) > 0$, $j(L) > 0$ such that if $j > j(L)$ we must have that $d([p_j, q_j](t), [p_j, c_j]) \geq L$, $\forall t \geq R(L)$, where $[p_j, q_j](t)$ is an arclength parametrization of $[p_j, q_j]$ with $[p_j, q_j](0) = p_j$. And since $R(L)$ does not depend on j we have that $\ell[p_j, q_j] - R(L) \to +\infty$ if $j \to +\infty$ which completes the proof of the fact that the sequence $\Delta(p_j, q_j, c_j)$ is L-fat for every $L > 0$ and j big enough. □

Proof of Theorem 2.1. Assume that $\widehat{M}$ is not a visibility manifold. We are going to exibit a sequence of quadrilaterals $\square(a_n, b_n, c_n, d_n)$ satisfying the conditions in the statement of the theorem. Consider the sequence of triangles $\Delta(p_n, q_n, r_n)$ defined above. Let $h_n \in [p_n, r_n]$ be as before the point such that $d(h_n, p_n) = d(p_n, q_n)$. According to lemma 2.2 we have two feasible behaviors for subsequences of $\Delta(p_n, q_n, r_n)$.

Case 1. There exist $M > 0$, $P > 0$ and a subsequence $n_k \to +\infty$ such that for every $k \in \mathbb{N}$ there exists $x_k \in [h_{n_k}, r_{n_k}]$ satisfying

i) $d(x_k, h_{n_k}) \leq Pd(h_{n_k}, q_{n_k})$

ii) $d(x_k, [r_{n_k}, q_{n_k}]) \leq M$

Consider the triangles $\Delta(p_{n_k}, q_{n_k}, x_k)$. From lemma 2.2 we also know that for every $L > 0$ there exists $R(L) > 0$, $k(L) > 0$ such that $\forall k \geq k(L)$ we have

i) $d([p_{n_k}, q_{n_k}](t), [p_{n_k}, x_k]) \geq L + 1$, $\forall t \geq R(L)$

ii) $\ell[q_{n_k}, x_k] \leq D\ell[p_{n_k}, q_{n_k}]$

Let

$$s_0 = \inf\{s \in \ell[p_{n_k}, q_{n_k}], d([p_{n_k}, q_{n_k}](s), [p_{n_k}, x_k]) \geq L\}.$$

Let $\overline{p}_{n_k} = [p_{n_k}, q_{n_k}](s_0)$ and let $\overline{x}_k \in [p_{n_k}, x_k]$ be such that $d(\overline{p}_{n_k}, \overline{x}_k) = L$. We claim that the sequence of quadrilaterals $\square(a_k = \overline{p}_{n_k}, b_k = q_{n_k}, c_k = \overline{x}_k, d_k = x_k)$ with sides

$$[\overline{p}_{n_k}, q_{n_k}], [q_{n_k}, x_k], [x_k, \overline{x}_k], [\overline{x}_k, \overline{p}_{n_k}]$$

satisfies the statement of theorem 2.1. Indeed, from (i) above we have that $\ell[p_{n_k}, \overline{p}_{n_k}] \leq R(L)$ which implies that

$$\frac{\ell[p_{n_k}, \overline{p}_{n_k}]}{\ell[p_{n_k}, q_{n_k}]} \leq \frac{R(L)}{n_k} \to 0$$

and $\ell[\overline{p}_{n_k}, q_{n_k}] \to +\infty$ when $k \to +\infty$, by the choice of the sequence p_n, q_n, r_n. Let $\overline{q}_{n_k} \in [r_{n_k}, q_{n_k}]$ such that $d(x_k, \overline{q}_{n_k}) \leq M$. By the quasi-convexity we get that

$$\overline{d}([x_k, \overline{q}_{n_k}], [q_{n_k}, \overline{q}_{n_k}]) \leq KM + C$$

$$\Downarrow$$

$$\begin{aligned} d(p_{n_k}, [x_k, q_{n_k}]) &\geq d(p_{n_k}, [q_{n_k}, \overline{q}_{n_k}]) - KM - C \\ &\geq d(p_{n_k}, q_{n_k}) - KM - C \\ &\geq (1 - \delta)d(p_{n_k}, q_{n_k}) \\ &= (1 - \delta)\ell[p_{n_k}, q_{n_k}] \end{aligned} \qquad (*)$$

where δ is as close to zero as we wish and $k \geq k(\delta)$ big enough. Since

$$\overline{d}(p_{n_k}, [\overline{p}_{n_k}, \overline{x}_k]) \leq R(L) + M$$

we obtain

$$\begin{aligned} \inf_{t,s\in[0,1]} \{d([\overline{p}_{n_k}, \overline{x}_k](t), [q_{n_k}, x_k](s))\} &\geq d(p_{n_k}, [q_{n_k}, x_k]) - \overline{d}(p_{n_k}, [\overline{p}_{n_k}, \overline{x}_k]) \\ &\geq (1 - \delta)d(p_{n_k}, q_{n_k}) - R(L) - M \\ &\geq (1 - \overline{\delta})d(p_{n_k}, q_{n_k}) \\ &\geq (1 - \overline{\delta})d(\overline{p}_{n_k}, q_{n_k}) \end{aligned}$$

for $\overline{\delta}$ close to zero and $k \geq k(\overline{\delta})$ suitably large. This, together with the assumptions on the sequence p_n, q_n, r_n clearly imply statements (i) and (iii) of Theorem 2.1. To show statement (ii) of theorem 2.1 we must show that the lengths of the sides of the quadrilateral $\square(\overline{p}_{n_k}, q_{n_k}, \overline{x}_k, x_k)$

are D-bounded by the length of $[\overline{p}_{n_k}, q_{n_k}]$ for some constant D. By lemma 2.2 we already had that

$$\begin{aligned}\ell[q_{n_k}, x_k] &\leq D\ell[p_{n_k}, q_{n_k}]\\ &\leq D(\ell[\overline{p}_{n_k}, q_{n_k}] + \ell[\overline{p}_{n_k}, p_{n_k}])\\ &\leq D\ell[\overline{p}_{n_k}, q_{n_k}]\left(1 + \frac{\ell[\overline{p}_{n_k}, p_{n_k}]}{\ell[\overline{p}_{n_k}, q_{n_k}]}\right)\\ &\leq D\ell[\overline{p}_{n_k}, q_{n_k}]\left(1 + \frac{R(L)}{\ell[\overline{p}_{n_k}, q_{n_k}]}\right)\\ &\leq 2D\ell[\overline{p}_{n_k}, q_{n_k}]\end{aligned}$$

if k is large enough. On the other hand, since $\ell[\overline{p}_{n_k}, x_k] = M$ and $\ell[\overline{p}_{n_k}, q_{n_k}] \to +\infty$ with k we clearly have for k large that

$$\ell[\overline{p}_{n_k}, x_k] \leq \ell[\overline{p}_{n_k}, q_{n_k}]$$

Finally, we can estimate the length of $[\overline{x}_k, x_k]$ as follows:

$$\begin{aligned}\ell[\overline{x}_k, x_k] &\leq M + d(\overline{p}_{n_k}, q_{n_k}) + d(q_{n_k}, x_k)\\ &\leq M + d(\overline{p}_{n_k}, q_{n_k}) + Dd(\overline{p}_{n_k}, q_{n_k})\\ &\leq (D+2)d(\overline{p}_{n_k}, q_{n_k})\end{aligned}$$

for k big. This concludes the proof of the claim.

So it remains to analize.

Case 2. Here we have from lemma 2.2 that for every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that for every $n \geq n_m$ the sequence of triangles $\Delta(r_n, h_n, q_n)$ is m-fat with respect to $[r_n, h_n]$ and $[h_n, q_n]$ is $\frac{1}{m-1}$-bounded by $[r_n, h_n]$. From lemma 2.2 we can take $x(n_m) \in [r_{nm}, h_{nm}], y(n_m) \in [r_{nm}, q_{nm}]$ such that

1) $d(y(n_m), [r_{nm}, q_{nm}]) = d(x(n_m), y(n_m)) = m$
2) $d([r_{nm}, h_{nm}](t), [r_{nm}, q_{nm}]) \geq m, \forall t \in [0, \ell[r_{nm}, h_{nm}]]$ with $[r_{nm}, h_{nm}](t) \in [y(n_m), h_{nm}]$.
3) $\ell[h_{nm}, q_{nm}] \to +\infty$ and $\ell[h_{nm}, q_{nm}] \leq \frac{1}{m-1}\ell[x(n_m), h_{nm}]$

We claim that the sequence of quadrilaterals $\square(a_m = x(n_m),\ b_m = h_{nm},\ c_m = y(n_m),\ d_m = q_{nm})$ with sides $[x(n_m), h_{nm}]$, $[h_{nm}, q_{nm}]$, $[q_{nm}, y(n_m)]$, $[y(n_m), x(n_m)]$ satisfies Theorem 2.1. First notice that since

$$d(x, [r_{nm}, q_{nm}]) \geq m = \ell[x(n_m), y(n_m)]$$

for every $x \in [x(n_m), h_{nm}]$ we have that

$$\ell[h_{nm}, q_{nm}] \geq \ell[x(n_m), y(n_m)]$$

so from (3) above we get

$$\ell[x(n_m), y(n_m)] \leq \frac{1}{m-1}\ell[x(n_m), h_{nm}]$$

This implies that

$$\begin{aligned}d(y(n_m), q_{nm}) &\leq \ell[y(n_m), x(n_m)] + \ell[x(n_m), h_{nm}] + \ell[h_{nm}, q_{nm}]\\ &\leq \frac{2}{m-1}\ell[x(n_m), h_{nm}] + \ell[x(n_m), h_{nm}]\\ &\leq (\frac{2}{m-1} + 1)\ell[x(n_m), h_{nm}]\end{aligned}$$

which clearly implies theorem 2.1 (ii). On the other hand

$$\begin{aligned}d(x(n_m), [h_{nm}, q_{nm}]) &\geq \ell[x(n_m), h_{nm}] - \ell[h_{nm}, q_{nm}]\\ &\geq \ell[x(n_m), h_{nm}] - \frac{1}{m-1}\ell[x(n_m), h_{nm}]\\ &\geq (1 - \frac{1}{m-1})\ell[x(n_m), h_{nm}]\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}\underline{d}([x(n_m), y(n_m)], [h_{nm}, q_{nm}]) &\geq d(x(n_m), [h_{nm}, q_{nm}]) - d(x(n_m), y(n_m))\\ &\geq (1 - \frac{1}{m-1})\ell[x(n_m), h_{nm}] - d(x(n_m), y(n_m))\\ &\geq (1 - \frac{2}{m-1})\ell[x(n_m), h_{nm}]\end{aligned}$$

where we recall that $\underline{d}([x(n_m), y(n_m)], [h_{nm}, q_{nm}])$ is the infimum of the distances between the points of $[x(n_m), y(n_m)]$ and $[h_{nm}, q_{nm}]$. From this we deduce theorem 2.1 (i) and (iii) and we finish the proof of the theorem. □

3. Preparation lemmas

As in the previous section, if S, T are subsets of $\widehat{M}$ then $d(S, T)$ will be the Hausdorff distance between them.

Lemma 3.1. *Let M be a compact manifold with no conjugate points such that $\widehat{M}$ is a K, C-quasi-convex manifold. Suppose we have constants d_1, $d_2 > 0$ and a sequence of geodesics segments $[x_n, y_n], [z_n, p_n]$ such that*

i) $d(x_n, z_n) \leq d_1, d(y_n, p_n) \leq d_1$, $\forall n$

ii) *There exists $\alpha_n \in [x_n, y_n]$ such that*

$$\inf\{d(\alpha_n, x_n), d(\alpha_n, y_n)\} \to +\infty, \quad \textit{if} \quad n \to +\infty,$$

and

$$d(\alpha_n, [z_n, p_n]) \geq d_2$$

for every $n \in \mathbb{N}$. Then there exists a pair of geodesics γ, τ in $\widehat{M}$ such that $d(\gamma, \tau) \leq Kd_1 + C$ and $d(\gamma(0), \tau) \geq d_2$.

Proof. Condition (ii) in the statement of the lemma clearly implies that $\lim_{n\to+\infty} \ell[x_n, y_n] = +\infty$, and then by condition (i) we get $\lim_{n\to+\infty} \ell[z_n, p_n] = +\infty$, $\lim_{n\to+\infty} \ell[\alpha_n, z_n] = +\infty$, $\lim_{n\to+\infty} \ell[\alpha_n, p_n] = +\infty$. Fix a fundamental domain M_0 of M in $\widehat{M}$ and let φ_n be a sequence of covering maps such that $\psi_n(\alpha_n) \in M_0$, $\forall n$. Also, by the quasi-convexity of the metric we have that

$$d([x_n, y_n], [z_n, p_n]) \leq Kd_1 + C$$

for every n. Consider the geodesic segments $\psi_n[x_n, y_n] = [\psi_n(x_n), \psi_n(y_n)]$ and $\psi_n[z_n, p_n] = [\psi_n(z_n), \psi_n(p_n)]$. Let $[x_n, y_n](t)$ be an arclength parametrization of $[x_n, y_n]$ such that $[x_n, y_n](0) = \alpha_n$ and $[x_n, y_n](-\ell[x_n, \alpha_n]) = x_n$. Since the maps ψ_n are isometries we have

a) $d([\psi_n(x_n), \psi_n(y_n)], [\psi_n(z_n), \psi_n(p_n)]) \leq Kd_1 + C$, $\forall n$

b) $\psi_n(\alpha_n) = \psi_n([x_n, y_n](0)) \in M_0$, $\forall n$ and

$$\inf\{d(\psi_n(\alpha_n), \psi_n(x_n)), d(\psi_n(\alpha_n), \psi_n(y_n)), d(\psi_n(\alpha_n), \psi_n(p_n)),$$
$$d(\psi_n(\alpha_n), \psi_n(p_n))\} \to +\infty$$
$$d(\psi_n(\alpha_n), [\psi_n(z_n), \psi_n(p_n)]) \geq d_2 \,.$$

So by taking convergent subsequences $[\psi_n(x_n), \psi_n(y_n)] \to \gamma$, $[\psi_n(z_n), \psi_n(p_n)] \to .\tau$ we get two geodesics $\gamma\colon(-\infty, +\infty) \to M$, $\tau\colon(-\infty, +\infty) \to M$ satisfying $d(\gamma, \tau) \leq Kd_1 + C$ and $d(\gamma(0), \tau) \geq d_2$ thus proving lemma 3.1. $\square$

Let us recall the conclusions of theorem 2.1. Let $\overline{M}$ be a quasi-convex manifold without conjugate points. If $\overline{M}$ is not a visibility manifold then there exist constants $B > 0$, $D > 0$ such that for a given $m \in \mathbb{N}$ there are sequences a_n, b_n, c_n, d_n of points in $\widehat{M}$ such that

i) $d(a_n, b_n) \to +\infty, d(b_n, d_n) \to +\infty, d(c_n, d_n) \to +\infty, d(a_n, c_n) = m$ and

$$\inf_{t\in[0,1]} \{d([a_n, b_n](t), [c_n, d_n])\} \geq m, \ \forall n \geq n_0,$$

where $[a_n, b_n]\colon [0,1] \to \widehat{M}$ is a parametrization of the segment $[a_n, b_n]$.

ii) $d(b_n, d_n) \leq Bd(a_n, b_n)$ and $d(c_n, d_n) \leq Bd(a_n, b_n)$

iii) $\inf_{t,s\in[0,1]}\{d([a_n, c_n](t), [b_n, d_n](s))\} \geq Dd(a_n, b_n)$.

In other words, for every fixed $m \in \mathbb{N}$ we obtain a sequence of quadrilaterals $\square(a_n, b_n, c_n, d_n)$ having sides $[a_n, b_n], [b_n, d_n], [d_n, c_n], [c_n, a_n]$ such that the lengths of all sides go to $+\infty$ with n in a controlled way, their 'widths' are comparable to $\ell[a_n, b_n]$ and their 'hights' with respect to $[a_n, b_n]$ are m. Let us say that $x, y \in [p, q]$ satisfy $x < y$ if $d(p, x) < d(p, y)$. Define the points $y_{ni} \in [c_n, d_n]$ by

$$y_{ni} = \sup\{x \in [c_n, d_n] \backslash d(t, [a_n, b_n]) \leq m^i, \ \ \forall t < x, t \in [c_n, d_n]\}$$

Notice that by the construction of the point c_n we deduce that $y_{n1} = c_n$, $\forall n \in \mathbb{N}$. We shall state some properties of the points y_{ni} in the following lemma.

Lemma 3.2.

i) $d(y_{ni}, [a_n, b_n]) = m^i$

ii) $d(y_{ni}, y_{ni+1}) \geq m^i(m-1)$

iii) *For every $x \in [y_{ni}, y_{ni+1}]$ we have*

$$d(x, [a_n, b_n]) \geq \frac{1}{K}(m^i - C)$$

iv) *Let $i(n)$ be the number of points y_{ni} in the segment $[c_n, d_n]$. Then there exists a constant $L > 0$ independent of n such that*

$$i(n) \leq L\log(\ell[a_n, b_n])$$

Proof. Assertion (i) follows easily from the very definition of the y_{ni}'s. To show assertion (ii) notice that $d(y_{ni+1}, [a_n, b_n]) = m^{i+1}$ so the length of every path from y_{ni+1} to $[a_n, b_n]$ must be greater or equal than m^{i+1}. Let $q_{ni} \in [a_n, b_n]$ be such that $d(y_{ni}, q_{ni}) = m^i$. This last remark implies that the path formed by $[y_{ni}, y_{ni+1}] \cup [y_{ni}, q_{ni}]$ has length $\geq m^{i+1}$. But this means that

$$d(y_{ni}, y_{ni+1}) \geq m^{i+1} - d(y_{ni}, q_{ni}) \geq m^{i+1} - m^i = m^i(m-1)$$

which proves (ii). For the proof of (iii) we use the K, C-quasi-convexity. Indeed, if $x > y_{ni}$ let $z \in [a_n, b_n]$ be defined by $d(x, z) = d(x, [a_n, b_n])$. Then we have that

$$m^i = d(y_{ni}, [a_n, b_n]) \leq d(y_{ni}, z) \leq Kd(x, z) + C$$

and this is just assertion (iii). To show (iv) let $i(n)$ be the number of points y_{ni}'s in $[a_n, b_n]$. By the quasi-convexity and Theorem 2.1 we deduce that

$$m^{i(n)} = d(y_{ni(n)}, [a_n, b_n]) \leq Kd(b_n, d_n) + C \leq KB\ell(a_n, b_n) + C$$

where B is the constant defined in Theorem 2.1. So we get

$$i(n) \leq \frac{1}{\log m}(\log K + \log B + \log \ell(a_n, b_n))$$

and since m is fixed and $\ell(a_n, b_n)$ goes to $+\infty$ assertion (iv) follows. □

Recall as before that the points $q_{ni} \in [a_n, b_n]$ are defined by $d(y_{ni}, [a_n, b_n]) = d(y_{ni}, q_{ni})$. The points q_{ni} may not be uniquely defined, nevertheless our reasoning is independent of this fact. From now on we just associate to each i one and only one q_{ni}. Remark that once q_{ni} and q_{nj} are interior points of $[a_n, b_n]$ then $q_{ni} = q_{nj}$ if and only if $i = j$. This is because in this case both points are perpendicular to $[a_n, b_n]$.

Lemma 3.3. *Let $m \geq 2C$, and suppose that there exists a subsequence $n_s \to +\infty$ such that for every s there is a $1 \leq i_s \leq i(n_s)$ satisfying*

$$\lim_{s \to +\infty} \frac{d(y_{i_s}, [a_{n_s}, b_{n_s}])}{d(q_{i_s}, q_{i_s+1})} = 0$$

Then there exists a pair γ, β of geodesics in $\widehat{M}$ such that

$$d(\gamma, \beta) \leq Km^2 + C$$

and $d(\gamma(0), \beta) \geq \frac{m}{2K}$.

Proof. If there exists an infinite subsequence $s_p \to +\infty$ such that $i_{s_p} = 1$, $\forall p \in \mathbb{N}$ then we can apply lemma 3.1 to the sequence of pairs $[a_n, q_{n2}] = [q_{n1}, q_{n2}], [c_n, y_{n2}] = [y_{n1}, y_{n2}]$ with $d_1 = Km + C$ and $d_2 = m$. Indeed, in this case we would have that $d(a_n, c_n) = m$, $d(y_{n2}, q_{n2}) = m$, $\underline{d}([a_n, q_{n2}], [c_n, y_{n2}]) \geq m$ by Theorem 2.1, and by the hypotheses of the lemma we would also get that $\ell[a_n, q_{n2}] \to +\infty$, $\ell[c_n, y_{n2}] \to +\infty$. So let us assume that $i_s \geq 2$ for $s \geq s_0$ sufficiently big. For a given $i \leq i(n)$ let $p_j \in [y_{ni+1}, q_{ni+1}]$ be defined by $p_1 = y_{ni+1}$ and $d(p_j, p_{j+1}) = m^2$, where $1 \leq j \leq m^{i-1}$. Let $x_j \in [y_{ni}, q_{ni}]$ be defined by $x_1 = y_{ni}$, $d(x_j, x_{j+1}) = m$. So the number of points x_j's is $\frac{d(y_{ni}, q_{ni})}{m} = \frac{m^1}{m} = m^{i-1}$ which is the same number of p_j's. By the quasi-convexity we have that

$$d([x_j, p_j], [x_{j+1}, p_{j+1}]) \leq Km^2 + C$$

for every j. Let $z_1 \in [y_{ni}, y_{ni+1}] = [x_1, p_1]$ be the point whose distance to y_{ni+1} is $\frac{1}{2}d(y_{ni}, y_{ni+1})$, and define $z_{j+1} \in [x_{j+1}, p_{j+1}]$ by

$$d(z_j, z_{j+1}) = d(z_j, [x_{j+1}, p_{j+1}]).$$

In that way we get a path $\Gamma = \bigcup_{j=1}^{m^{i-1}-1} [z_j, z_{j+1}]$ from $z_1 \in [y_{ni}, y_{ni+1}]$ to $z_{m^{i-1}} \in [a_n, b_n]$. It is clear that

$$\ell(\Gamma) \leq (m^{i-1} - 1)(Km^2 + C) \leq (\ell[y_{ni}, q_{ni}] - m)\left(Km + \frac{C}{m}\right)$$

since $\ell[y_{ni}, q_{ni}] = m^i$.

Claim. The distance from every point $x \in \Gamma$ to either $[y_{ni}, q_{ni}]$ or $[y_{ni+1}, q_{ni+1}]$ goes to infinity with s if $i = i_s$.

In fact, from the last inequality we obtain that

$$\frac{\ell(\Gamma)}{\ell[q_{ni}, q_{ni+1}]} \leq \left(\frac{\ell[y_{ni}, q_{ni}] - m}{\ell[q_{ni}, q_{ni+1}]}\right)(Km + C)$$

for $m \geq 1$, so if $i = i_s$ the left side of the inequality goes to zero if s goes to $+\infty$ by the hypotheses of the lemma. So for every $x \in [y_{ni}, q_{ni}]$ we get that

$$d(x,\Gamma) \ge d(y_{ni}, z_1) - \ell(\Gamma) - \ell[y_{ni}, q_{ni}]$$
$$\ge \frac{1}{2}\ell[y_{ni}, y_{ni+1}] - \ell(\Gamma) - \ell[y_{ni}, q_{ni}]$$
$$\ge \frac{1}{2}(\ell[q_{ni}, q_{ni+1}] - \ell[y_{ni}, q_{ni}] - \ell[y_{ni+1}, q_{ni+1}]) - \ell(\Gamma) - \ell[y_{ni}, q_{ni}]$$
$$\Downarrow$$
$$\frac{d(x,\Gamma)}{\ell[q_{ni}, q_{ni+1}]} \ge \frac{1}{2}\left(1 - 2\frac{\ell[y_{ni+1}, q_{ni+1}]}{\ell[q_{ni}, q_{ni+1}]}\right) - \frac{\ell(\Gamma)}{\ell[q_{ni}, q_{ni+1}]} - \frac{\ell[y_{ni}, q_{ni}]}{\ell[q_{ni}, q_{ni+1}]}$$

where, by the hypotheses, if $i = i_s$ all the terms in the right hand side of the inequality which are divided by $\ell[q_{ni}, q_{ni+1}]$ go to zero if $s \to +\infty$. Therefore, given $\delta > 0$ there exists s_0 such that $\forall s \ge s_0$ we have

$$\underline{d}(\Gamma, [y_{ni_s}, q_{ni_s}]) \ge \frac{1}{2}(\ell[q_{ni_s}, q_{ni_s+1}] - \delta)\,.$$

Analogously we deduce that there exists s_1 such that $\forall s \ge s_1$ we get

$$\underline{d}(\Gamma, [y_{ni_s+1}, q_{ni_s+1}]) \ge \frac{1}{2}(\ell[q_{ni_s}, q_{ni_s+1}] - \delta)$$

These last two relations imply the claim.

To conclude the proof of the lemma, remark that from lemma 3.2 (iii) we must have

$$\ell(\Gamma) = \sum_{j=1}^{m^{i_s-1}-1} \ell[z_j, z_{j+1}] \ge d(z_1, [a_{n_s}, b_{n_s}]) \ge \frac{1}{K}(m^{i_s} - C)$$

since $z_1 \in [y_{ni_s}, y_{ni_s+1}]$. Thus, there exists $j_0 = j_0(s) \le m^{i_s-1}$ such that

$$\ell[z_{j_0}, z_{j_0+1}] \ge \frac{1}{K}\left(\frac{m^{i_s} - C}{m^{i_s-1} - 1}\right) = \frac{m}{K}\left(\frac{1 - C/m^{i_s}}{1 - 1/m^{i_s-1}}\right) \ge \frac{m}{2K}$$

since $m \ge 2C$ and $i_s > 1$. Joining the claim and the last inequality we conclude that the sequence of pairs of geodesic segments $[x_{j_0(n)}, p_{j_0(n)}]$, $[x_{j_0(n)+1}, p_{j_0(n)+1}]$ satisfies the hypotheses of lemma 3.1 with $d_1 = Km^2 + C$ and $d_2 = \frac{m}{2K}$ from which follows lemma 3.3. □

Corollary 3.1. *Let $m \ge 2C$. Assume that there exist $\alpha > 0$ and a sequence $n_j \to +\infty$ such that for every $j \in \mathbb{N}$ there exists $0 \le i_j \le n_j$ satisfying*

i) $\frac{\ell[a_{n_j}, y_{n_j i_j}]}{\ell[a_{n_j}, b_{n_j}]} \ge \alpha$, $\forall j \in \mathbb{N}$

ii) *Either* $\lim_{j\to+\infty} \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, b_{n_j}]} = 0$ *or* $\lim_{j\to+\infty} \frac{\ell[b_{n_j}, d_{n_j}]}{\ell[a_{n_j}, b_{n_j}]} = 0$

Then there exists a pair of geodesics $\gamma\colon(-\infty,\infty)\to\widehat{M}$, $\beta\colon(-\infty,\infty)\to\widehat{M}$ such that $d(\gamma,\beta) \le Km^2 + C$ and $d(\gamma(0),\beta) \ge \frac{m}{2K}$.

Proof. We shall check that the hypotheses of Corollary 3.1 imply the assumptions of either lemma 3.1 or lemma 3.3. Again, if there exist $L > 0$ and a subsequence $j_p \to +\infty$ such that $i_{j_p} \le L$, $\forall p \in \mathbb{N}$ we can apply lemma 3.1 to the sequence of pairs $[a_{n_{j_p}}, q_{n_{j_p}L}]$, $[c_{n_{j_p}}, y_{n_{j_p}L}]$ with $d_1 = Km^L + C$ and $d_2 = m$. So suppose that i_j goes to infinity with j. Observe first that from lemma 3.2 (iv) we have that

$$\frac{1}{K}(m^{i(n)} - C) \le \ell[b_n, d_n] \le m^{i(n)+1}$$
$$\Downarrow$$
$$\frac{1}{K} - \frac{C}{m^{i(n)}} \le \frac{\ell[b_n, d_n]}{m^{i(n)}} \le m$$

We also have that

$$\frac{\ell[y_{ni}, q_{ni}]}{\ell[y_{ns}, q_{ns}]} = \frac{m^i}{m^s} = m^{i-s}$$

and

$$\frac{\ell[y_{ni}, q_{ni}]}{\ell[b_n, d_n]} = \frac{m^i}{\ell[b_n, d_n]} = \frac{m^{i(n)}}{\ell[b_n, d_n]} m^{i-i(n)}$$
$$\le m^{-(i(n)-i)}\left(\frac{1}{K} - Cm^{-i(n)}\right)$$
$$\le Am^{-(i(n)-i)}$$

for every $i \le i(n)$, where $A \ge 1$ is some constant depending on m, C and K. So we get that

$$\sum_{i=1}^{i(n)} \frac{\ell[y_{ni}, q_{ni}]}{\ell[b_n, d_n]} \le A\sum_{i=1}^{i(n)} m^{-i(n)-i)} \le A(1 - \frac{1}{m})^{-1} = A\frac{m}{m-1}$$

and similarly

$$\sum_{i=1}^{s} \frac{\ell[y_{ni}, q_{ni}]}{\ell[y_{ns}, q_{ns}]} \leq \sum_{i=1}^{s} m^{s-i} \leq \frac{m}{m-1}$$

for every $s \leq i(n)$. To simplify the notation let us define $y_{n(i(n)+1)} = d_n$ and $q_{n(i(n)+1)} = b_n$. So in the remaining of this proof we consider $1 \leq i_j \leq n_j + 1$.

Claim. Given $p \in \mathbb{N}$ there exist $j(p) \in \mathbb{N}$ and $1 \leq \nu_p \leq i_{j(p)}$ such that

$$\frac{\ell[q_{n\nu p}, q_{n(\nu_p+1)}]}{\ell[y_{n\nu_p}, q_{n\nu_p}]} \geq p\,.$$

Indeed, otherwise there would exist constants $E > 0$ and $j_0 > 0$ such that for every $j \geq j_0$

$$\frac{\ell[q_{n_j i}, q_{n_j(i+1)}]}{\ell[y_{n_j i}, q_{n_j i}]} \leq E$$

for every $1 \leq i \leq i_j$. Notice that in general we have that

$$d(a_n, q_{ni}) \leq \sum_{s=1}^{i-1} \ell[q_{ns}, q_{n(s+1)}]$$

because the points q_{ni} may not be well-ordered in $[a_n, b_n]$, i.e., it may exists i such that $q_{ni} > q_{ni+1}$. Nevertheless this implies that

$$d(a_{n_j}, q_{n_j i_j}) \leq \sum_{i=1}^{i_j-1} \ell[q_{n_j i}, q_{n_j(i+1)}] \leq E \sum_{i=1}^{i_j-1} \ell[y_{n_j i}, q_{n_j i}]$$

$$\leq E\, A \frac{m}{m-1} \ell[y_{n_j i_j}, q_{n_j i_j}]$$

$$\Downarrow$$

$$\frac{\ell[a_{n_j}, q_{n_j i_j}]}{\ell[a_{n_j}, y_{n_j i_j}]} \leq E\, A \frac{m}{m-1} \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, y_{n_j i_j}]}$$

$$\leq E\, A \frac{m}{m-1} \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, b_{n_j}]} \frac{\ell[a_{n_j}, b_{n_j}]}{\ell[a_{n_j}, b_{n_j}]} \ell[a_{n_j}, y_{n_j i_j}]$$

$$\leq \frac{E\, Am}{\alpha(m-1)} \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, b_{n_j}]} \to 0$$

if $j \to +\infty$, where in the last two steps we used the hypotheses in the statement. But this leads to a contradiction since

$$d(a_{n_j}, y_{n_j i_j}) \leq d(a_{n_j}, q_{n_j i_j}) + d(y_{n_j i_j}, q_{n_j i_j})$$

$$\Downarrow$$

$$\frac{\ell[a_{n_j}, q_{n_j i_j}]}{\ell[a_{n_j}, y_{n_j i_j}]} \geq 1 - \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, y_{n_j i_j}]} \geq \frac{1}{2}$$

for j big enough since

$$\frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, y_{n_j i_j}]} = \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, b_{n_j}]} \frac{\ell[a_{n_j}, b_{n_j}]}{\ell[a_{n_j}, y_{n_j i_j}]} \leq \frac{1}{\alpha} \frac{\ell[y_{n_j i_j}, q_{n_j i_j}]}{\ell[a_{n_j}, b_{n_j}]}$$

which goes to zero if j goes to infinity by hypotheses. This finishes the proof of the claim.

To conclude the proof of corollary 3.1 just notice that the claim implies that

$$\frac{\ell[y_{n\nu_p}, q_{n\nu_p}]}{\ell[q_{n\nu_p}, q_{n(\nu_p+1)}]} \leq \frac{1}{p} \to 0$$

as p increases. But this allows us to apply lemma 3.3 to the subsequence $n_p = n_{j(p)}$, with $1 \leq i = \nu_p \leq i_{j(p)}$ as the appropriate $i \leq i(n_p)$ satisfying the hypotheses of lemma 3.3. □

4. The proof of the main theorem

Actually, the preparation lemmas of the last section provide a proof of a particular case of Theorem B. Indeed, the statement of corollary 3.1 says that if $\ell[b_n, d_n]$ is very small compared with $\ell[a_n, b_n]$ then Theorem B holds. So let us assume throughout this section that there exists $\beta > 0$ such that

$$\frac{\ell[b_n, d_n]}{\ell[a_n, b_n]} \geq \beta$$

for every $n \in \mathbb{N}$. Let $K > 0$, $C > 0$ be as before the quasi-convexity constants of themetric in $\widehat{M}$ and fix $m \geq 2C$. Let $D > 0$ be the constant defined in Theorem 2.1, i.e.,

$$\underline{d}([a_n, c_n], [b_n, d_n]) \geq D\ell[a_n, b_n]$$

for every n, where recall that $\underline{d}(S,T)$ is the infimum of the distances from points in S to points in T. Define $z_n \in [c_n, d_n]$ by $d(z_n, c_n) = \frac{1}{3} D\ell[a_n, b_n]$. Let $i(z_n)$ be defined by

$$z_n \in [y_{ni(z_n)}, y_{n(i(z_n)+1)}]$$

where y_{nj} comes from the last section, i.e., $y_{nj} \in [c_n, d_n]$ is the supremum of the points $t \in [c_n, d_n]$ such that $d(x, [a_n, b_n]) \leq m^j$, $\forall x \in [c_n, t]$.

Lemma 4.1. *Suppose that there exists a subsequence $n_s \to +\infty$ such that $j_s = i(n_s) - i(z_{n_s}) \to +\infty$ if $s \to +\infty$. Then Theorem B holds.*

Proof. In this case we have

$$\begin{aligned} d(y_{n_s(i(z_{n_s})+1)}, [a_{n_s}, b_{n_s}]) = d(y_{n_s(i(z_{n_s})+1)}, q_{n_s(i(z_{n_s})+1)}) &= m^{i(z_n)+1} \\ &= m^{i(n_s)-j_s+1} \\ &= \frac{m^{i(n_s)}}{m^{j_s-1}} = \frac{\ell[y_{n_s i(n_s)}, q_{n_s i(n_s)}]}{m^{j_s-1}} \\ &\leq \frac{K\ell[b_{n_s}, d_{n_s}] + C}{m^{j_s-1}} \\ &\leq \frac{KB\ell[a_{n_s}, b_{n_s}] + C}{m^{j_s-1}} \end{aligned}$$

where B is the constant defined in Theorem 2.1. This implies that

$$\lim_{s\to+\infty} \frac{\ell[y_{n_s(i(z_{n_s})+1)}, q_{n_s(i(z_{n_s})+1)}]}{\ell[a_{n_s}, b_{n_s}]} = 0 \tag{i}$$

On the other hand

$$\begin{aligned} \frac{\ell[y_{n_s(i(z_{n_s})+1)}, a_{n_s}]}{\ell[a_{n_s}, b_{n_s}]} &\geq \frac{1}{\ell[a_{n_s}, b_{n_s}]}(\ell[z_{n_s}, c_{n_s}] - \ell[a_{n_s}, c_{n_s}]) \\ &\geq \frac{1}{\ell[a_{n_s}, b_{n_s}]}(\ell[z_{n_s}, c_{n_s}] - m) \geq \frac{D}{4} \end{aligned} \tag{ii}$$

for s suitably big. Therefore, the number $\alpha = \frac{D}{3}$ and the subsequences n_s, $i_s = i(z_{n_s})$ satisfy the hypotheses of corollary 3.1, from which we conclude that Theorem B holds in this case. □

End of proof of Theorem B. According to lemma 4.1 we can assume that there exists $w > 0$ such that

$$j_n = i(n) - i(z_n) \leq w$$

for every $n \in \mathbb{N}$. Let $p_j \in [b_n, d_n]$, $j = 0, 1, \ldots, r_n$ be defined by $p_0 = d_n$, $p_{r_n} = b_n$ and

$$d(p_j, p_{j+1}) = m$$

for every $j < r_n - 1$ and $d(p_{r_n-1}, p_{r_n}) = (\ell[b_n, d_n]/m) - [[\ell[b_n, d_n]/m]]$, where $[[a]]$ is the integer part of $a \in \mathbb{R}$. Clearly r_n is either $[[\ell[b_n, d_n]/m]]$ or $[[\ell[b_n, d_n]/m]]+1$ and the number of intervals $[p_j, p_{j+1}]$ is r_n. Consider the geodesic segments $[a_n, p_j]$ for n fixed and $j = 0, 1, \ldots r_n$. By the K, C quasi-convexity we have that $d([a_n, p_n], [a_n, p_{j+1}]) \leq Km + C$, $\forall j$. Now for a given $N \in \mathbb{N}$ let

$$A_j(N) = \{x \in [a_n, p_j] \backslash d(x, [a_n, p_{j+1}]) > \frac{m}{N}\}$$

Then two possibilities may occur:

1. There exists $N_0 > 0$ and a subsequence $n_r \to +\infty$ such that there is some $0 \leq j_r \leq j(n_r)$ satisfying the following property: there exists a point $x \in A_{j_r}(N_0)$ such that $\lim_{r\to+\infty} d(x, a_{n_r}) = +\infty$ and $\lim_{r\to+\infty} d(x, p_{j_r}) = +\infty$.

In this case we can apply lemma 3.1 to the sequence of pairs $[a_{n_r}, p_{j_r}]$, $[a_{n_r}, p_{j_r+1}]$, with constants $d_1 = Km + C$ and $d_2 = \frac{m}{N_0}$ so Theorem B holds.

2. There is no such N_0. Or in other words, for every $N \in \mathbb{N}$ there exist $H = H(N)$ and $n_N > 0$ such that for every $n \geq n_N$ we have that every point $x \in A_j(N)$ is at distance at most H from either a_n or p_j for every $0 \leq j \leq r(n)$. Under this condition the following holds:

Claim. There exists $\overline{N}$ depending on m, D such that for every $N \geq \overline{N}$ there exists $n(N)$ such that for every $n \geq n(N)$ there is a continuous path Γ from z_n to $[a_n, b_n]$ whose length satisfies

$$\frac{\ell(\Gamma)}{\ell[b_n, d_n]} \leq \frac{L}{N}$$

where $L > 0$ is some constant depending on m, B and β. Here $B > 0$ is the constant defined in Theorem 2.1.

For the proof of this statement consider the sequence of points $x_j \in [a_n, p_j]$ defined by $x_0 = z_n$, x_{j+1} a point in $[a_n, p_{j+1}]$ such that

$d(x_j, [a_n, p_{j+1}]) = d(x_j, x_{j+1})$. Since every point of $A_0(N)$ is at distance at most H from either a_n or $p_0 = d_n$ and $\ell[a_n, b_n] \to +\infty$ with n, we have that $d(z_n, [a_n, p_1]) = d(x_0, x_1) < \frac{m}{N}$. In this way we construct a path

$$\Gamma = \bigcup_{j=0}^{b_n} [x_j, x_{j+1}]$$

beginning at $x_0 = z_n$ such that $\ell[x_j, x_{j+1}] \leq \frac{m}{N}$, $\forall 0 \leq j \leq b_n$, where $b_n \leq r(n)$. We affirm that $b_n = r(n)$. To see this remark that

$$\ell(\Gamma) = \ell\left(\bigcup_{j=0}^{b_n} [x_j, x_{j+1}]\right) = \sum_{j=0}^{b_n} \ell[x_j, x_{j+1}] \leq b_n \frac{m}{N} \leq r(n) \frac{m}{N}.$$

On the other hand, since

$$r(n) \leq [[\ell[b_n, d_n]/m]] + 1 \leq \ell[b_n, d_n]/m + 1 \leq B\ell[a_n, b_n]/m + 1$$

this implies that

$$\ell(\Gamma) \leq \frac{m}{N}(B\ell[a_n, b_n]/m + 1)$$
$$\Downarrow$$
$$\frac{\ell(\Gamma)}{\ell[a_n, b_n]} \leq \frac{m}{N}B + \delta_n \leq 2B\frac{m}{N}$$

for n sufficiently large. But this implies that

$$\begin{aligned} \underline{d}(\Gamma, [a_n, c_n]) &\geq d(c_n, z_n) - d(a_n, c_n) - \ell(\Gamma) \\ &\geq \frac{D}{3}\ell[a_n, b_n] - m - 2B\frac{m}{N}\ell[a_n, b_n] \\ &\geq \ell[a_n, b_n]\left(\frac{D}{3} - \frac{2Bm}{N} - \frac{m}{\ell[a_n, b_n]}\right) \qquad (*) \\ &\geq \frac{D}{4}\ell[a_n, b_n] \end{aligned}$$

for $N > \frac{Bm}{6D}$ and n suitably large. Also we have

$$\begin{aligned} \underline{d}(\Gamma, [b_n, d_n]) &\geq \underline{d}([a_n, c_n], [b_n, d_n]) - d(\Gamma, [a_n, c_n]) \\ &\geq D\ell[a_n, b_n] - d(\Gamma, [a_n, c_n]) \end{aligned}$$

where $d(\Gamma, [a_n, c_n])$ is, as usual, the Hausdorff distance between the sets.

And since

$$\begin{aligned} d(\Gamma, [a_n, c_n]) &\leq d(a_n, c_n) + d(c_n, z_n) + \ell(\Gamma) \\ &\leq m + \frac{D}{3}\ell[a_n, b_n] + 2B\frac{m}{N}\ell[a_n, b_n] \\ &\leq \frac{D}{3}\ell[a_n, b_n](1 + \delta(n, N)) \\ &\leq \frac{D}{2}\ell[a_n, b_n] \end{aligned}$$

for $N > \frac{12Bm}{D}$ and n large we deduce that

$$\underline{d}(\Gamma, [b_n, d_n]) \geq \left(D - \frac{D}{2}\right)\ell[a_n, b_n] \geq \frac{D}{2}\ell[a_n, b_n] \qquad (**)$$

for n large enough. Thus, from (*) and (**) we can conclude that the endpoint of Γ, $x_{b_n+1} \in [a_n, p_{b_n+1}]$ does not belong to $A_{b_n+1}(N)$ for $N \geq \frac{12Bm}{D}$ and n sufficiently large, so $d(x_{b_n+1}, [a_n, p_{b_n+1}]) < \frac{m}{N}$. This shows that we can construct a continuous path Γ from $z_n \in [c_n, d_n]$ to some point in $[a_n, b_n]$ such that $\Gamma \cap [a_n, p_j]$ never belongs to $A_j(N)$. We already knew that $\ell(\Gamma) < \frac{2Bm}{N}\ell[a_n, b_n]$ so we have

$$\frac{\ell(\Gamma)}{\ell[b_n, d_n]} = \frac{\ell[a_n, b_n]}{\ell[b_n, d_n]}\frac{\ell(\Gamma)}{\ell[a_n, b_n]} < \frac{1}{\beta}\left(\frac{2Bm}{N}\right)$$

where $\beta > 0$ comes from the assumption in the beginning of the section. Taking $N = 12Bm/D$ and $L = 2Bm/\beta$ we finish the proof of the claim.

To conclude the proof of Theorem B we shall show that case (2) leads to a contradiction. For, on one hand we have that

$$m^{i(z_n)} = d(y_{ni(z_n)}, [a_n, b_n]) \leq Kd(z_n, [a_n, b_n]) + C \leq K\frac{L}{N}\ell[b_n, d_n] + C$$

where in the last inequality we used the previous claim. So we get

$$\frac{m^{i(z_n)}}{\ell[b_n, d_n]} \leq \frac{KL}{N} + \lambda_n$$

where $\lambda_n \to 0$ if $n \to +\infty$ from the assumptions on $[b_n, d_n]$. And on the other hand we have

$$m^{i(z_n)} = m^{i(n)-j_n} = \frac{m^{i(n)}}{m^{j_n}} \geq \frac{\ell[b_n, d_n]}{m}\left(\frac{1}{m^w}\right) = \frac{\ell[b_n, d_n]}{m^{w+1}}$$

where $w \geq j_n$, $\forall n$ according to the hypotheses of case (2). So

$$\frac{m^{i(z_n)}}{\ell[b_n, d_n]} \geq m^{-w-1}$$

which contradicts the conclusion of the claim above.

5. The geometry of the fundamental group

In this section we discuss some geometrical features concerning the fundamental group of a compact manifold with no conjugate points whose geodesic flow is expansive. Our purpose is to show that there exists a strong ressemblance between the fundamental group of such a manifold and the fundamental group of compact manifolds with negative curvature. The references we shall follow for the preliminaries of geometric group theory are [3], [8], [9].

Definition 5.1. Given a metric space (X, d) and two points p, $q \in X$, a *geodesic segment* joining p to q is an isometry $g: [0, d(p,q)] \to X$ such that $g(0) = p$, $g(d(p,q)) = q$.

(X, d) is said to be a *geodesic space* if for every pair of points in X there exists a geodesic segment joining them.

A *geodesic triangle* with vertices x, y, z is a union of three geodesic segments joining respectively x to y, y to z and z to x.

A complete Riemannian manifold is an example of a geodesic space. Another family of examples of such spaces is provided by the so-called *Cayley graphs* of finitely generated groups. Given a finitely generated group Γ and a finite, symmetric set S of generators (i.e., if $\sigma \in \Gamma$ then $\sigma^{-1} \in \Gamma$) let $l_s(\tau)$ be the length of $\tau \in \Gamma$ with respect to S, i.e., the smallest number of elements of S giving τ as a product of generators. We can endow Γ with a metric d_s defined by $d_s(\tau, \sigma) = l_s(\tau^{-1}\sigma)$.

Definition 5.2. The *Cayley graph* of Γ, $\mathcal{G}(\Gamma, S)$ is a 1-dimensional simplicial complex such that the endpoints of each interval in the complex are elements of Γ and such that every edge of the complex having endpoints γ, β satisfies $d_s(\gamma, \beta) = 1$.

We can endow $\mathcal{G}(\Gamma, S)$ with a metric which gives to each edge the

standard metric of the interval $[0, 1]$. Every curve can be decomposed into a disjoint union of segments, each one contained in some edge and then the length of this curve is the sum of the lengths of these segments. With this metric the Cayley graph is a geodesic space and there is a natural embedding of (Γ, d_s) into $\mathcal{G}(\Gamma, S)$ which is an isometry.

Definition 5.3. A geodesic space (X, d) is said to be *δ-thin* for $\delta > 0$ if every geodesic triangle enjoys the following property: the distance of every point of one given side of the triangle is at distance at most δ from the union of the other two sides.

The hyperbolic space is an example of a δ-thin space. In general, Riemannian universal coverings of compact manifolds whose geodesic flows are Anosov flows are δ-thin spaces.

Definition 5.4. A finitely generated group Γ is said to be *hyperbolic* if there exist a finite set S of symmetric generators and a constant $\delta_0 > 0$ such that $\mathcal{G}(\Gamma, S)$ is δ_0-thin.

Although the number δ_0 may depend on the choice of S the hyperbolicity of Γ does not depend on the given set of generators in the following sense.

Definition 5.5. Two geodesic spaces (X, d), (X', d') are said to be *quasi-isometric* if there exist maps $f: X \to X'$, $g: X' \to X$ and constants $\gamma > 0$, $C > 0$ such that

i) $d(f(x), f(y)) \leq \lambda d(x, y) + C$, $\forall x, y \in X$
ii) $d(g(x'), g(y')) \leq \lambda d'(x', y') + C$, $\forall x', y' \in X'$
iii) $d(g(f(x)), x) \leq C$, $\forall x \in X$
iv) $d(f(g(x')), x') \leq C$, $\forall x' \in X'$

Lemma 5.1. *Let (X, d), (X', d') be quasi-isometric spaces. Then (X, d) is δ-thin if and only if (X', d') is δ'-thin for a certain $\delta' > 0$.*

Lemma 5.2. *Let Γ be a finitely generated group and let S, S' be two finite sets of generators. Then $\mathcal{G}(\Gamma, S)$ and $\mathcal{G}(\Gamma, S')$ are quasi-isometric.*

So the hyperbolicity of a group is well defined up to quasi-isometries. The fundamental groups of compact manifolds with negative curvature are well known examples of hyperbolic groups. In general, we have the following relation between the geometry of a manifold M and $\pi_1(M)$.

Lemma 5.3. *Let M be a compact, Riemannian manifold. Then $\mathcal{G}(\pi_1(M), S)$ and $\widehat{M}$ are quasi-isometric for every finite symmetric set S of generators.*

Now we show that the collection of hyperbolic geodesic spaces includes the Visibility manifolds.

Proposition 5.1. *Let N be a simply connected Riemannian manifold with no conjugate points. Then N is a uniform Visibility manifold if and only if N is δ-thin for some $\delta > 0$.*

Proof. Suppose that N is a uniform Visibility manifold and suppose by contradiction that N is not δ-thin for any $\delta > 0$. Given two points x, $y \in N$ let $[x, y]$ be the geodesic segment joining x and y. Then there exist sequences x_n, y_n, z_n, t_n of points in N such that

i) $t_n \in [x_n, y_n]$
ii) $d(t_n, [y_n, z_n] \cup [z_n, x_n]) \geq n$

This means that $d(t_n, [y_n, z_n]) \to +\infty$ and $d(t_n[z_n, x_n]) \to +\infty$ if $n \to +\infty$. Consider the geodesic triangles

$$T_n = [t_n, x_n] \cup [x_n, z_n] \cup [z_n, t_n]$$
$$T'_n = [t_n, y_n] \cup [y_n, z_n] \cup [z_n, t_n]$$

From the Axiom of Visibility we get that

i) $\measuredangle_{t_n}([t_n, z_n], [t_n, x_n]) \to 0$
ii) $\measuredangle_{t_n}([t_n, z_n], [t_n, y_n]) \to 0$

where $\measuredangle_p([p, b], [p, d])$ is the angle at p formed by $[p, b]$ and $[p, d]$. This implies that

$$\pi = \measuredangle_{t_n}([t_n, x_n], [t_n, y_n]) \leq \measuredangle_{t_n}([t_n, x_n], [t_n, z_n]) + \measuredangle_{t_n}([t_n, y_n], [t_n, z_n]) \to 0$$

which shows that N must be a hyperbolic geodesic space.

Now suppose that N is δ-thin for some $\delta > 0$. Consider a sequence of geodesic triangles

$$T_n = [x_n, y_n] \cup [y_n, z_n] \cup [z_n, x_n]$$

such that $d(x_n, [y_n, z_n]) \geq n$. Parametrizing the segment $[y_n, z_n]$ as

$\gamma\colon [0, l_n] \to \widehat{M}$ with $\gamma(0) = y_n$ we get a point $e_n = \gamma(t) \in [y_n, z_n]$ defined by

$$t = \sup\{s \in [0, l_n], d(\gamma(s), [x_n, y_n]) \leq \delta\}.$$

By the δ-thin condition in N this implies that there exist points $\overline{e}_n \in [y_n, z_n]$ arbitrarily near to e_n such that $d(\overline{e}_n, [x_n, z_n]) \leq \delta$. Let us suppose that $d(e_n, \overline{e}_n) \leq \delta$. Let $d_n \in [x_n, y_n]$ be such that $d(e_n, d_n) \leq \delta$ and let $q_n \in [x_n, z_n]$ be such that $d(\overline{e}_n, q_n) \leq \delta$. Then we get

$$d(d_n, q_n) < d(d_n, e_n) + d(e_n, \overline{e}_n) + d(\overline{e}_n, q_n) < 3\delta.$$

So the geodesic triangle

$$S_n = [x_n, d_n] \cup [d_n, q_n] \cup [q_n, x_n]$$

has two sides of length $\geq n - 2\delta$, and one side of bounded length $\leq 3\delta$. Since N has no conjugate points we get from lemma 2.2 that the angle formed by the big sides of S_n at the point x_n goes to zero if n goes to $+\infty$ which proves that N satisfies the Axiom of Visibility. □

Corollary 5.1. *Let M be a compact Riemannian manifold with no conjugate points. If $\widehat{M}$ is a Visibility manifold, then $\pi_1(M)$ is a hyperbolic group.*

Proof. The proof is a straightforward consequence of Proposition 5.1 and lemmas 5.1 and 5.3. □

As a consequence of the results of this section and Theorem B we get that if M is a compact Riemannian manifold with no conjugate points, the expansivity of the geodesic flow implies that the fundamental group of the manifold is hyperbolic.

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