Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 25, N. 2, 181-206

Simultaneous Uniformization for the Leaves of Projective Foliations by Curves

Alcides Lins Neto

Abstract. In this paper we prove that, given a holomorphic foliation by curves on $\mathbb{C}P^n$, of degree ≥ 2, whose singularities have nondegenerate linear part, then there exists a hermitian metric g on $\mathbb{C}P^n - S$ (S = singular set) which is complete and induces strictly negative Gaussian curvature on the leaves of the foliation (Theorem B). This implies, in particular, that all leaves of the foliation are uniformized by the unit disc and that the set of uniformizations of the leaves is paracompact (Theorem A). We obtain also some consequences concerning the non existence of vanishing cycles in the sense of Novikov, the equivalence of the existence of a parabolic element in the group of deck transformations of the leaf and of a separatrix in the leaf, etc...

1. Introduction

Let $\mathcal{F}$ be a singular foliation by curves on M, a compact connected complex manifold of dimension $n \geq 2$. We denote by $S(\mathcal{F})$ the singular set of $\mathcal{F}$. Concerning the analytic structure of the leaves of $\mathcal{F}$ (outside $S(\mathcal{F})$), several questions arise naturally:

I - When all leaves of $\mathcal{F}$ are covered by the unit disk, $\mathbb{D}$?

If this is the case we will use the following notation:

$$\mathcal{U}(\mathcal{F}) = \{\alpha\colon \mathbb{D} \to M - S(\mathcal{F}) | \alpha \text{ is an uniformization of some leaf of } \mathcal{F}\}$$

II - Suppose that all leaves of $\mathcal{F}$ are covered by the unit disk. What is the nature of $\mathcal{U}(\mathcal{F})$, if we consider it as a subset of $\mathcal{O}(\mathbb{D}, M) = \{f\colon \mathbb{D} \to M | f \text{ is holomorphic}\}$ with the topology of uniform convergence in the compact parts of $\mathbb{D}$? More precisely, what is $\overline{\mathcal{U}(\mathcal{F})}$, the closure of $\mathcal{U}(\mathcal{F})$?

III - In the same hypothesis of II, how is the Poincaré metric of the leaves of $\mathcal{F}$, if we consider it in $M - S(\mathcal{F})$? Is it continuous? How does

Received 15 December 1993.

it behaves in a neighborhood of $S(\mathcal{F})$? Is there a $C^r (r \geq 0)$ hermitian metric μ in $M - S(\mathcal{F})$ such that μ induces the Poincaré metric in the leaves of $\mathcal{F}$?

IV - In the same hypothesis of II, given a leaf L, how is $G(L)$, the Fuchsian group of deck transformations associated to a universal covering of L? How does $G(L)$ varies with L? What can be said about the topology of the leaves of $\mathcal{F}$?

V - If $\mathcal{F}$ is as in I, how are the neighbours foliations? Are their leaves uniformized by the unit disc? If yes, how the uniformizations vary with the foliation?

In this paper we intend to answer some of these questions when $M = \mathbb{C}P^n$, $n \geq 2$, and $\mathcal{F}$ is a foliation of degree ≥ 2, such that its singularities have Milnor's number 1 (that is nondegenerate linear part). The main technique will be the construction of a C^2 hermitian metric g on $\mathbb{C}P^n - S(\mathcal{F})$ which induces negative Gaussian curvature on the leaves of $\mathcal{F}$. Before state the precise results we have, let us fix some notations.

Let g be a hermitian metric on $M - S(\mathcal{F})$ of class C^r, $r \geq 2$. Define $K_g \colon M - S(\mathcal{F}) \to \mathbb{R}$ by $(K_g \in C^{r-2})$.

$K_g(p) =$ Gaussian curvature of L_p at p, where L_p is the leaf of $\mathcal{F}$ through p.

If $K_g(p) \leq -a$, $a > 0$, for all $p \in M - S(\mathcal{F})$, then all leaves of $\mathcal{F}$ are uniformized by the unit disc $\mathbb{D}$. In this case we can consider the set $\mathcal{U}(\mathcal{F})$ as defined in I. Our first result is of a general nature and will be used in the case of $\mathbb{C}P^n$.

Theorem A. *Let M, $\mathcal{F}$ and $S(\mathcal{F})$ be as above. Suppose that there exists a hermitian metric g on $M - S(\mathcal{F})$ such that:*

a) *g is complete (that is, the distance induced by g on $M - S(\mathcal{F})$ is complete) and C^r, $r \geq 2$,*

b) $K_g(p) \leq -a^2, a > 0, \quad \forall\, p \in M - S(\mathcal{F})$.

c) *All connected components of $S(\mathcal{F})$ have a neighbourhood hyperbolic, in the sense of* [**K**].

Then $\mathcal{U}(\mathcal{F})$ is relatively compact in the topology of uniform convergence in the compact parts of $\mathbb{D}$. Moreover, its closure $\overline{\mathcal{U}(\mathcal{F})} \subset$

$\mathcal{U}(\mathcal{F}) \cup \mathcal{O}(\mathbb{D}, S(\mathcal{F}))$, where $\mathcal{O}(\mathbb{D}, S(\mathcal{F})) = \{\alpha \colon \mathbb{D} \to M | \alpha$ is holomorphic and $\alpha(\mathbb{D}) \subset S(\mathcal{F})\}$. In particular, if $(\alpha_m)_{m \geq 1}$ is a sequence in $\mathcal{U}(\mathcal{F})$, then (α_m) has a convergent subsequence, which converges to $\alpha \colon \mathbb{D} \to M$, where either $\alpha \in \mathcal{U}(\mathcal{F})$ or $\alpha \in \mathcal{O}(\mathbb{D}, S(\mathcal{F}))$.

Since a point $p \in M$ has a hyperbolic neighbourhood (for instance a polydisc), we get the following:

Corollary 1. *Let M, $\mathcal{F}$ and $S(\mathcal{F})$ be as above. Suppose that there exists a hermitian metric g on $M - S(\mathcal{F})$ which satisfies hypothesis a) and b) of Theorem A, and c') $S(\mathcal{F})$ is finite. Then all conclusions of Theorem A are true. Moreover $\overline{\mathcal{U}(\mathcal{F})} = \mathcal{U}(\mathcal{F}) \cup S$, where $S = \{\alpha \colon \mathbb{D} \to S(\mathcal{F}) | \alpha$ is constant$\}$.*

Concerning the Poincaré metric on the leaves of $\mathcal{F}$, we have the following:

Corollary 2. *Let M, $\mathcal{F}$ and $S(\mathcal{F})$ be as before, and suppose that there exists a hermitian metric g which satisfies a) and b) of Theorem A. Then there exists a (unique) continuous function $h \colon M - S(\mathcal{F}) \to (0, +\infty)$ such that the continuous hermitian metric $\mu = h \cdot g$ is complete and induces the Poincaré metric on the leaves of $\mathcal{F}$. Moreover, if g satisfies also*

d) $K_g \geq -b^2, b > 0$

then μ is equivalent to g.

Definition. We say that a foliation by curves $\mathcal{F}$ has a *vanishing cycle outside* $S(\mathcal{F})$ (in the sense of Novikov [**N**]), if $\mathcal{F}$ has a leaf L with the following properties:

i) L contains a closed curve γ, which is not homotopic to a constant in L.

ii) The holonomy of γ (in some transversal section Σ through $p \in \Sigma \cap \gamma$), $h \colon (\Sigma, p) \to (\Sigma, p)$ has an infinite number of distinct fixed points $(p_n)_{n \geq 1}$ where $\lim_{n \to \infty} p_n = p$, so that for $n \geq n_0$, we can lift γ to a closed curve γ_n, contained in L_{p_n}, the leaf of $\mathcal{F}$ through p_n, where $p_n \in \gamma_n$ and $\gamma_n \to \gamma$ uniformly.

iii) $(\gamma_n)_{n \geq n_0}$, has a subsequence $(\gamma_{n_k})_{k \geq 1}$, such that γ_{n_k} is homotopic to a constant in the leaf $L_{p_{n_k}}$.

An interesting consequence of Theorem A is the following:

Corollary 3. *Let M, $\mathcal{F}$ and $S(\mathcal{F})$ be as before and suppose that there exists a hermitian metric like in Theorem A. Then $\mathcal{F}$ has no vanishing cycles outside $S(\mathcal{F})$.*

Example. Let $\mathcal{F}$ be a foliation by curves in $\mathbb{C}P(n)$ whose singularities have Milnor's number 1. In this case $\mathcal{F}$ has a finite number of singularities, say $p_1, \dots, p_N$. For each j let B_j be a small ball around p_j, in such a way that $\overline{B}_i \cap \overline{B}_j = \phi$ if $i \neq j$. Set

$$M = \mathbb{C}P(n) - \bigcup_{j=1}^{N} \overline{B}_j$$

and $\mathcal{G} = \mathcal{F}|M$. If $\mathcal{F}$ has a singularity, say p_1, in the Siegel domain, then $\mathcal{G}$ has a vanishing cycle outside $S(\mathcal{G})$, because there is some leaf L of $\mathcal{F}$ which is tangent to ∂B_1 at some point $q_1 \in L \cap \partial B_1$, in such a way that L is locally outside B_1 (see Figure 1). In this case it is not possible to find a metric g in M which satisfies a) and b) of Theorem A, although it is possible to find one which satisfies b).

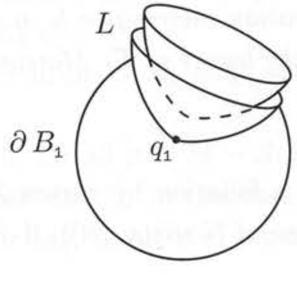


Figure 1

The main result of this paper is the following:

Theorem B. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P^n$, $n \geq 2$, such that:*

a) $\deg(\mathcal{F}) \geq 2$.

b) *All singularities of $\mathcal{F}$ are isolated and have Milnor's number* 1.

Then there exists a C^2 hermitian metric g on $\mathbb{C}P^n - S(\mathcal{F})$ which is complete and such that $K_g(p) \leq -a$, $a > 0$, $\forall\ p \in \mathbb{C}P^n - S(\mathcal{F})$. In particular all leaves of $\mathcal{F}$ are uniformized by $\mathbb{D}$ and $\overline{u(\mathcal{F})} = u(\mathcal{F}) \cup S$, where $S = \{\alpha : \mathbb{D} \to S(\mathcal{F}) | \alpha \text{ is constant}\}$.

It is convenient to observe here that a holomorphic foliation $\mathcal{F}$ on $\mathbb{C}P^n$, $n \geq 2$, with isolated singularities can be defined in an affine coordinate system $\mathbb{C}^n \subset \mathbb{C}P^n$ by a polynomial differential equation of the form:

$$\dot{z} = P(z), \quad z = (z_1, \dots, z_n), \quad P = (P_1, \dots, P_n) \qquad (*)$$

where $\mathbb{C}^n \cap S(\mathcal{F}) = \{P = 0\}$ and the leaves of $\mathcal{F}|\mathbb{C}^n$ are the complex solutions of $(*)$. If the affine coordinate system is such that the hyperplane at infinity is not invariant for $\mathcal{F}$, then P can be written as

$$P(z) = p_0 + \cdots + p_k(z) + g_k(z) \cdot z$$

where $p_j : \mathbb{C}^n \to \mathbb{C}^n$ is homogeneous of degree j and $g_k : \mathbb{C}^n \to \mathbb{C}$ is homogeneous of degree k and $g_k \not\equiv 0$. The degree of $\mathcal{F}$ is by definition k. If the hyperplane at infinity is invariant then P has the same form, but in this case $g_k \equiv 0$, $p_k \not\equiv 0$ and p_k can not be written as $g_{k-1}(z) \cdot z$, where g_{k-1} is homogeneous of degree $k-1$. Conversely, if $\mathcal{F}'$ is a foliation on $\mathbb{C}^n$, given by a differential equation as in $(*)$, then it can be extended to a unique foliation $\mathcal{F}$ on $\mathbb{C}P(n)$.

An interesting consequence of Theorem B, which is a foliated version of Picard's small Theorem, is the following:

Corollary 4. *Consider a differential equation in $\mathbb{C}^n$ like in $(*)$ and let $\mathcal{F}$ be the foliation on $\mathbb{C}P^n$ whose leaves extend the solutions of $(*)$. Suppose that $\mathcal{F}$ has degree ≥ 2 and that all singularities of $\mathcal{F}$ have Milnor's number 1. Then the unique solutions $z(T)$ of $(*)$ which can be extended to all $\mathbb{C}$ are the constant solutions $z \equiv z_0 \in S(\mathcal{F}) \cap \mathbb{C}^n$.*

The corollary is true because all non singular leaves of $\mathcal{F}$ are covered by the unit disc.

Next we will state a result which can be considered as a foliated version of big Picard's Theorem. Let $\mathcal{F}$ be a foliation on $\mathbb{C}P^n$ with degree ≥ 2 and whose singularities have Milnor's number 1.

Corollary 5. *Let $h : \mathbb{D}^* \to \mathbb{C}P^n$ be a holomorphic map ($\mathbb{D}^* = \mathbb{D} - \{0\}$), such that $h(\mathbb{D}^*) \subset L$, where L is some leaf of $\mathcal{F}$. Then h extends to a holomorphic map $\tilde{h} : \mathbb{D} \to \mathbb{C}P^n$ in such a way that either $\tilde{h}(0) \in L$ or $\tilde{h}(0) \in S(\mathcal{F})$.*

Let us state an interesting consequence of Corollary 5. Let $\mathcal{F}$ be as in Corollary 5 and L be a leaf of $\mathcal{F}$. We will say that L contains a *local separatrix* of $\mathcal{F}$ if there exists $p \in S(\mathcal{F})$ and an irreducible (germ) of analytic curve $\overline{\Gamma}$, eventually singular at p, with the following properties:

a) $\Gamma = \overline{\Gamma} - \{p\}$ is a smooth curve.

b) $\Gamma \subset L$.

In this case, it is known that $\overline{\Gamma}$ admits a Puiseaux's parametrization in a neighbourhood of p, that is, there exists a neighbourhood V of p and a holomorphic map $\delta: \mathbb{D} \to V$ such that:

c) $\delta(\mathbb{D}) = \overline{\Gamma} \cap V$ and $\delta(\mathbb{D}^*) = \Gamma \cap V$.

d) $\tilde{\delta} = \delta|\mathbb{D}^*: D^* \to V$ is an embedding.

Now, consider $\tilde{\delta}: D^* \to L$ and the induced map in the homotopy $\tilde{\delta}_*: \pi_1(\mathbb{D}^*) \to \pi_1(L)$. Let $\gamma \in \pi_1(L)$ be the image of a generator of $\pi_1(\mathbb{D}^*) \simeq \mathbb{Z}$, by $\tilde{\delta}_*$. Let $\alpha: \mathbb{D} \to L$ be an uniformization of L, $G(L) \subset$ PSL$(2, \mathbb{R})$, be the group of deck transformations associated to α and $d: \pi_1(L) \to G(L)$ be an isomorphism (associated to α). We will see in §3 that $d(\gamma)$ is parabolic. We have the following result:

Corollary 6. *Let $\mathcal{F}$ be as in Corollary 5, L be a leaf of $\mathcal{F}$, $\alpha: \mathbb{D} \to L$, $G(L)$ and $d: \pi_1(L) \to G(L)$ be as above. Then $G(L)$ contains a parabolic element h if, and only if, L contains a local separatrix Γ of $\mathcal{F}$. Moreover, if this is the case, then $d^{-1}(h) = \gamma^k$, for some $k \in \mathbb{Z} - \{0\}$, where γ is obtained by using a Puiseaux's parametrization of $\overline{\Gamma}$ as indicated before.*

Let us state now a result related to question V.

We will see in the proof of Theorem B that the metric g can be constructed in such a way that it is also C^2 with respect to $\mathcal{F}$. Let N_k^n be the set of foliations on $\mathbb{C}P^n$, $n \geq 2$, with degree $k \geq 2$ and whose singularities are isolated and have Milnor's number 1. Observe that N_k^n is an open and dense subset of the set of all foliations of degree k in $\mathbb{C}P^n$ (in fact it is a Zariski's open set). Let

$$\mathcal{U}_k^n = \{(\mathcal{F}, \alpha) | \mathcal{F} \in N_k^n \text{ and } \alpha \in \mathcal{U}(\mathcal{F}) \text{ or } \alpha: \mathbb{D} \to S(\mathcal{F}) \text{ is constant}\}.$$

We have the following consequence:

Theorem C. *$\mathcal{U}_k^n$ is locally compact. In particular, if $(\alpha_m)_{n \geq 1}$ is a sequence, such that for all $m \geq 1$, α_m is an uniformization of some leaf of $\mathcal{F}_m \in N_k^n$, where $\mathcal{F}_m \to \mathcal{F}_0 \in N_k^n$, then $(\alpha_m)_{m \geq 1}$ has a subsequence which converges to α_0, where either $\alpha_0 \in \mathcal{U}(\mathcal{F}_0)$ or $\alpha_0(\mathbb{D})$ is a constant in $S(\mathcal{F}_0)$.*

This result can be considered as a kind of "continuous dependence of the solution with parameters and initial conditions".

Remarks.

1 - In **[C,G]**, Candel and Gómez-Mont prove a result similar to the conclusion of Theorem B, for $n = 2$. They assume that all singularities of $\mathcal{F}$ are hyperbolic and that $\mathcal{F}$ has no algebraic leaf, so that our result is a little bit more general. Nevertheless, the technique that they use to prove that all leaves of $\mathcal{F}$ are uniformized by $\mathbb{D}$ is different and could be interesting in itself. They prove that, under their hypothesis, $\mathcal{F}$ has no transverse invariant measure and from this they conclude that the leaves of $\mathcal{F}$ can not be covered by $\mathbb{C}$. Also, their proof of the fact that a convergent sequence of uniformization which does not converge to an uniformization, converges to a constant in $S(\mathcal{F})$ is different. They prove first in the "local case", and in their proof they use that the foliation is equivalent to a linear vector field in a neighbourhood of each singularity. We would like to observe here that our technique works well in this case too. More specifically we will consider the following situation: let X be a holomorphic vector field defined in a bounded polydisc $P \subset \mathbb{C}^n$ which has a unique singularity at $0 \in P$. Let $\mathcal{F}$ be the singular foliation whose leaves are the non constant solutions of $\dot{z} = X(z)$. Since P is a hyperbolic manifold (cf. ch. IV of **[K]**), it follows that all leaves of $\mathcal{F}$ are uniformized by the unit disc. Let $\mathcal{U}(\mathcal{F})$ be as before.

Proposition 7. *In the above situation, suppose that $DX(0)$ is non singular and that $|X(z)| \geq c > 0$ if $z \in P$ and $|z| \geq r$, for some $r > 0$ such that $\overline{B_r(0)} \subset P$. Then there exists a hermitian metric g on $P - \{0\}$ such that:*

a) g is complete.

b) $K_g(p) \leq -a, a > 0, \quad \forall\, p \in P - \{0\}$.

In particular, let $(\alpha_m)_{m \geq 1}$ be a sequence in $\mathcal{U}(\mathcal{F})$ such that there *exists* $\lim_{m\to\infty} \alpha_m(0) = q$, $q \in P$. *Then, if* $q = 0$, $\alpha_m \to 0$ *uniformly in the compact parts of* $\mathbb{D}$, *and if* $q \neq 0$ *then* (α_m) *has a subsequence which converges to some* $\alpha \in \mathcal{U}(\mathcal{F})$, *which uniformizes* L_q.

A natural problem is the following:

Problem. To find the "more general" singularity of a holomorphic germ of vector field for which the conclusions of Proposition 7 are true.

2 - We would like to observe here that Corollaries 5 and 6 could be stated in the more general setting of Theorem A. However we have preferred to state them in the case of $M = \mathbb{C}P^n$ because this is the unique case in which we have a concrete metric g like in Theorem A.

3 - We think that our construction can be extended to singular foliations defined in more general projective manifolds. For instance in the case of a foliation on $M \subset \mathbb{C}P^n$, whose singularities have Milnor's number 1 and which is the restriction of some foliation on $\mathbb{C}P^n$. However in this case there are some difficulties, according to the nature of the extended foliation.

4 - The metric g of Theorem B was inspired in a metric constructed in §3 of **[CLS]**.

2. Proofs of Theorem A and of Corollaries 2 and 3

2.1 Proof of Theorem A

Let $M, \mathcal{F}, S(\mathcal{F})$ and g be as in the hypothesis of Theorem A. Suppose that

$$K_g(p) \leq -a^2 \quad \forall\, p \in M - S(\mathcal{F}),$$

where $a^2 > 0$. Ahlfor's lemma (cf. ch. I of **[K]**) implies that for any $\alpha \in \mathcal{U}(\mathcal{F})$ we have:

$$d(\alpha(z_2), \alpha(z_1)) \leq d_L(\alpha(z_2), \alpha(z_1)) \leq \frac{1}{a} d_P(z_2, z_1), \tag{1}$$

where d is the distance on $M - S(\mathcal{F})$ induced by g, d_L the distance induced by g on the leaf L uniformized by α and d_P is the Poincaré distance on $\mathbb{D}$. In particular $\mathcal{U}(\mathcal{F})$ is equicontinuous.

Let $(\alpha_m)_{m\geq 1}$ be a sequence in $\mathcal{U}(\mathcal{F})$. By taking a subsequence, if necessary, we can suppose that $(\alpha_m(0))_{m\geq 1}$ converges, say

$$\lim_{m\to\infty} \alpha_m(0) = p \in M.$$

Suppose first that $p \in S(\mathcal{F})$. We will prove that in this case (α_m) has a subsequence which converges to $\alpha\colon \mathbb{D} \to M$, where $\alpha(\mathbb{D}) \subset S(\mathcal{F})$.

Let C be the connected component of $S(\mathcal{F})$ which contains p and let V be a hyperbolic neighbourhood of C. Fix $p_0 \in M - S(\mathcal{F})$ and let

$$\overline{B}_r = \{q \in M - S(\mathcal{F}) | d(q, p_0) \leq r\}.$$

Since g is complete we have:

i) $\overline{B}_r$ is compact for any $0 < r < +\infty$.

ii) $\bigcup_{r>0} \overline{B}_r = M - S(\mathcal{F})$.

For $r > 0$, let W_r be the connected component of $M - \overline{B}_r$ which contains p. It follows from (i) and (ii) that:

iii) $\bigcap_{r>0} W_r = C$ and $W_{r'} \subset W_r$ if $r' > r$.

iv) If r is big enough then $\overline{W}_r \subset V$.

Now, fix $0 < \rho < 1$ and r_0 such that $\overline{W}_r \subset V$ if $r \geq r_0$. Let $c(\rho) = a^{-1} \cdot d_P(0, \rho)$. Fix also $m_0 \geq 1$ such that $\alpha_m(0) \in W_{r'}$ for $m \geq m_0$, where $r' = r + c(\rho)$. Inequality (1) implies that if $z \in \overline{D}_\rho = \{z||z| \leq \rho\}$, then

$$\begin{aligned} d(\alpha_m(z), p_0) &\geq d(\alpha_m(0), p_0) - d(\alpha_m(z), \alpha_m(0)) \\ &> r' - a^{-1} d_P(z, 0) \geq r' - c(\rho) = r. \end{aligned}$$

Therefore $\alpha_m(\overline{D}_\rho) \subset M - \overline{B}_r$. From connectedness, it follows that $\alpha_m(\overline{D}_\rho) \subset \overline{W}_r$ for $m \geq m_0$. Now, since $\overline{W}_{r_0}$ is compact and V is hyperbolic, the sequence, $(\alpha_m|\overline{D}_\rho)_{m\geq m_0}$ is normal, and so it has a subsequence which converges uniformly, say $\alpha_{m_k}|\overline{D}_\rho \to \alpha_\rho$ as $k \to \infty$, where $\alpha_\rho\colon \overline{D}_\rho \to \overline{W}_{r_0}$. From the above argument, it is clear that

$$\alpha_\rho(\overline{D}_\rho) \subset \bigcap_{r\geq r_0} \overline{W}_r = C.$$

We have proved that for any $0 < \rho < 1$, the sequence $(\alpha_m|\overline{D}_\rho)_{m\geq 1}$ has a subsequence $(\alpha_{m_\rho(k)}|\overline{D}_\rho)_{k\geq 1}$ which converges uniformly to $\alpha_\rho\colon \overline{D}_\rho \to C$. By using this and the diagonal Cantor's method, it is not difficult to see that $(\alpha_m)_{m\geq 1}$ has a subsequence $(\alpha_{m_k})_{k\geq 1}$, which converges to $\alpha: \mathbb{D} \to C$ in the compact parts of $\mathbb{D}$.

Suppose now that $\alpha_m(0) \to p \in M - S(\mathcal{F})$. Fix $0 < \rho < 1$ and consider the sequence $(\alpha_m|\overline{D}_\rho)_{m\geq 1}$. Let

$$r_0 = \sup\{d(\alpha_m(0), p_0)|m \geq 1\} < +\infty$$

and $c(\rho) = a^{-1} d_P(0, \rho)$. It follows from (1) that for all $m \geq 1$:

$$d(\alpha_m(z), p_0) \leq d(\alpha_m(z), \alpha_m(0)) + d(\alpha_m(0), p_0) \leq c(\rho) + r_0 = r',$$

so that $\alpha_m(\overline{D}_\rho) \subset \overline{B}_{r'}$ for all $m \geq 1$. Inequality (1) implies also that $(\alpha_m|\overline{D}_\rho)_{m\geq 1}$ is equicontinuous. Since $\overline{B}_{r'}$ is compact, it follow that $(\alpha_m|\overline{D}_\rho)_{m\geq 1}$ has a subsequence which converges uniformly to some $\alpha_\rho: \overline{D}_\rho \to \overline{B}_{r'}$. This fact and diagonal Cantor's method imply once again that $(\alpha_m)_{m\geq 1}$ has a subsequence which converges to some $\alpha: \mathbb{D} \to M - S(\mathcal{F})$ in the compact parts of $\mathbb{D}$. Let us prove that $\alpha \in \mathcal{U}(\mathcal{F})$. From now on we suppose that $(\alpha_m)_{m\geq 1}$ converges to $\alpha: \mathbb{D} \to M - S(\mathcal{F})$ in the compact parts of $\mathbb{D}$ and $\lim_{m\to\infty} \alpha_m(0) = p = \alpha(0)$.

Define $\lambda: M - S(\mathcal{F}) \to (0, +\infty)$ by

$$\lambda(q) = g_q(\gamma'(0)) \tag{2}$$

where $\gamma: \mathbb{D} \to L_q$ is an uniformization of L_q, the leaf of $\mathcal{F}$ through q, such that $\gamma(0) = q$.

It is well known that

$$\lambda(q) = \sup\{g_q(\delta'(0))|\delta: \mathbb{D} \to L_q \text{ and } \delta(0) = q\}$$

and that if $g_q(\beta'(0)) = \lambda(q)$, where $\beta: \mathbb{D} \to L_q$ is such that $\beta(0) = q$, then β is an uniformization of L_q (cf. [**V**]). It is known also that λ is lower semicontinuous (cf. [**C**] and [**V**]). Let $\gamma: \mathbb{D} \to L_p$ be an uniformization such that $\gamma(0) = p$ and $\alpha_m \to \alpha$ as above, where $\alpha(0) = p$. Since $\alpha_m: \mathbb{D} \to L_{pm}$, $p_m = \alpha_m(0)$, it is not difficult to see that $\alpha: \mathbb{D} \to L_p$. Therefore

$$g_p(\alpha'(0)) \leq \lambda(p) = g_p(\gamma'(0)).$$

On the other hand, since λ in lower semicontinuous we must have

$$\lim_{m\to\infty} \lambda(p_m) = \lim_{m\to\infty} g_{pm}(\alpha'_m(0)) = g_p(\alpha'(0)) \geq \lambda(p),$$

so that, $g_p(\alpha'(0)) = \lambda(p)$, which implies that $\alpha \in \mathcal{U}(\mathcal{F})$. This proves Theorem A. □

2.2 Proof of Corollary 2

Observe first that the last argument in the proof of Theorem A, proves that λ is continuous and does not depend on the hypothesis (c) of Theorem A. On the other hand, a direct computation shows that, if $\alpha \in \mathcal{U}(\mathcal{F})$, $\alpha(0) = p$, then

$$\lambda(\alpha(z)) = (1 - |z|^2)^2 g_{\alpha(z)}(\alpha'(z)) \tag{3}$$

Relation (3) can be proved by taking

$$\beta(w) = \alpha\left(\frac{w + z}{1 + \overline{z}w}\right),$$

where $\beta(0) = \alpha(z)$ and

$$\beta'(0) = (1 - |z|^2)\alpha'(z).$$

It implies that $\mu = 4\lambda^{-1} \cdot g$ induces the Poincaré metric on L_p. From Ahlfor's lemma it follows that

$$g_{\alpha(z)}(\alpha'(z))|dz|^2 = \alpha^*(g) \leq \frac{1}{a^2} \frac{4|dz|^2}{(1 - |z|^2)^2}.$$

Hence $\lambda(p) = g_{\alpha(0)}(\alpha'(0)) \leq \frac{4}{a^2}$, so that $0 < \lambda \leq 4a^{-2}$. This implies that

$$\mu = 4\lambda^{-1} \cdot g \geq a^2 g.$$

Since g is complete, μ is also complete.

Now, suppose that $K_g \geq -b^2$, $b > 0$. Since g is complete, if α is as before, a Theorem of Bland and Kalka (cf. [**B-K**]) implies that

$$\alpha^*(g) \geq \frac{1}{b^2} \frac{4|dz|^2}{(1 - |z|^2)^2},$$

and so $\lambda(p) = g_{\alpha(0)}(\alpha'(0)) \geq 4/b^2$, and we get that,

$$a^2 g \leq \mu \leq b^2 g.$$

This implies that μ and g are equivalent, which proves Corollary 2. □

2.3 Proof of Corollary 3

Let $M, \mathcal{F}$, and $S(\mathcal{F})$ be as in the hypothesis of Corollary 3. Let L be a leaf of $\mathcal{F}$, $p \in L$ and Σ be an embedded transversal section to $\mathcal{F}$ such that $p \in \Sigma$ and Σ is biholomorphic to a polydisc in $\mathbb{C}^{n-1}$. We can take Σ small in such a way that there exists a holomorphic vector field X defined in some neighbourhood U of Σ, without singularities and which represents $\mathcal{F}/U$. For each $q \in U$ there exists a unique uniformization $\alpha_q: \mathbb{D} \to L_q$, the leaf of $\mathcal{F}$ through q, such that $\alpha_q(0) = q$ and $\alpha_q'(0) = \beta(q) \cdot X(q)$ where $\beta(q) > 0$. It follows from Theorem A that $\beta: U \to \mathbb{R}_+$ is continuous. In fact, since $\beta > 0$, we get

$$\beta(q) = \sqrt{\frac{g_q(\alpha'(0))}{g_q(X(q))}} = \sqrt{\frac{\lambda(q)}{g_q(X(q))}}$$

where λ is as in (2).

Now, consider the function $A: U \times \mathbb{D} \to M - S(\mathcal{F})$, defined by $A(q,z) = \alpha_q(z)$. Let us prove that A is continuous. Let $((q_m, z_m))_{m \geq 1}$ be a sequence in $U \times \mathbb{D}$ such that $\lim_{m\to\infty}(q_m, z_m) = (q_o, z_o) \in U \times \mathbb{D}$ and $\alpha_m = \alpha_{q_m}$. Observe first that $\alpha_m \to \alpha_{q_o}$ in the compact parts of $\mathbb{D}$. In fact, if $(\alpha_{m_k})_{k\geq 1}$ is any convergent subsequence of $(\alpha_m)_{m\geq 1}$, $\alpha_{m_k} \to \alpha_o$, then $\alpha_{m_k}(0) = q_{m_k} \to q_o = \alpha_o(0) \notin S(\mathcal{F})$, so that Theorem A implies that $\alpha_o \in \mathcal{U}(\mathcal{F})$. Since,

$$\alpha'_{m_k}(0) = \beta(q_{m_k}) \cdot X(q_{m_k}) \to \beta(q_o) \cdot X(q_o) = \alpha'_o(0)$$

we get that $\alpha_o = \alpha_{q_o}$. Therefore $\alpha_m \to \alpha_{q_o}$. Now, since $z_o = \lim_{m\to\infty} z_m \in \mathbb{D}$, we have $\alpha_m(z_m) \to \alpha_{q_o}(z_o)$, and so $\lim_{m\to\infty} A(q_m, z_m) = A(q_o, z_o)$. We need the following:

Lemma 1. (Lifting lemma.) *Let $V = A(U \times \mathbb{D})$, $I = [0,1]$ and $\Sigma_1 \subset \Sigma$ be a connected open set in Σ with $p \in \Sigma_1$. Let $F: \Sigma_1 \times I \to V$ be a continuous map such that:*

a) $F(q,0) = q \quad \forall\, q \in \Sigma_1$.

b) *For any fixed $q \in \Sigma_1$ we have $F(q \times I) \subset L_q$, the leaf of $\mathcal{F}$ through q.*

Then there exists a unique $\hat{F}: \Sigma_1 \times I \to U \times \mathbb{D}$ such that $A \circ \hat{F} = F$ and $\hat{F}(q,0) = (q,0) \in \Sigma_1 \times \mathbb{D}$, for all $q \in \Sigma_1$.

Proof. Observe that $A(q,0) = q$ and $A(q,z) = \alpha_q(z)$, where α_q is an uniformization of L_q. Moreover $F_q(t) = F(q,t)$ is a path in L_q with $F_q(0) = q$. It follows from the lifting theorem for covering maps that there exists a unique path $\hat{F}_q: I \to \mathbb{D}$ such that $\hat{F}_q(0) = 0$. If we set $\hat{F}(q,t) = \hat{F}_q(t)$, then clearly $A \circ \hat{F} = F$. The continuity of $\hat{F}$ follows from the continuity of A, as the reader can check. □

Now, let $\gamma: I \to L_p$ be a closed path with $\gamma(0) = p$. By taking a C^∞ fibration transversal to $\mathcal{F}$ in a neibhbourhood of $\gamma(I)$, such that Σ contains a fiber, we can lift γ to the leaves neighbours to L_p, in such a way that we obtain a continuous map $F: \Sigma_1 \times I \to V$ satisfying (a), (b) of Lemma 1 and

c) $F(p,t) = \gamma(t), t \in I$.

d) $F(q,t)$ belongs to the fiber through $\gamma(t)$.

In this case $f(q) = F(q,1) \in \Sigma$ is the holonomy transformation of L_p associated to $[\gamma] \in \pi_1(L_p)$. Let $\hat{F}$ be as in the lifting lemma. If $[\gamma] \neq 0$ in $\pi_1(L_p)$ then $\hat{F}_p: I \to \mathbb{D}$ is not a closed path in $\mathbb{D}$. Since $\hat{F}$ is continuous there exists a neighbourhood Σ_2 of p in Σ_1 such that $\hat{F}_q$ is also not closed, $q \in \Sigma_2$. This implies that if $q \in \Sigma_2$ is such that $f(q) = q$, so that the path $F_q: I \to L_q$ is closed, then it can not be homotopic to a constant in L_q, because $A_q = \alpha_q: \mathbb{D} \to L_q$ is a universal covering. Therefore $\mathcal{F}$ cannot have a vanishing cycle outside $S(\mathcal{F})$. □

3. Proofs of Theorems B, C and of Proposition 7

3.1 Proof of Theorem B

Let $\mathcal{F}$ be a foliation by curves in $\mathbb{C}P^n$, $n \geq 2$, such that $S(\mathcal{F})$ is finite and the singularities of $\mathcal{F}$ in $S(\mathcal{F})$ have Milnor's number 1. In an affine coordinate system $\mathbb{C}^n \subset \mathbb{C}P^n$, the leaves of $\mathcal{F}$ are the solutions of a differential equation

$$\frac{dz}{dt} = \dot{z} = P(z), \quad z = (z_1, \ldots, z_n) \tag{1}$$

where $S(\mathcal{F}) \cap \mathbb{C}^n = P^{-1}(0)$ and $P = (P_1, \dots, P_n)$ where the $P_{j's}$ are polynomials. We choose the affine coordinate system $\mathbb{C}^n$ in such a way that $S(\mathcal{F}) \subset \mathbb{C}^n$. In this case, the fact that $\deg(\mathcal{F}) = k \geq 2$ means that $P_j = p_j + z_j \cdot h$ where p_j is a polynomial of degree $\leq k$ and h is an homogeneous polynomial of degree k. The fact that $S(\mathcal{F}) \subset \mathbb{C}^n$ implies that $h \not\equiv 0$. The fact that all singular points of $\mathcal{F}$ have Milnor's number 1 is equivalent to the following:

$$\# S(\mathcal{F}) = N = 1 + k + \dots + k^n \tag{2}$$

or

$$\text{if } p \in S(\mathcal{F}) \text{ then } A = DP(p) \text{ is non singular.} \tag{3}$$

Let $S(\mathcal{F}) = \{p_1, \dots, p_N\}$ and $A_j = DP(p_j), j = 1, \dots, N$. Set $Z_j(z) = z - p_j$, so that (1) can be written as

$$\frac{dZ_j}{dt} = \dot{Z}_j = A_j \cdot Z_j + Q_j(Z_j), \quad j = 1, \dots, N \tag{1j}$$

where $\lim\limits_{z \to 0} \frac{Q_j(z)}{|z|} = 0$. The hermitian metric g will be of the form

$$g = \varphi_0 \dots \varphi_N \mu \tag{4}$$

where

$$\mu = \frac{\sum\limits_{j=1}^{n} |dz_j|^2 + \sum\limits_{i<j} |z_i dz_j - z_j dz_i|^2}{\left(\sum\limits_{j=1}^{n} |P_j|^2 + \sum\limits_{i<j} |R_{ij}|^2\right)}$$

$$R_{ij} = z_i P_j - z_j P_i = z_i p_j - z_j p_i$$

$$\varphi_0(z) = \left(1 + \sum_{j=1}^{n} |z_j|^2\right)^{\alpha}, \quad \alpha = \frac{(k-1)}{(N+1)}.$$

and

$$\varphi_j(z) = \varphi(|Z_j(z)|^2), |Z_j(z)|^2 = \sum_{e=1}^{n} |z_e - p_{je}|^2, \quad p_j = (p_{j1}, \dots, p_{jn}),$$

where $\varphi\colon (0, +\infty) \to (0, +\infty)$ is defined by

$$\varphi(t) = \begin{cases} (\ell n t)^{-2} & \text{for } 0 < t \leq t_0, \text{ where } t_0 = e^{-(1+2/\alpha)} < 1. \\ a(t+b)^{\alpha} & \text{for } t > t_0, \text{ where } a = \dfrac{2^{\alpha} \alpha^2 e^{2+\alpha}}{(\alpha+2)^{\alpha+2}} \\ & \text{and } b = \frac{\alpha}{2} e^{-(1+2/\alpha)}. \end{cases} \tag{5}$$

We observe that φ was obtained by solving in t, a and b the system

$$\varphi_1(t) = \varphi_2(t), \varphi_1'(t) = \varphi_2'(t), \varphi_1''(t) = \varphi_2''(t),$$

where $\varphi_1(t) = (\ell n t)^{-2}$ and $\varphi_2(t) = a(t+b)^{\alpha}$. This implies that φ is C^2. Observe that the denominator of μ vanishes exactly on $S(\mathcal{F})$ and $\varphi > 0$, $\varphi_0 > 0$, so that, g defines a hermitian metric on $\mathbb{C}^n - S(\mathcal{F})$. We have to check the following facts:

I - g extends to $\mathbb{C}P^n - S(\mathcal{F})$. Let us call g again the extension.

II - g is complete.

III - $K_g \leq -A$, where $A > 0$.

Proof of I. Let us consider a change of affine coordinates in $\mathbb{C}P^n$, for instance, $w_1 = 1/z_1$, $w_j = z_j/z_1$, $2 \leq j \leq n$. From (5), it is easy to check that there exists $r > 0$ such that if $|z| \geq r$ then

$$\varphi_0\left(\frac{1}{w_1}, \frac{w_2}{w_1}, \dots, \frac{w_n}{w_1}\right) = \frac{(1 + \sum\limits_{j=1}^{n} |w_j|^2)^{\alpha}}{|w_1|^{2\alpha}},$$

and

$$\varphi_j\left(\frac{1}{w_1}, \frac{w_2}{w_1}, \dots, \frac{w_n}{w_1}\right) = \\ = \frac{a(|1 - p_{j1} w_1|^2 + \sum\limits_{e=2}^{n} |w_e - p_{je} w_1|^2 + b|w_1|^2)^{\alpha}}{|w_1|^{2\alpha}}.$$

Moreover, in the new coordinate system,

$$\mu = \frac{\sum\limits_{j=1}^{n} |dw_j|^2 + \sum\limits_{i<j} |w_i dw_j - w_j dw_i|^2}{\sum\limits_{j=1}^{n} \left|\widetilde{P}_j\right|^2 + \sum\limits_{i<j} \left|\widetilde{R}_{ij}\right|^2} \times |w_1|^{2k-2}.$$

where $\widetilde{P} = (\widetilde{P}_1, \dots, \widetilde{P}_n)$ is such that the leaves of $\mathcal{F}$ in $\mathbb{C}^n = \mathbb{C}P^n - \overline{\{z_1 = 0\}}$ are the solutions of the differential equation $\frac{dw}{d\tau} = \widetilde{P}(w)$ and $\widetilde{R}_{ij} = w_i\widetilde{P}_j - w_j\widetilde{P}_i$. We leave this computation for the reader (see also [**C-L-S**]). Since $\alpha = (k-1)/(N+1)$, g can be written in a neighbourhood of $\{w_1 = 0\}$ in this new coordinates system as

$$g = \varphi_0(w)\tilde{\varphi}_1(w)\dots\varphi_N(w)\,\frac{\sum\limits_{j=1}^{n}|dw_j|^2 + \sum\limits_{i<j}|w_i dw_j - w_j dw_i|^2}{\sum\limits_{j=1}^{n}\left|\widetilde{P}_j\right|^2 + \sum\limits_{i<j}\left|\widetilde{R}_{ij}\right|^2} \tag{6}$$

where

$$\tilde{\varphi}_j(w) = a(|1 - p_{j1}w_1|^2 + \sum_{\ell=2}^{n}|w_\ell - p_{j\ell}w_1|^2 + b|w_1|^2)^\alpha,\ j = 1,\dots,N.$$

This proves that g extends to all $\mathbb{C}P^n - S(\mathcal{F})$ as a C^2 hermitian metric.

Proof of II. It is sufficient to analyze g in a neighbourhood of a $p_j \in S(\mathcal{F})$. Without lost of generality we will suppose $p_j = p_1 = 0$ (that is $j = 1$). In this case, we have for $|z|^2 < t_0 = e^{-(1+2/\alpha)}$, that $\varphi_1(z) = (\ell n|z|^2)^{-2}$. We have also $\varphi_0(0) > 0$, $\varphi_j(0) > 0$ for $j \geq 2$, so that there exists $r \leq \sqrt{t_0}$ and $c > 1$ such that

$$c^{-1} \leq \varphi_0(z)\varphi_2(z)\dots\varphi_N(z) \leq c \text{ if } |z| \leq r.$$

On the other hand, since $A_1 = DP(0)$ is non singular, by taking a smaller r if necessary, there exists $m > 1$ such that if $|z| \leq r$, then

$$m^{-1}|z|^2 \leq \sum_{j=1}^{n}|P_j(z)|^2 + \sum_{i<j}|R_{ij}(z)|^2 \leq m|z|^2.$$

Moreover, it is not difficult to see that if $\gamma\colon(\alpha,\beta) \to \overline{B(0,r)} = \{z; |z| \leq r\}$, is a C^1 curve, then there exists another constant $D > 1$ such that

$$D^{-1}|\gamma'(t)|^2 \leq \sum_{j=1}^{n}\left|\gamma_j'(t)\right|^2 + \sum_{i<j}\left|\gamma_i(t)\gamma_j'(t) - \gamma_j(t)\gamma_i'(t)\right|^2 \leq D|\gamma'(t)|^2.$$

This implies that if $E = cmD$, then

$$E^{-1}\,\frac{|\gamma'(t)|^2}{|\gamma(t)|^2(\ell n|\gamma(t)|^2)^2} \leq g_{\gamma(t)}(\gamma'(t)) \leq E\,\frac{|\gamma'(t)|^2}{|\gamma(t)|^2(\ell n|\gamma(t)|^2)^2}$$

Therefore g is equivalent to the hermitian metric $\frac{|dz|^2}{|z|^2(\ell n|z|^2)^2}$ in $B(0,r) - \{0\}$. Since this last metric is complete in $B(0,1)$, g is also complete.

Proof of III. Let us compute K_g. Fix $p \in \mathbb{C}^n - S(\mathcal{F})$, where $\mathbb{C}^n$ is the first affine coordinate system considered. Let $z\colon D_\varepsilon = \{T||T| < \varepsilon\}$ be the solution of (1) such that $z(0) = p$. It is easy to see that

$$z^*(g) = \varphi_0(z(T))\dots\varphi_N(z(T))|dT|^2 = \psi(T)|dT|^2$$

This implies that

$$K_g(p) = -\frac{2}{\psi(0)}\,\frac{\partial^2}{\partial T\partial\overline{T}}\,\ell n(\psi(T))|_{T=0}$$
$$= -\frac{2}{\psi(0)}\sum_{j=0}^{N}\frac{\partial^2}{\partial T\partial\overline{T}}\,[\ell n(\varphi_j(z(T))]_{T=0}$$

If we set $K_j(p) = \frac{1}{\varphi_j(p)}\frac{\partial^2}{\partial T\partial\overline{T}}[\ell n(\varphi_j(z(T))]_{T=0}$, we get

$$K_g(p) = -2\sum_{j=0}^{N}\,[\varphi_0(p)\dots\widehat{\varphi_j(p)}\dots\varphi_N((p)]^{-1}K_j(p)$$
$$= -2\sum_{j=0}^{N}\left(\prod_{i\neq j}\varphi_i(p)\right)^{-1} K_j(p).$$

We will prove the following facts:

III.1 - $K_0(p) > 0$ for all $p \in \mathbb{C}^n - S(\mathcal{F})$.

III.2 - $K_j(p) \geq 0$ for all $p \in \mathbb{C}^n - S(\mathcal{F})$.

III.3 - $\liminf\limits_{p\to pj} K_j(p) \geq a_j > 0,\quad j = 1,\dots,N.$

III.4 - $\lim\limits_{p\to pj} K_i(p) = 0$, if $i \neq j$.

III.5 - $K_g(p) > 0$ if $p \in \mathbb{C}P^n - \mathbb{C}^n$, the hyperplane at infinity.

It is not difficult to see that these five assertions imply that $K_g \leq -A$ for some $A > 0$.

III.1 - A direct computation shows that

$$K_0(z) = \frac{\alpha\left(\sum_{j=1}^{n} |P_j(z)|^2 + \sum_{i<j} |R_{ij}(z)|^2\right)}{(1+|z|^2)^{2+\alpha}}$$

This implies III.1 and that $\lim_{p\to pj} K_0(p) = 0$.

III.2 - For $|z-p_j|^2 \geq t_0$ we have

$$\ell n(\varphi_j(z)) = \ell n a + \alpha \ell n(b + |z-p_j|^2).$$

Since $b > 0$, $\ell n(b+|z-p_j|^2)$ is pluri subharmonic, and so $K_j(z) \geq 0$ for $|z-p_j|^2 \geq t_0$. On the other hand, for $|z-p_j|^2 \leq t_0$, we have

$$\ell n(\varphi_j(z)) = \ell n[(\ell n|z-p_j|^2)^{-2}].$$

Hence

$$\frac{\partial}{\partial T}[\ell n(\varphi_j(z(T)))] = -2\frac{\partial}{\partial T}\frac{[\ell n|z(T)-p_j|^2]}{\ell n|z(T)-p_j|^2} = -2\frac{\langle \dot z, Z_j\rangle}{|Z|^2 \ell n|Z_j|^2}$$

where $\dot z = P(z)$, $Z_j = z - p_j$ and $\langle u, v\rangle = \sum_{\ell=1}^{n} u_\ell \overline{v}_\ell$.

$$\frac{\partial^2}{\partial T\partial\overline{T}}[\ell n(\varphi_j(z(T)))] = -2\frac{\partial}{\partial\overline{T}}\left[\frac{\langle \dot z, Z_j\rangle}{|Z_j|^2\ell n|Z_j|^2}\right]$$
$$= -2|Z_j|^{-2}(\ell n|Z_j|^2)^{-1}|\dot z|^2 + 2\{|Z_j|^{-4}(\ell n|Z_j|^2)^{-1} + |Z_j|^{-4}(\ell n|Z_j|^2)^{-2}\}|\langle \dot z, Z_j\rangle|^2.$$

Dividing the above expression by $\varphi_j(z) = (\ell n|Z_j|^{-2})^{-2}$, we get:

$$K_j(z) = -2\frac{\ell n|Z_j|^2}{|Z_j|^4}[|\dot z|^2|Z_j|^2 - |\langle \dot z, Z_j\rangle|^2] + 2\frac{|\langle \dot z, Z_j\rangle|^2}{|Z_j|^4}.$$

Now, a direct computation shows that

$$|\dot z|^2|Z_j|^2 - |\langle \dot z, Z_j\rangle|^2 = \sum_{\ell<m} |(z_\ell - p_{j\ell})\dot z_m - (z_m - p_{jm})\dot z_\ell|^2 = |H|^2$$

Since $|Z_j|^2 < t_0 < 1$, we get

$$K_j(z) = 2\frac{|\ell n|Z_j|^2|}{|Z_j|^4}|H|^2 + 2\frac{|\langle \dot z, Z_j\rangle|^2}{|Z_j|^4} \geq 0$$

III.3 - Observe first that

$$\dot z = \dot Z_j = P(p_j + Z_j) = A_j \cdot Z_j + Q_j(Z_j)$$

where the order of Q_j at $Z_j = 0$ is ≥ 2. Since we are fixing j, we will omit from now on the index j. Let $A\cdot Z = (B_1\cdot Z, \ldots, B_n\cdot Z)$, $Z = (Z_1, \ldots, Z_n)$ and set

$$\widehat{K}(z) = 2\frac{|\ell n|Z|^2|}{|Z|^4}\left|\widehat{H}\right|^2 + 2\frac{|\langle A\cdot Z, Z\rangle|^2}{|Z|^4}$$

where

$$\left|\widehat{H}\right|^2 = \sum_{\ell<m} |Z_\ell B_m\cdot Z - Z_m B_\ell \cdot Z|^2.$$

It is not difficult to see that

$$K_j(z) = \widehat{K}(z) + R(z)$$

where,

$$|R(z)| \leq c_1|Z|\left|\ell n|Z|^2\right| + c_2|Z| \Rightarrow \lim_{Z\to 0}\mathbb{R}(z) = 0$$

so that,

$$\liminf_{z\to p_j} K_j(z) = \liminf_{z\to p_j}\widehat{K}(z).$$

On the other hand, since $|Z|^2 < t_0$, we get

$$\widehat{K}(z) \geq 2\frac{(|\ell n t_0|\left|\widehat{H}\right|^2 + |\langle A\cdot Z, Z\rangle|^2)}{|Z|^4} = k_1(Z) \geq 0 \tag{7}$$

Now, if $s > 0$ we have $k_1(sZ) = k_1(Z)$, so that it is sufficient to prove that $k_1(Z) \neq 0$ for $Z \neq 0$. But $k_1(Z) = 0$ implies that $\left|\widehat{H}\right|^2 = 0$ and $\langle A\cdot Z, Z\rangle = 0$. Now,

$$\left|\widehat{H}\right|^2 = 0 \Rightarrow A\cdot Z = \lambda Z \Rightarrow Z \text{ is an eigenvector of } A \Rightarrow \lambda \neq 0,$$

because A is non singular. In this case $\langle A\cdot Z, Z\rangle = \lambda|Z|^2 = 0 \Rightarrow Z = 0$.

Therefore $\liminf_{z\to p_j} K_j(z) > 0$, which proves III.3.

We leave the proof of III.4 for the reader. For the proof of III.5, just use (6) and III.1. This proves Theorem B. □

3.2 Proof of Theorem C

Let $\mathcal{F}_0 \in N_k^n$ and $s \mapsto \mathcal{F}_s$ be a parametrization of a neighbourhood $\mathcal{N}$ of $\mathcal{F}_0$ in N_k^n. Let $\{p_1, \dots, p_N\} = S(\mathcal{F}_0)$. Since $\mathcal{F}_0$ has Milnor's number 1 and all $p_{j's}$, it is well known that we can choose the domain of the parametrization, say B, in such a way that there are holomorphic functions $p_1, \dots, p_N: B \to \mathbb{C}P^n$ such that $p_j(0) = p_j$ and $S(\mathcal{F}_s) = \{p_1(s), \dots, p_N(s)\}$. Fix also an affine coordinate system $\mathbb{C}^n \subset P^n$, such that $S(\mathcal{F}_0) \subset \mathbb{C}^n$. If we take B small enough, we can suppose that $S(\mathcal{F}_s) \subset \mathbb{C}^n$ for all $s \in B$. In this case we can define for each $s \in B$ a hermitian metric g^s as in (4) by,

$$g^s = \varphi_0 \varphi_1^s \cdots \varphi_N^s \mu^s \tag{4'}$$

where φ_0 is as before, $\varphi_j^s(z) = \varphi(|z - p_j(s)|^2)$ and

$$\mu^s = \frac{\sum_{j=1}^n |dz_j|^2 + \sum_{i<j} |z_j dz_j - z_j dz_j|^2}{\left(\sum_{j=1}^n \left|P_j^s\right|^2 + \sum_{i<j} \left|R_{ij}^s\right|^2\right)}$$

where $\mathcal{F}_s|\mathbb{C}^n$ is given by the differential equation $\dot{z} = P^s(z)$, $P^s = (P_1^s, \dots, P_n^s)$ and $R_{ij}^s = z_i P_j^s - z_j P_i^s$. Observe that $(s,p) \mapsto g_p^s$ is C^2.

Let $S \subset B \times \mathbb{C}P^n$ be defined by

$$S = \{(s, p_j(s)) | s \in B, j = 1, \dots, n\}.$$

Then $U = B \times \mathbb{C}P^n - S$ is open and we can define $K: U \to \mathbb{R}$ by $K(s,p) = K_{gs}(p)$. Clearly K is continuous.

Now, if $\mathcal{U}_k^n$ is defined as in §1 and $C \subset B$ is a compact set let

$$C' = \{(\mathcal{F}_s, \alpha) \in \mathcal{U}_k^n | s \in C\}.$$

It is enough to prove that C' is compact.

It follows from the proof of Theorem B that

$$\sup\{K(s,p) | (s,p) \in U, s \in C\} = -a, \ a > 0.$$

Let $\mathcal{F}$ be the foliation on $B \times \mathbb{C}P^n$, whose singular set is S and whose leaf through $(s,p) \in U$ is L_p^s, the leaf of $\mathcal{F}_s$ through p. Consider the metric g on U defined by $g_{(s,p)} = g^s + |ds|^2$. It is easy to see that $K_g(s,p) = K(s,p)$.

Let $((\mathcal{F}_n, \alpha_n))_{n\geq 1}$ be a sequence in C', where $\mathcal{F}_n = \mathcal{F}_{s_n}$. Since C is compact we can suppose that $\mathcal{F}_n \to \mathcal{F}_s$ for some $s \in C$. We can suppose also that the α_n's are not constants. From the proof of Theorem A, it follows that $(\alpha_n)_{n\geq 1}$ has a convergent subsequence, so that we will suppose that $\alpha_n \to \alpha$ in the compact parts of $\mathbb{D}$, where either α is the uniformization of some leaf of $\mathcal{F}_s$, or $\alpha(\mathbb{D}) \subset S$. Now, since the leaves of $\mathcal{F}$ are contained in the fibers of $\pi_1: B \times \mathbb{C}P^n \to B$, it is easy to see that in the second case $\alpha(\mathbb{D}) \subset S \cap \pi_1^{-1}(s) = S(\mathcal{F}_s)$. Hence $\alpha(\mathbb{D})$ is a constant in $S(\mathcal{F}_s)$. This proves Theorem C. □

3.3 Proof of Proposition 7

Let P and $X = (X_1, \dots, X_n)$ be as in the statement of the proposition. Since P is bounded we can assume that

$$P \subset B_{1/2} = \{z \in \mathbb{C}^n | |z| < \frac{1}{2}\}.$$

Consider the hermitian metric g_1 in $P - \{0\}$, defined by:

$$g_1 = \frac{\sum_{j=1}^n |dz_i|^2}{(\ell n|z|^2)^2 \sum_{j=1}^n |X_j(z)|^2} \tag{8}$$

If we compute K_{g_1} by the method of the proof of III.2 in §3.1, we get

$$K_{g_1}(z) = -4\frac{|\langle X(z), z\rangle|^2}{|z|^4} + \frac{4\ell n|z|^2}{|z|^4}\,[|X(z)|^2|z|^2 - |\langle X(z), z\rangle|^2].$$

Since $|z| \leq 1/2$ on P, we have $\ell n|z|^2 \leq -\ell n 4 < -1$. Therefore

$$\begin{aligned} |K_{g_1}(z)| &= \frac{4|\langle X(z), z\rangle|^2}{|z|^4} + \frac{4\left|\ell n|z|^2\right|}{|z|^4}\,[|X(z)|^2|z|^2 - |\langle X(z), z\rangle|^2] \\ &\geq \frac{4|X(z)|^2}{|z|^2} > 0. \end{aligned} \tag{9}$$

Since $DX(0)$ is non singular, by an argument similar to the proof of III.3 of §3.1, we get from (9) that:

$$\liminf_{z\to 0} K_{g_1}(z) = -a_1, \quad a_1 > 0. \tag{10}$$

Moreover, since $|X(z)| \geq c > 0$ for $|z| \geq r, z \in P$, we get from (9) that $|K_{g_1}(z)| \geq 16c^2$ for $z \in P$ with $|z| \geq r$. This implies that $K_{g_1} \leq -a_2$ in $P - \{0\}$, for some $a_2 > 0$.

Now, let $P = D_{r_1} \times \cdots \times D_{r_n}$, where

$$D_r = \{z \in \mathbb{C} | |z| < r\}$$

and g_2 be the hermitian metric on P defined by

$$g_2 \equiv \sum_{j=1}^{n} \frac{4r_j^2|dz_j|^2}{(r_j^2 - |z_j|^2)^2}.$$

The metric g_2 has the following properties:
1 - g_2 is complete (on P).
2 - There exists $a_3 > 0$ such that for any holomorphic embedding $\gamma: \mathbb{D} \to P$, then g_2 induces Gaussian curvature $\leq -a_3$ on $\gamma(\mathbb{D})$ (cf. ch. III of [**K**]).

It follows from Proposition 3.1 of Chapter I of [**K**] that g induces Gaussian curvature $K_g \leq -a$ on the leaves of $\mathcal{F}$, where

$$a^{-1} = a_2^{-1} + a_3^{-1}.$$

Finally, since g_2 is complete on P, by the same computations already done in the proof of II of §3.1 it follows that g is complete on $P - \{0\}$.

The proof of the remaining part of the proposition is analogous to the proof of Theorem A, so we leave it for the reader. □

4. Proof of Corollaries 5 and 6

4.1 Proof of Corollary 5

Let $\mathcal{F}$ be a foliation on $\mathbb{C}P^n$ like in Theorem B, and $h: \mathbb{D}^* \to L$ be a holomorphic map, where L is some leaf of $\mathcal{F}$. Fix an uniformization $\alpha: \mathbb{D} \to L$. Let γ be a generator of $\pi_1(\mathbb{D}^*)$ and consider $h_*(\gamma) \in \pi_1(L)$. We have two possibilities:

a) $h_*(\gamma) = 1$ in $\pi_1(L)$.
b) $h_*(\gamma) \neq 1$ in $\pi_1(L)$.

Let us suppose that $h_*(\gamma) = 1$. In this case, it follows from the theory of covering spaces that h can be lifted to $\hat{h}: \mathbb{D}^* \to \mathbb{D}$, where $\hat{h}$ is holomorphic and $\alpha \circ \hat{h} = h$. Therefore big Picard's Theorem (cf. ch. V of [**K**]) implies that $\hat{h}$ extends to a holomorphic map, which we call again $\hat{h}$, $\hat{h}: \mathbb{D} \to \mathbb{D}$. From this we can conclude that h extends to a holomorphic map (h again), $h: \mathbb{D} \to \mathbb{D}$, where $h(0) = \alpha(\hat{h}(0)) \in L$.

Let us suppose now that $h_*(\gamma) \neq 1$. We will prove that there exists $\lim_{z\to 0} h(z) = p$, where $p \in S(\mathcal{F})$. Consider the set $K = \bigcap_{0<r<1} \overline{h(D_r^*)}$, where

$$D_r^* = \{z \in \mathbb{D} | 0 < |z| < r\}.$$

Then K is compact, connected and non empty, because

$$K = \bigcap_{n=1}^{\infty} \overline{h(D_{1/n}^*)}.$$

It is enough to prove that $K = \{p\}$, $p \in S(\mathcal{F})$. Suppose by contradiction that this is not true. Then, there exists $q_o \in K - S(\mathcal{F})$. Clearly $q_o = \lim_{n\to\infty} h(z_n)$, where $z_n \in \mathbb{D}^*$ and $\lim_{n\to\infty} z_n = 0$. Set $r_n = |z_n|$ and

$$\gamma_n = \{z \in \mathbb{D}^* | |z| = r_n\}.$$

We will show that, for large n, $h(\gamma_n)$ is homotopic to a constant in L, which contradicts the fact that $h_*(\gamma) \neq 1$. Let g be the hermitian metric constructed in Theorem B, where $K_g \leq -a^2$, $a > 0$, and μ_P be the Poincaré metric in $\mathbb{D}^*$. It follows from Ahlfor's lemma that

$$h^*(g) \leq \frac{1}{a^2} \mu_P. \tag{1}$$

Let ℓn be the length of γ_n in the metric μ_P, and $\overline{\ell_n}$, d_n be the length and the diameter of $h(\gamma_n)$ in the metric g, respectively. Inequality (1) implies that:

$$\overline{\ell_n} \leq \frac{1}{a} \ell_n. \tag{2}$$

Since $\ell_n = 2\pi/|\ell n r_n|$, we have $\lim_{n\to\infty} \ell_n = 0$, and so $\lim_{n\to\infty} \overline{\ell_n} = 0$. Therefore $\lim_{n\to\infty} d_n = 0$, because $d_n \le \overline{\ell_n}$.

Now, $q_o \notin S(\mathcal{F})$, implies that there exists a local trivialization (φ, Q) of $\mathcal{F}$, where

$$q_o \in Q,\ \varphi(Q) = B^{n-1} \times \mathbb{D},\ 0 \in B^{n-1},\ B^{n-1}$$

is a ball in $\mathbb{C}^{n-1}$, $\varphi(q_o) = (0,0)$ and the leaves of $\mathcal{F}|Q$ are the disks $\{x\} \times \mathbb{D}$, $x \in B^{n-1}$. On the other hand,

$$\lim_{n\to\infty} h(z_n) = q_o, h(z_n) \in h(\gamma_n) \text{ and } \lim_{n\to\infty} d_n = 0.$$

Hence, there exists $n_o \ge 1$ such that for $n \ge n_o$ we have $h(\gamma_n) \subset Q$. Since $h(\gamma_n) \subset L$ we must have $h(\gamma_n) \subset \{0\}\times\mathbb{D} \subset L$, which implies that $h(\gamma_n)$ is homotopic to a constant in L. This proves that h extends to a continuous map (which we call again h), $h\colon \mathbb{D} \to \mathbb{C}P^n$, where $h(0) = p \in S(\mathcal{F})$. It follows from Riemann's extension theorem that h is holomorphic. □

4.2 Proof of Corollary 6

Let $\mathcal{F}$ be a foliation on $\mathbb{C}P^n$ like in Theorem B. Let L be a leaf of $\mathcal{F}$, $\alpha\colon \mathbb{D} \to L$ be an uniformization with $\alpha(0) = q \in L$, and $d\colon \pi_1(L,q) \to G(L)$ be the isomorphism associated to α.

Suppose first that L contains a separatrix Γ, where $\overline{\Gamma} = \Gamma \cup \{p\}$ and $\overline{\Gamma}$ has a Puiseaux's parametrization $\delta\colon \mathbb{D} \to \overline{\Gamma}$, where $\delta(0) = p \in S(\mathcal{F})$ and $\delta(\mathbb{D}^*) = \Gamma$. We will assume that $\delta(1/2) = q = \alpha(0)$, so that if

$$\gamma = \{z \in \mathbb{D}^* \,||z| = \frac{1}{2}\},$$

we have $\delta_*(\gamma) \in \pi_1(L,q)$. Observe that Corollary 5 implies $\delta_*(\gamma) \ne 1$, because $p = \delta(0) \notin L$. Consider L with its structure of Riemann surface, induced by the complex structure of $\mathbb{C}P^n$. The idea is to prove that there exists a Riemann surface $\widetilde{L}$ and a point $\tilde{p} \in \widetilde{L}$ such that $\widetilde{L} - \{\tilde{p}\}$ is biholomorphic to L in such a way that Γ is a punctured neighbourhood of $\tilde{p}$ in $\widetilde{L}$. This will imply that $d(\delta_*(\gamma))$ is parabolic (cf. [**M**]). In order to prove this fact we use the blowing-up resolution of $\overline{\Gamma}$. It is know that $\overline{\Gamma}$ can be solved by a sequence of blowing-ups, in such a way that if we denote the composition of all these blowing-ups by π, then $\pi\colon M \to$

$\mathbb{C}P(n)$ is a proper analytic map with the following properties (cf. [**F**]):

a) M is a complex manifold.

b) $\pi^{-1}(p) = \mathcal{D}$ is a union of projective spaces $\mathbb{C}P^{n-1}$.

c) $\pi|M - \mathcal{D}\colon M - \mathcal{D} \to \mathbb{C}P^n - \{p\}$ is a biholomorphism.

d) $\pi^{-1}(\overline{\Gamma}) = \widetilde{\Gamma} \cup \mathcal{D}$, where $\widetilde{\Gamma}$ is smooth and $\pi(\widetilde{\Gamma}) = \overline{\Gamma}$.

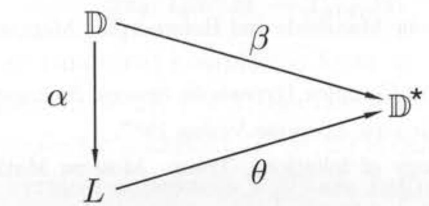
e) $\widetilde{\Gamma}$ is transverse to $\mathcal{D}$ and $\widetilde{\Gamma} \cap \mathcal{D} = \{\tilde{p}\}$, where $\tilde{p}$ belongs to a unique projective space of the decomposition of $\mathcal{D}$.

The curve $\widetilde{\Gamma}$ is called the strict transform of $\overline{\Gamma}$. The Riemann surface $\widetilde{L}$ is obtained from L as follows: $\widetilde{L} = (L - \Gamma) \cup \widetilde{\Gamma}$. It is clear from the construction that $\widetilde{L} - \{\tilde{p}\} = L$ and that $\Gamma = \widetilde{\Gamma} - \{\tilde{p}\}$ is a punctured neighbourhood of $\tilde{p}$ in $\widetilde{L}$, as claimed.

Now, suppose that $G(L)$ contains a parabolic element f. Let

$$C(f) = \{h \in G(L) | hf = fh\}.$$

Since f is parabolic, $C(f)$ is isomorphic to $\mathbb{Z}$ and $f = h^n$, $n \ge 1$, where h is a generator of $C(f)$. Consider the quotient space $\mathbb{D}/C(f)$, of $\mathbb{D}$ by the equivalence relation which identifies points in the same orbit of $C(f)$. Then $\mathbb{D}/C(f)$ is biholomorphic to $\mathbb{D}^*$, and the projection of the equivalence relation can be identified with a holomorphic covering map $\beta\colon \mathbb{D} \to \mathbb{D}^*$. Now, for any $w \in \mathbb{D}^*$, $\beta^{-1}(w)$ is a $C(f)$-orbit and so $\alpha(\beta^{-1}(w))$ is a point in L. In this way we obtain a holomorphic covering map $\theta\colon \mathbb{D}^* \to L$, which makes the diagram below commute



If $d\colon \pi_1(L,q) \to G(L)$ is as before, then there exists $q^* \in \mathbb{D}^*$ such that $\theta_*(\pi_1(\mathbb{D}^*, q^*)) = d^{-1}(C(f))$, so that if γ is a generator of $\pi_1(\mathbb{D}^*, q^*)$, then $\theta_*(\gamma) = d^{-1}(h) \ne 1$. It follows from Corollary 5 that θ extends to a holomorphic map $\tilde{\theta}\colon \mathbb{D} \to \mathbb{C}P^n$ such that $\tilde{\theta}(0) = p \in S(\mathcal{F})$.

Let (z, U) be a local coordinate chart such that $z(p) = 0 \in \mathbb{C}^n$ and

$r > 0$ be such that $\tilde{\theta}(D_r) \subset U$. Then, for $|T| < r$, we have

$$\tilde{\theta}(T) = (T^{k_1} u_1(T), \dots, T^{k_n} u_n(T))$$

where either $u_j(0) \neq 0$, or $u_j \equiv 0$, $1 \leq j \leq n$, and there exists $j_0 \in \{1, \dots, n\}$ such that $u_{j_o} \not\equiv 0$. After a linear change of variables in $\mathbb{C}^n$ we can suppose that $u_1, \dots, u_n \not\equiv 0$ and that $k_1 = \dots = k_n = k$. After a change of variables in a neighbourhood of $0 \in \mathbb{D}$, of the form $s = T \sqrt[k]{u_1(T)}$, we can suppose that $\tilde{\theta}|D_{r'} = \varphi$ is of the form

$$\varphi(s) = (s^k, s^k u_2(s), \dots, s^k u_n(s)),$$

where $u_j(0) \neq 0$, $j \geq 2$. This implies that if $\overline{\Gamma} = \tilde{\theta}(D_{r'})$ and $\Gamma = \tilde{\theta}(D^*_{r'})$, then $\varphi|D^*_{r'}: D^*_{r'} \to \Gamma$ is a finite covering and $\varphi'(s) \neq 0$ if $s \neq 0$. In fact $\varphi|D^*_{r'}$ will be an embedding, because $d^{-1}(\theta_*(\gamma))$ is a generator of $C(f)$. Therefore L contains a separatrix Γ, which proves Corollary 6. □

References

[**B-K**] J. Bland and M. Kalka, Complete metrics conformal to the hyperbolic disc, Proc. A.M.S., 97 (1986), 128-132.

[**C**] A. Candel, Uniformization of Surface Laminations. Ann. Sc. Ec. Norm. Sup. Field. Preprint.

[**C-G**] A. Candel and X. Gómez-Mont, Uniformization of the leaves of a Rational Vector Field. Preprint.

[**CLS**] C. Camacho, A. Lins Neto and P. Sad, Minimal Sets of Projective Foliations, Publ. Math. de I'IHES, #68.

[**F**] W. Fulton, Algebraic Curves. An Introduction to Algebraic Geometry, N.Y., W.A. Benjamin 1969.

[**K**] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker Inc. N.Y. 1970.

[**L**] S. Lang, Introduction to Complex Hyperbolic Spaces, Springer Verlag 1987.

[**M**] B. Maskit, Kleinian Groups, Springer Verlag 1987.

[**N**] S. P. Novikov, Topology of foliations, Trans. Moscow Math. Soc. 1965, pg. 268-304.

[**V**] A. Verjovsky, An Uniformization theorem for Holomorphic Foliations, Contemp. Math. 58(III) (1987) 233-253.

Alcides Lins Neto
IMPA - Estrada Dona Castorina, 110
22460-320, Rio de Janeiro, RJ
Brazil