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Asymptotic Stability at Infinity of Planar Vector Fields

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Abstract. Let $\rho > 0$ and X be a C^1 vector field on the plane such that: (i) for all $q \in \mathbb{R}^2$, $\mathrm{Det}(DX(q)) > 0$; and (ii) for all $p \in \mathbb{R}^2$, with $||p|| \geq \rho$, $\mathrm{Trace}(D(X(p)) < 0$. If X has a singularity and $\int_{\mathbb{R}^2} \mathrm{Trace}(DX)\, dx \wedge dy$ is less than 0 (resp. greater or equal than 0), then the point at infinity of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ is a repellor (resp. an attractor) of X.

I. Introduction

This paper extends the results obtained by the first author in [**Gu2**]. Concerning Global Asymptotic Stability, there is a rich literature on the subject; see for instance, [**Aiz**], [**GLS**], [**Gu1**], [**MY**], [**Ole**]. We suggest the reader to see [**Gu2**] and [**MO**] for further references and connections with other problems.

Let us proceed to state the main result of this paper. For definitions and basic results used here, we suggest the reader the book of W. de Melo and J. Palis [**MP**].

Let $X: \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field. We say that "∞" *is an attractor of* X, if there is an $R > 0$ such that for any $p \in \mathbb{R}^2$, with $|p| > R$ we have that $\omega(p)$ (the ω–limit set of p) is empty. We say that "∞" *is a repellor of* X, if it is an attractor of $-X$.

In this paper we prove:

Theorem A. *Let* $\rho > 0$ *and* $X: \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *vector field satisfying the following conditions:*

(i) X *has at least one singularity, say* S;

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(ii) $\mathrm{Det}(DX(p)) > 0$, *for every* $p \in \mathbb{R}^2$;

(iii) *For all* $p \in \mathbb{R}^2$, *with* $||p|| \geq \rho$, $\mathrm{Trace}(D(X(p)) < 0$.

If

$$\int_{\mathbb{R}^2} \mathrm{Trace}(DX)\, dx \wedge dy$$

is less than 0 *(resp. greater of equal than* 0*), then "∞" is a repellor (resp. an attractor) of* X. *In particular if* X *has no closed trajectories, then either the stable manifold,* $W^s(S)$, *of* S *is equal to* $\mathbb{R}^2$ *or the unstable manifold,* $W^u(S)$, *of* S *is equal to* $\mathbb{R}^2$

Concerning examples of vector fields satisfying the conditions of Theorem A, there are linear vector fields such that $W^s(0) = \mathbb{R}^2$. In section IV, we show an example of a vector field for which $W^u(0) = \mathbb{R}^2$.

N. V. Chau studied in **[Cha]** the same problem as this work for polynomial vector fields, in which case $\int_{\mathbb{R}^2} \mathrm{Trace}(DX)\, dx \wedge dy = -\infty$ and therefore "∞" is always a repellor. See also **[Kra]**.

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II. Preliminaries

The proof of the following proposition is done in Lemmas 2.2–2.9 below.

Proposition 2.1. *Let* $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$, *be a* C^1 *vector field satisfying the following conditions:*

(i) *There exist* $\delta > 0$ *and an open disc* V *bounded by a closed trajectory of* X *such that, for all* $p \in \mathbb{R}^2 \setminus V$, $||X(p)|| \geq \delta$;

(ii) *There exists* $\rho > 0$ *such that, for all* $p \in \mathbb{R}^2$, *with* $||p|| \geq \rho$, $\mathrm{Trace}(DX(p)) < 0$.

Then "∞" is either an attractor or a repellor of X.

Let $X^* = (-g, f)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field orthogonal to X.

Observe that $||X|| = ||X^*||$, and $\langle X, X^* \rangle = 0$ on $\mathbb{R}^2$. Denote by $\gamma(p,t)$ and $\gamma^*(p,t)$ the flows induced by X and X^*, respectively. If $p \in R^2$ and $q = \gamma(p, t_0)$ (resp $q = \gamma^*(p, t_0)$), for some $t_0 \in \mathbb{R}$, then (pq) (resp. $(pq)^*$) will denote the arc of trajectory of X (resp. of X^*) joining the points p and q. For any arc of trajectory $(pq)^*$ of X^*, let

$$L(p,q) = \Big| \int_{(pq)^*} ||X|| ds \Big|,$$

where ds denotes the arc length element. It follows from Green's formula that

Lemma 2.2. *Let* A *be a compact region the boundary of which is made up of two arcs of trajectory* $(p_1q_1)^*$, $(p_2q_2)^*$ *of* X^* *and two arcs of trajectory of* X. *If the flow induced by* X *goes into* A *by* $(p_1q_1)^*$ *and exits* A *by* $(p_2q_2)^*$, *then*

$$L(p_2, q_2) - L(p_1, q_1) = \int_A \mathrm{Trace}(DX)\, dx \wedge dy.$$

This lemma implies that:

Corollary 2.3. *Let* $A \subset \mathbb{R}^2 \setminus V$ *be as in Lemma 2.2. Let*

$$\Delta(p_1, q_1) = \sup\{||X(p)||\colon p \in (p_1q_1)^*\}$$

and $l(p_i, q_i)$ *be the arc length of* $(p_iq_i)^*$. *Then*

$$l(p_2, q_2) < \frac{\tau}{\delta} + l(p_1, q_1)\frac{\Delta(p_1, q_1)}{\delta},$$

where $\tau = \pi\rho^2(\sup\{\mathrm{Trace}(DX(p))\colon p \in \mathbb{R}^2\})$.

The proof of next lemma can be found in **[Ole]**.

Lemma 2.4. *The set* $A_\infty = \{p \in \mathbb{R}^2\colon w(p) = \varnothing\}$ *is open.*

In the following, given a trajectory γ of X, the α-limit set of γ will be denoted by $\alpha(\gamma)$.

If $\gamma \subset \mathbb{R}^2 \setminus V$ is a closed orbit of X, denote by $D(\gamma)$ the compact disc in $\mathbb{R}^2$ such that $\partial D(\gamma) = \gamma$. Notice that, as X has no singularities in $\mathbb{R}^2 \setminus V$, the disc $D(\gamma)$ contains $\overline{V}$. Moreover, given two discs $D(\gamma_1)$ and $(D\gamma_2)$, one of them contains the other. Let $\mathcal{D}$ be the union of such discs.

Lemma 2.5. *The set $\mathcal{D}$ is nonempty and compact. Moreover, $\partial\mathcal{D} = \Gamma$ is a periodic orbit of X.*

Proof. First, we claim that $\mathcal{D} \neq \mathbb{R}^2$ (and so $\partial\mathcal{D} \neq \varnothing$). Otherwise, there would be a sequence of closed orbits γ_n converging to "∞". So that for n large enough γ_n would be in the region where $\mathrm{Trace}(DX) < 0$. If we took two successive orbits, say γ_n, γ_{n+1}, the Green formula applied to the annulus between them, would give a contradiction.

Now suppose, by contradiction, that the frontier $\partial\mathcal{D}$ of $\mathcal{D}$ is not a closed trajectory of X. As $X|_{\mathbb{R}^2 \setminus V}$ has no singularities (hypothesis (i)), it follows from the Poincaré Bendixson Theorem that if $p \in \partial\mathcal{D}$ then, $\omega(p)$ is either (1) a closed trajectory, or (2) the empty set, or else (3) made up of regular trajectories γ such that $\omega(\gamma) \cup \alpha(\gamma)$ is the empty set. However, (2) and (3) are excluded by Lemma 2.4 (because ∂D is accumulated by periodic orbits). As $\partial(\mathcal{D})$ is invariant, it must be that $\omega(p) = \partial(\mathcal{D})$ is a closed trajectory. This contradiction finishes the proof. □

Denote by

$$W_1^s = W_1^s(\Gamma) = \{p \in \mathbb{R}^2 \setminus \mathcal{D} \colon \omega(p) = \Gamma\}$$

and by

$$W_1^u = W_1^u(\Gamma) = \{p \in \mathbb{R}^2 \setminus \mathcal{D} \colon \alpha(p) = \Gamma\}.$$

We claim that, between W_1^s and W_1^u, one is an empty set and the other is a nonempty open set whose boundary contains Γ. In fact, as $\mathbb{R}^2 \setminus \mathcal{D}$ contains no closed orbit, the return map along Γ on a small half segment pointing outside $\mathcal{D}$ has no fixed point outside Γ. This implies that Γ is locally attracting or repelling. The set $W_1^s \cup \Gamma$ (resp. $W_1^u \cup \Gamma$) is a one–sided stable (resp. unstable) manifold of Γ.

Lemma 2.6. *If $W_1^s(\Gamma)$ is nonempty, then $W_1^s(\Gamma) = \mathbb{R}^2 \setminus \mathcal{D}$. In particular, "$\infty$" is a repellor of X.*

Proof. Assume, by contradiction, that there is a point $p \in \mathbb{R}^2 \setminus \mathcal{D}$ belonging to the frontier of $W_1^s(\Gamma) \cup \Gamma$. By the Poincaré–Bendixson Theorem and the fact that $\mathbb{R}^2 \setminus \mathcal{D}$ contains neither a singular point (hypothesis (i)) nor a closed orbit, $\omega(p)$ is either (1) the empty set or

(2) made up of regular trajectories γ such that $\omega(\gamma) \cup \alpha(\gamma)$ is the empty set. However, (1) is excluded by Lemma 2.4 (because $W_1^s(\Gamma)$ and A_∞ are open sets); (2) is also excluded by Lemma 2.4 (because A_∞ is an open set and so $\omega(p)$ cannot contain points belonging to A_∞). This proves the lemma. □

Given $p \in \mathbb{R}^2$, on each trajectory $\gamma(p,t)$ and $\gamma^*(p,t)$ of the vector fields X and X^*, respectively, we consider the order induced by the time t. Take a 2π-periodic, regular global parameterization $s \to \Gamma(s)$ of Γ. Denote by $\gamma_s^* = \gamma_s^*(t)$, with $t > 0$, the positive half-trajectory of X^* starting at $\Gamma(s)$ (with $\Gamma(s)$ excluded). Let T^* be the union of all half-trajectories γ_s^* and T^{**} be the set whose elements are the trajectories γ_s^*.

Lemma 2.7. *Suppose that $W_1^u(\Gamma)$ is nonempty. Given $q \in W_1^u(\Gamma)$ and $\gamma^* \in T^{**}$, there exists $t_0 > 0$ such that $\gamma(q, t_0)$ meets γ^*.*

Proof. Suppose, by contradiction that there exist $q \in W_1^u(\Gamma)$ and $\gamma_{s_0}^* \in T^{**}$ such that, for every $t > 0$ (belonging to its domain of definition). $\gamma(q,t)$ does not meet $\gamma_{s_0}^*$.

Consider only the case in which X^* points outside $\mathcal{D}$. We know, due to the local properties of X around the closed trajectory Γ, that we may find $t_1 > t_2 \geq 0$ such that: (1) the points $p_1 = \gamma(q, -t_1)$ and $p_2 = \gamma(q, -t_2)$ belong to $\gamma_{s_0}^*$, and (2) the compact arc of trajectory (qp_1) (contained in $\gamma(q)$) meets $\gamma_{s_0}^*$ exactly at $\{p_1, p_2\}$. It is clear that the compact arc of trajectory (p_1, p_2) (contained in $\gamma(q)$) meets every element γ_s^* of T^{**}. Let M be the set made up of the $s \in (s_0, s_0 + 2\pi]$ such that, for all $r \in (0, s]$, there exists $t = t(r) > 0$ satisfying $\gamma(p_2, t) \cap \gamma_r^* \neq \varnothing$. Certainly, M is non empty and connected. Let $s' = \mathrm{Sup}\, M$ and let $\{s_n\}$ be an increasing sequence in $M \cap (s_0, s_0 + 2\pi)$ converging to s'. Denote by P_n the intersection point between $\gamma_{s_n}^*$ and the compact arc (p_1, p_2). Let Q_n be the first intersection point of the positive half-trajectory $\{\gamma(p_2, t) : t > 0\}$ with $\gamma_{s_n}^*$. Compare now the arc lengths $l(p_1, p_2)$ (of the sub arc $(p_1, p_2)^*$ of $\gamma_{s_0}^*$) and $l(P_n Q_n)$ (of the sub arc $(P_n, Q_n)^*$ of $\gamma_{s_n}^*$). By Corollary 2.3, there exists a constant $K > 0$ such that, for every n, $l(P_n, q_n) < K$. It follows that the orbit $\gamma_{s'}^*$ cuts the half-trajectory $\{\gamma(p_2,t) : t>0\}$ at some $Q' = \lim Q_n$. But then $s' \in M$ and so $s' = s_0 + 2\pi$. However, this is impossible because $\{\gamma(q,t) : t>0\}$ (and therefore $\{\gamma(p_2,t)\, t>0\}$) is assumed to be disjoint of $\gamma^*_{s_0} = \gamma^*_{s_0+2\pi}$. This contradiction proves the lemma. □

Lemma 2.8. *If $W_1^u = W_1^u(\Gamma)$ is not empty, $W_1^u(\Gamma) = T^*$.*

Proof. If follows from Lemma 2.7 that, for each pair (p,γ^*) in $W_1^u(\Gamma)\times T^{**}$, there exists an infinite sequence of real numbers (in the domain of definition of $\gamma(p,\cdot)$) $0<t_1<t_2<\cdots<t_n<\cdots$ such that:

(a) the sequence $p_i = \gamma(p,t_i)$ belongs to γ^* and, on this oriented half–trajectory, we have that $p_1 < p_2 < \cdots < p_n < \cdots$;

(b) if $t_i < t < t_{i+1}$ then $\gamma(p,t) \notin \gamma^*$ and the compact arc of trajectory (p_ip_{i+1}) of $\gamma(p)$ meets all half–trajectories belonging to T^{**}.

(c) If A_i denotes the open annulus bounded by Γ and the arcs of trajectory (p_ip_{i+1}) and $(p_ip_{i+1})^*$, then the increasing union $\cup_i A_i$ is precisely W_1^u.

This implies that $W_1^u \subset T^*$. Now we claim that $W_1^u = T^*$. In fact, suppose by contradiction, that there exists $q\in T^*\setminus W_1^u$ belonging to the frontier of W_1^u. First observe that by the Poincaré–Bendixson Theorem (using a similar argument to that of the proof of Lemma 2.5) $\omega(q)$ is empty. Let $\gamma^*\in T^{**}$ be the half–trajectory containing q. Choose a point $p\in W_1^u$ and take the sequence $\{p_i\}$ of $\gamma(p)\cap\gamma^*$ as in (a)–(b) above. It follows from (a)–(c) above that $\lim_i p_i = q$. Therefore (as $\omega(q)$ is empty) by using Lemma 2.4 we obtain that, for i large enough, $\omega(p_i)$ is also empty. This is a contradiction because $q\in\omega(p_i)$. □

Lemma 2.9. *If $W_1^u(\Gamma)$ is nonempty, then $W_1^u(\Gamma) = \mathbb{R}^2\setminus\mathcal{D}$. In particular, "$\infty$" is an attractor of X.*

Proof. As T^* is an open subset of $\mathbb{R}^2\setminus\mathcal{D}$, it is enough to prove that it is a closed one. Suppose, by contradiction, that there exists $p\in\mathbb{R}^2\setminus(T^*\cup\mathcal{D})$ belonging to the frontier of $T^*\cup\mathcal{D}$. Take a small neighborhood V of p which is a flow box for both X and X^*. By the assumptions, there is a trajectory σ of $X^*|_V$ which is contained in T^* and is a global cross section for $X|_V$. As V is the union of the trajectories of $X|_V$ meeting σ, it follows from Lemma 2.8 that V is contained in T^*. This contradiction proves the lemma. □

III. Proof of the main result

In this section we prove the following:

Theorem A. *Let $\rho>0$ and $X=(f,g)\colon\mathbb{R}^2\to\mathbb{R}^2$ be a C^1 vector field satisfying the following conditions:*

(i) *X has at least one singularity, say S;*

(ii) $\mathrm{Det}(DX(p))>0$, *for every $p\in\mathbb{R}^2$;*

(iii) *For all $p\in\mathbb{R}^2$, with $||p||\ge\rho$,* $\mathrm{Trace}(D(X(p)))<0$.

If

$$\int_{\mathbb{R}^2} \mathrm{Trace}(DX)\, dx\wedge dy$$

is less than 0 (resp. greater of equal than 0), then "∞" is a repellor (resp. an attractor) of X. In particular if X has no closed trajectories, then either $W^s(S)=\mathbb{R}^2$ or $W^u(S)=\mathbb{R}^2$.

The proof of the following result can be found in **[Gu2]**

Lemma 3.1. *$X\colon\mathbb{R}^2\to\mathbb{R}^2$ is an injective map.*

Proof of Theorem A. Suppose that X has a closed trajectory. Let V be the open disc bounded by this trajectory. As, by the assumptions and Lemma 3.1, X takes diffeomorphically $\mathbb{R}^2$ onto an open subset of $\mathbb{R}^2$ containing 0, there exists $\delta>0$ such that, for all $p\in\mathbb{R}^2\setminus V$, $||X(p)||\ge\delta$. Therefore, by Proposition 2.1, "∞" is either an attractor or a repellor of X.

Now, suppose that X has no closed trajectories. By the assumption (ii) of this theorem and by Lemma 3.1, S is the only singularity of X and it is either a local attractor or a local repellor.

Let us proceed by considering only the case in which S is a local repellor. Let $W^u(s)$ denote the unstable manifold of S. Take two small compact discs D_1 and D_2 such that

$$S\in\mathrm{Int}(D_1)\subset D_1\subset\mathrm{Int}(D_2)\subset D_2\subset W^u(s)$$

and both $\partial D_1, \partial D_2$ are transversal to X.

By the assumption (ii) and Lemma 3.1, there exists $\delta_1 > 0$ such that, for all $p \in \mathbb{R}^2 \setminus \mathrm{Int}(D_1)$, $||X(p)|| \geq \delta_1$

It is not difficult to see that there exists a C^1 vector field $Y: \mathbb{R}^2 \to \mathbb{R}^2$ such that

a) $Y = X$ in the set $\mathbb{R}^2 \setminus \mathrm{Int}(D_2)$;
b) ∂D_1 is a closed orbit of Y;
c) there exists $0 < \delta < \delta_1$ such that, for all $p \in D_2 \setminus \mathrm{Int}(D_1)$, $||Y(p)|| \geq \delta$.

Under these circumstances we may apply Proposition 2.1 to obtain that "∞" is either an attractor or a repellor of Y and, in this way, of X too (see (a) above).

Now suppose that "∞" is a repellor and suppose, by contradiction, that $\mathcal{I}(\mathbb{R}^2) \geq 0$, where

$$\mathcal{I}(W) = \int_W \mathrm{Trace}(DX)\, dx \wedge dy.$$

Take a circle C contained in $\{p \in \mathbb{R}^2 : ||p|| \geq \rho + 1\}$ and such that $X_{|C}$ points inside to the compact disc D bounded by C. This implies, by the Green's formula, that $\mathcal{I}(D) < 0$. By the assumption (iii) of the theorem $\mathcal{I}(\mathbb{R}^2 \setminus \mathrm{Int}(D)) < 0$. Therefore

$$\mathcal{I}(\mathbb{R}^2) = \mathcal{I}(D) + \mathcal{I}(\mathbb{R}^2 \setminus \mathrm{Int}(D)) < 0$$

which is the required contradiction.

In a similar way it can be seen that, if "∞" is an attractor, $\mathcal{I}(\mathbb{R}^2) \geq 0$. The last claim of the theorem follows from the Poincaré–Bendixson Theorem. □

IV. An Example

The purpose of this section is to exhibit a vector field X satisfying the conditions of Theorem A and such that the unstable manifold $W^u(0)$, of 0, is $\mathbb{R}^2$. In particular "∞" is an attractor of X. The required vector field is given by:

$$X(x, y) = g(r)(e^{-r}x - y, x + e^{-r}y)$$

where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad g(r) = \frac{1 - e^{-r}}{r\sqrt{1 + e^{-2r}}}.$$

By direct computation, we may obtain the following expressions:

$$\mathrm{Det}(DX) = \frac{e^r - 1}{re^{2r}}, \tag{a}$$

$$\mathrm{Trace}(DX) = \frac{e^r + (r-1)(1 + 2e^{2r} - e^{3r})}{re^{4r}(1 + e^{-2r})^{3/2}}. \tag{b}$$

This implies that $\mathrm{Det}(DX) > 0$ everywhere and that $\mathrm{Trace}(DX) < 0$ in the region $\{r > K\}$ for some large K. Therefore, using Theorem A, we conclude that "∞" is either an attractor or a repellor of X. To obtain a stronger conclusion, we may observe that the inner product

$$\langle (x, y), X(x, y) \rangle = g(r) r^2 e^{-r} \tag{c}$$

is greater than 0, for all $r > 0$; in this way, the vector field X points outside all discs whose boundary has the form $\{r = \text{constant}\}$. This implies that $W^u(0) = \mathbb{R}^2$ and, in particular, that "∞" is an attractor of X. Hence, Theorem A implies that

$$\int_{\mathbb{R}^2} \mathrm{Trace}(DX)\, dx \wedge dy \geq 0.$$

To obtain the precise value of this integral we may use (c) above and Green's formula (on the disc bounded by $\{r = \text{constant}\}$) as follows:

$$\int_{\{\sqrt{x^2+y^2} \leq r\}} \mathrm{Trace}(DX)\, dx \wedge dy = \int_0^{2\pi r} g(r) e^{-r} ds = 2\pi r g(r) e^{-r}.$$

As $2\pi r g(r) e^{-r}$ goes to zero as r goes to ∞, we conclude that

$$\int_{\mathbb{R}^2} \mathrm{Trace}(DX)\, dx \wedge dy = 0.$$

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