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Deformations of Branched Lefschetz Pencils

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Abstract. Let M be a projective manifold of dimension ≥ 3 and $H^1(M, \mathbb{C}) = 0$. We will show that a deformation of a codimension one singular foliation $\mathcal{F}$ arising from the fibers of a generic meromorphic map of the form $f^p/g^q, p, q > 0$ has a meromorphic first integral of the same type.

0. Introduction

Recently, Gómez-Mont and Lins [**GM-L**] have shown the following result, which is an extension to codimension one holomorphic foliations with singularities of the Thurston-Reeb stability theorem:

Theorem. [**GM-L**] Let M be a projective manifold:

(1) If $H^1(M, \mathbb{C}) = 0$ and $\dim_{\mathbb{C}} M \geq 3$, then Lefschetz Pencils are $\mathcal{C}^0$-structurally stable foliations.

(2) If $\pi_1(M) = 0$ and $\dim_{\mathbb{C}} M \geq 4$, then Branched Lefschetz Pencils are $\mathcal{C}^0$-structurally stable foliations.

The aim of this work, is to improve the second part of this theorem.

Let L_1 and L_2 be positive holomorphic line bundles on M with holomorphic sections f_1, f_2. Assume that $L_1^{\otimes p} = L_2^{\otimes q}$, where p and q are relatively prime positive integers. The fibers of the meromorphic map $\phi = f_1^p/f_2^q$ define a codimension one holomorphic foliation with singularities represented by the twisted one-form

$$\omega = pf_2 df_1 - qf_1 df_2 \in Fol(M, L_1 \otimes L_2).$$

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In what follows, we shall say that ϕ is a *meromorphic first integral* of the foliation represented by the section ω.

By a *generic* meromorphic map $\phi = f_1^p/f_2^q$ we mean the following:

(1) The sets $\{f_1 = 0\}_{i=1,2}$ are smooth irreducible and meet transversely on a codimension two submanifold K.

(2) The subvarieties defined by $\lambda f_1^p - \mu f_2^q = 0$ with $(\lambda\colon\mu) \in \mathbb{P}^1$ are smooth on $M - K$ except for a finite set of values $\{(\lambda_i\colon\mu_i)\}_{\{i=1,\dots,k\}}$, where the corresponding subvariety has only one non-degenerate critical point.

A meromorphic map satisfying conditions (1) and (2) is called a *Lefschetz Pencil* if $p = q = 1$ and a *Branched Lefschetz Pencil* otherwise.

Our main results are the following:

Theorem A. *Let M be a projective manifold whose complex dimension is at least 3 and with $H^1(M,\mathrm{C}) = 0$. Let $\omega = pf_2df_1 - qf_1df_2$ with $p \neq q$ where f_i are holomorphic sections of the positive line bundles L_i and $L_1^p = L_2^q$. If $\{f_i = 0\}_{i=1,2}$ are smooth irreducible and meet transversally, then any deformation ω' of the foliation represented by ω has a meromorphic first integral $\phi' = f_1'^p/f_2'^q$ where $f_i' \in H^0(M,\mathcal{O}(L_i))$.*

As consequence of this result, the Gómez-Mont Lins theorem may be stated as follows:

Theorem B. *Let M be a projective manifold whose complex dimension is at least 3 and with $H^1(M,\mathbb{C}) = 0$. Let $\mathcal{F}$ be a codimension one holomorphic foliation with singularities arising from the fibers of a Lefschetz or a Branched Lefschetz Pencil. Then $\mathcal{F}$ is a C^0-structurally stable foliation.*

Finally, we relate our results with universal families of foliations:

Theorem C. *Let M be a projective manifold whose complex dimension is at least 3 and with $H^1(M,\mathbb{C}) = 0$. Let $\mathcal{B}(c)$ be an irreducible component of $Fol(M,c)$ that contains a foliation which has a generic meromorphic first integral. Then there exists a Zariski dense open subset of $\mathcal{B}(c)$ parameterizing the C^0-structurally stable foliations; all of them are topologically equivalent, and have a generic rational first integral.*

In [M], Muciño analyses the tangent space of the space of foliations on a Lefschetz pencil. From these infinitesimal methods, he gives an

independent proof of part (1) of the Gómez-Mont-Lins theorem.

1. Codimension one foliations

A codimension one holomorphic foliation with singularities on a complex manifold M may be given by a family of 1-forms ω_α defined on an open cover $\{U_\alpha\}$ of M, satisfying the Frobenius integrability condition $\omega_\alpha \wedge d\omega_\alpha = 0$, and the cocycle condition $\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta$ in $U_\alpha \cap U_\beta$, where $\lambda_{\alpha\beta}$ are non-vanishing holomorphic functions. If L denotes the holomorphic line bundle on M obtained with the cocycles $\{\lambda_{\alpha\beta}\}$, then the 1-forms glue to give a holomorphic section of the bundle $T^*M \otimes L$.

1.1 Definition

A *codimension one holomorphic foliation* $\mathcal{F}$ with singularities in the complex manifold M, is an equivalence class of sections $\omega \in H^0(M,\Omega^1(L))$ where L is a holomorphic line bundle, such that ω does not vanish identically on any connected component of M and satisfies the integrability condition $\omega \wedge d\omega = 0$. The *singular set* of the foliation $\mathcal{F}$ is the set of points $S(\mathcal{F}) = \{p \in M | \omega(p) = 0\}$. The *leaves of the foliation* are the leaves of the non-singular foliation in $M - S(\mathcal{F})$.

When a leaf $\mathcal{L}$ of $\mathcal{F}$ is such that its closure $\overline{\mathcal{L}}$ is a closed analytic subspace of M of codimension 1, we will also call $\overline{\mathcal{L}}$ a leaf of $\mathcal{F}$.

We will use the following notions:

A *holomorphic family* $\{\mathcal{F}_t\}_{t\in T}$ of *codimension one holomorphic foliations* with singularities parameterized by a complex analytic space T consists of the following:

(1) A holomorphic family of complex manifolds $\{M_t\}$, given as a smooth map $\pi\colon\mathcal{M} \to T$ between complex spaces with $\pi^{-1}(t) = M_t$.

(2) A holomorphic foliation with singularities $\tilde{\mathcal{F}}$ on $\mathcal{M}$ such that its leaves are contained in the t-fibers and the restriction $\tilde{\mathcal{F}}|_{M_t} = \mathcal{F}_t$ is a codimension one holomorphic foliation with singularities on M_t.

A foliation $\mathcal{F}_1$ is a *deformation* of the foliation $\mathcal{F}_2$ if there exists a family of foliations $\{\mathcal{F}_t\}_{t\in T}$ such that $\mathcal{F}_1 = \mathcal{F}_t$ and $\mathcal{F}_2 = \mathcal{F}_s$, where t, $s \in T$.

Given a family of foliations $\{\mathcal{F}_t\}$, the *perturbed holonomy* of a leaf $\mathcal{L}$ of the foliation $\mathcal{F}_0$ is the holonomy of $\mathcal{L}$ as a leaf of the foliation $\tilde{\mathcal{F}}$. It is clear that the perturbed holonomy has the form:

$$H_\alpha(t,z) = (h_\alpha(t,z), t),$$

where h_α is a holomorphic function such that $h_\alpha(0,z)$ is the holonomy map associated to $\alpha \in \pi_1(\mathcal{L})$ as a leaf of the foliation $\mathcal{F}_0$.

We will assume that M is compact and has complex dimension ≥ 3. In this case, the set $Fol(M,c)$ of those foliations defined by an equivalence class of sections $w \in H^0(M, \Omega^1(L))$ where L is a line bundle with Chern class $c(L) = c$, is an algebraic variety [**GM-M**] p. 133.

1.2 Definition

Consider $\omega \in H^0(M, \Omega^1(L))$. A section $\varphi \in H^0(M, \mathcal{O}(L))$ is say to be an *integrating factor* of ω if the meromorphic one form

$$\Omega := \frac{\omega}{\varphi}$$

is closed.

The following result shows, that if a section $\omega \in H^0(M, \Omega^1(L))$ has an integrating factor, then it is integrable.

1.3 Theorem. *Let M be a projective manifold with $H^1(M,\mathbb{C}) = 0$, and let $\varphi = \varphi_1^{r_1} \cdots \varphi_k^{r_k} \in H^0(M, \mathcal{O}(L))$, $r_i \in \mathbb{N}$ be an integrating factor of the foliation represented by ω. Then:*

$$\Omega = \frac{\omega}{\varphi} = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d(\Psi)$$

where Ψ is a meromorphic function with poles at the divisor

$$\sum_{i=1}^{k} l_i\{\varphi_i = 0\} \; l_i \leq r_i - 1, \lambda_i \in \mathbb{C}.$$

Proof. We follow [**C-M**] p. 38.

Consider the meromorphic 1-form

$$\Omega_1 = \Omega - \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} \quad \text{where} \quad \lambda_i = \frac{1}{2\pi i}\int_{\gamma_i} \Omega,$$

the loop $\gamma_i \in \pi_i(M - \{\varphi_i = 0\})$ denotes the generator of the kernel of the map $i_* = \pi_1(M - \{\varphi_i = 0\}) \to \pi_1(M)$, where $i\colon M - \{\varphi_i = 0\} \hookrightarrow M$ is the inclusion map.

Integrating by paths the meromorphic 1-form Ω_1, gives a representation

$$H_1(M,\mathbb{Z}) \to \mathbb{C}$$
$$[\gamma] \mapsto \int_\gamma \Omega_1.$$

Now, since $H^1(M;\mathbb{C}) = 0$, this representation defines a holomorphic map

$$H\colon M - \{\varphi = 0\} \to \mathbb{C}.$$

We claim that this map extends meromorphically to M with a pole of order smaller or equal to $r_i - 1$ along the divisor $\{\varphi_i = 0\}$.

To prove this assertion, take a coordinate system $(z;w)$ on a neighborhood of a smooth point $x \in \{\varphi_i = 0\}$, such that $\{z = 0\} = \{\varphi_i = 0\}$ and span the function H as a Laurent series in the variable z.

$$H(z;w) = \sum_{n=-\infty}^{\infty} a_n(w) \cdot z^n,$$

then we have:

$$dH(z;w) = \sum_{n=-\infty}^{\infty} z^n \cdot da_n(w) + (n-1)z^{n-1} \cdot a_n(w)dz$$

$$z^{r_i}dH(z;w) = \sum_{n=-\infty}^{\infty} z^{n+r_i} \cdot da_n(w) + (n-1)z^{r_i+n-1} \cdot a_n(w)dz.$$

Since $z^{r_i} \cdot dH$ is holomorphic, we have that $a_n(w)$ vanishes identically whenever $n \leq -(r_i - 1)$. □

Remark. With the notation above, by the residue theorem we have that

$$\sum_{i=1}^{k} \lambda_i \cdot c_1[\{\varphi_i = 0\}] = 0 \in H^2(M;\mathbb{C}),$$

where $c_1[\{\varphi_i = 0\}]$ denotes the Chern class of the line bundle associated to the divisor $\{\varphi_i = 0\}$.

1.4 Corollary. *Let M be a projective manifold with $H^1(M;\mathbb{C}) = 0$, and let $\mathcal{F}$ be a foliation represented by a section $\omega \in H^0(M;\Omega^1(L))$ and having an integrating factor φ. Then $\mathcal{F}$ has at least a compact leaf.*

Proof. Let $\varphi = \varphi_1^{r_1} \cdots \varphi_k^{r_k}$ be an integrating factor, then the theorem above shows that the hypersurfaces defined by the equations $\{\varphi_i = 0\}$ are invariant by the foliation, and hence, they are compact leaves of the foliation represented by the section ω. □

As a final comment, when a foliation has two linearly independent integrating factors φ_1 and φ_2, it is not difficult to show that the foliation has the meromorphic first integral given by

$$\frac{\varphi_1}{\varphi_2} \colon M \to \mathbb{P}^1.$$

2. Kupka type singularities

This section is dedicated to describe the singular set of foliations with a generic meromorphic first integral.

2.1 Definition

Let $\mathcal{F}$ be a codimension-one holomorphic foliation with singularities represented by $\omega \in H^0(M, \Omega^1(L))$. The *Kupka singular set* denoted by $K(\mathcal{F}) \subset S(\mathcal{F})$ is defined by:

$$K(\mathcal{F}) = \{p \in M | \omega(p) = 0 \quad d\omega(p) \neq 0\}.$$

The local structure of the Kupka singular set is described by the following result. The proof may be found in [**Me**].

2.2 Theorem. *Let ω and $K(\mathcal{F})$ as above, then:*

(1) *$K(\mathcal{F})$ is a codimension two locally closed submanifold of M.*

(2) *For every connected component $K \subset K(\mathcal{F})$ there exist a holomorphic 1-form*

$$\eta = A(x,y)dx + B(x,y)dy$$

defined in a neighborhood V of $0 \in \mathbb{C}^2$ vanishing only at 0, an open covering $\{U_\alpha\}$ of a neighborhood of K in M and a family of submersions $\varphi_\alpha \colon U_\alpha \to \mathbb{C}^2$ such that $\varphi_\alpha^{-1}(0) = K \cap U_\alpha$ and $\omega_\alpha = \varphi_\alpha^* \eta$ defines $\mathcal{F}$ in U_α.

(3) *$K(\mathcal{F})$ is persistent under variations of $\mathcal{F}$; namely, for $p \in K(\mathcal{F})$ with defining 1-form $\varphi^*\eta$ as above, and for any foliation $\mathcal{F}^*$ sufficiently close to $\mathcal{F}$, there is a holomorphic 1-form η' defined on a neighborhood of $0 \in \mathbb{C}^2$ and a submersion φ' close to φ such that $\mathcal{F}'$ is defined by $(\varphi')^*\eta'$ on a neighborhood of p.*

Remark. The germ at $0 \in \mathbb{C}^2$ of η is well defined up to biholomorphism and multiplication by non-vanishing holomorphic functions. We will call it the *transversal type* of $\mathcal{F}$ at K. Let X be the dual vector field of η, since $d\omega \neq 0$, we have that $\operatorname{Div} X(0) \neq 0$, thus the linear part $D = DX(0)$ is well defined up to linear conjugation and multiplication by scalars. We will say that D is the *linear type* of K. Since $trD \neq 0$, it has at least one non-zero eigenvalue. Normalizing, we may assume that the eigenvalues are 1 and μ. We will distinguish three possible types of Kupka type singularities:

(a) Degenerate: If $\mu = 0$.

(b) Semisimple: If $\mu \neq 0$ and D is semisimple.

(c) Non-semisimple, $\mu = 1$ and D is not semisimple.

The topological properties of the embedding $K(\mathcal{F}) \hookrightarrow M$, which can be measure in terms of the normal bundle ν_K of $K \subset M$, and the transversal type, are strongly related. [**GM-L**] p. 320-324.

2.3 Theorem. *Let K be a compact connected component of $K(\mathcal{F})$ such that the first Chern class of the normal bundle of K in M is non-zero, then the linear transversal type is non-degenerate, semisimple with eigenvalues $\mu \in \mathbb{Q}$ and 1. Furthermore, if $0 < \mu$, then the transversal type is linearizable, and for any deformation $\{\mathcal{F}_t\}$ of $\mathcal{F} = \mathcal{F}_0$, the transversal type is constant through the deformation.*

Let f_1 and f_2 be holomorphic sections of the positive line bundles L_1 and L_2 respectively. Assume that the line bundles satisfy the relation $L_1^{\otimes p} = L_2^{\otimes q}$ for some p, q relatively prime positive integers. Also suppose that the hypersurfaces $\{f_1 = 0\}$ and $\{f_2 = 0\}$ are smooth, and meet transversely. The integrable holomorphic section of the bundle $T^*M \otimes$

$L_1 \otimes L_2$ given by

$$\omega = pf_2 df_1 - qf_1 df_2,$$

has the meromorphic first integral $\phi = f_1^p/f_2^q$. Any point $x \in \{f_1 = 0\} \cap \{f_2 = 0\}$ belongs to the Kupka set

$$\omega(x) = 0 \quad \text{and} \quad d\omega(x) = -(p+q)(df_1 \wedge df_2)(x) \neq 0.$$

In this case, $K(\omega) = \{f_1 = 0\} \cap \{f_2 = 0\}$, and the normal bundle is $\nu_K = (L_1 \otimes L_2)|_K$, thus it has non-vanishing first Chern class. Then, by Theorem 2.3, the transversal type is linearizable, and actually, it is given by the 1-form $\eta = pxdy - qydx$, moreover, it remains constant under deformations.

3. Proof of theorem A

In this section, we shall prove our main result:

Theorem A. *Let M be a projective manifold whose complex dimension is at least 3 and with $H^1(M,\mathbb{C}) = 0$. Let $\omega = pf_2df_1 - qf_1df_2$ with $p \neq q$ where f_i are holomorphic sections of the positive line bundles L_i and $L_1^p = L_2^q$. If $\{f_i = 0\}_{i=1,2}$ are smooth irreducible and meet transversally, then any deformation ω' of the foliation represented by ω has a meromorphic first integral $\phi' = f_1'^p/f_2'^q$ where $f_i' \in H^0(M, \mathcal{O}(L_i))$.*

Let $w = pf_2df_1 - qf_1df_2$ be as in theorem A. Observe that $f_1 \cdot f_2$ is an integrating factor for ω. Conversely, if $q/p,\ p/q \notin \mathbb{N}$ and ω' is an integrable section close to ω, we will show that the leaves $\{f_i = 0\}_{i=1,2}$ have non-trivial holonomy and are stable under deformations, namely, there are sections $f_i' \in H^0(M, \mathcal{O}(L_i)) i = 1,2$ such that $\{f_i' = 0\}_{i=1,2}$ are compact leaves of the foliation ω'. We will show that $f_1' \cdot f_2'$ is an integrating factor for ω' and the conclusion will follow from Theorem 1.3.

The following theorem may be found in [**C**].

3.1 Theorem. *Let M be a smooth projective manifold of complex dimension ≥ 3. If $\omega = pf_2df_1 - qf_1df_2 \in Fol(M, L_1 \otimes L_2)$ is a section as in theorem A, then at least one of the leaves $\{f_i = 0\}$ is stable under deformations.*

Proof. Let ω_t be a family of foliations such that $\omega_0 = \omega$. The idea is to find a fixed point of the perturbed holonomy. In order to do this, we will find a central element which has non-trivial linear holonomy.

Let V be a smooth algebraic manifold of complex dimension ≥ 2, and let $W \subset V$ be a smooth, positive divisor on V. Recall that, if $i \colon V - W \hookrightarrow V$ denotes the inclusion map, then the generator γ_W of the kernel of $i_* \colon \pi_1(V - W) \to \pi_1(V)$ is central in $\pi_1(V - W)$ (see [**N**] p. 315-316).

Since $K \subset \{f_i = 0\}$ $i = 1,2$ is a positive divisor, the loop $\gamma^i := \gamma_K^i \in \pi_1(\{f_i = 0\} - K)$ $i = 1,2$ is central, so, we must consider two cases:

a) If $p/q \notin \mathbb{N}$ and $q/p \notin \mathbb{N}$, then both leaves $\{f_i = 0\}_{i=1,2}$ are stable.

The perturbed holonomy of the element $\gamma_K^i \in \pi_1(\{f_i = 0\} - K)$, has the form:

$$H_{\gamma_K^1}(y,t) = (h_{\gamma_K^1}(y,t), t) \qquad \frac{\partial h_{\gamma_K^1}}{\partial y}(0,0) = \exp\left(2\pi i \frac{q}{p}\right) \neq 1$$
$$H_{\gamma_K^2}(y,t) = (h_{\gamma_K^2}(y,t), t) \qquad \frac{\partial h_{\gamma_K^2}}{\partial y}(0,0) = \exp\left(2\pi i \frac{p}{q}\right) \neq 1.$$

By the implicit function theorem, there are germs of analytic functions $t \mapsto p_t^i$ $i = 1,2$ such that

$$h_{\gamma_K^i}(p_t^i, t) = p_t^i \ i = 1,2.$$

Now, because $\gamma_K^i \in \pi_1(\{f_i = 0\} - K)$ is central, the unique fixed point (p_t^i, t), is fixed for any other element

$$\beta \in \pi_1(\{f_i = 0\} - K),$$

that is, $h_\beta(p_t^i, t) = p_t^i$, hence, it is fixed by the perturbed holonomy of the leaf $\{f_i = 0\}$.

By theorem 2.3, the transversal type of the Kupka set, remains constant through the deformation, and it is given by $\eta = pxdy - qydx$ [**GM-L**], thus, the leaves $\mathcal{L}_t^i$ through the points (p_t^i, t) contain the smooth separatrices of the Kupka set, and $\overline{\mathcal{L}_t^i}$ is a compact leaf of the foliation ω_t.

b) If $1 = p < q$, then the above argument, may be applied only to the leaf $\{f_2 = 0\}$. □

Remark. Observe that the fundamental class of the compact leaf $\{f_{it} = 0\}$ remains constant through the deformation.

We will use the following facts on holomorphic line bundles over Kähler manifolds:

It is well known, **[G-H]** p. 313, that a holomorphic line bundle L has Chern class zero, if there exists an open covering of M such that the transition functions are constants, and such line bundle is trivial, when it has a non-zero holomorphic section.

On the other hand, if $H^1(M,\mathbb{C}) = 0$, the Hodge decomposition theorem, **[G-H]** p. 116, implies that any holomorphic line bundle L over M is classified by its Chern class, hence, under this hypothesis, a *holomorphic line bundle L is holomorphically trivial if and only if* $c_1(L) = 0 \in H^2(M;\mathbb{Z})$.

3.2 Lemma. *Let $L_i i = 1,2$ be positive holomorphic line bundles, and let $\{f_i\}$ and ω be as in theorem (3.1). If $H^1(M,\mathbb{C}) = 0$, then any deformation of the foliation ω has an integrating factor.*

Proof. Let ω_t be an analytic family of foliations with $\omega_0 = (pf_2df_1 - qf_1df_2) \in Fol(M, L_1 \otimes L_2)$.

We will consider two cases:

(1) If $p/q \notin \{1,2,3,\dots,1/2,1/3,\dots\}$.

In this case, the leaves $\{f_1 = 0\}$ and $\{f_2 = 0\}$ are stable, thus there exists an analytic family of sections $f_{it} \in H^0(M, \mathcal{O}(L_i)) i = 1,2$ such that $\{f_{it} = 0\}_{i=1,2}$ are compact leaves of the foliation represented by ω_t.

We claim that the product $f_{1t} \cdot f_{2t} \in H^0(M, \mathcal{O}(L_1 \otimes L_2))$ is an integrating factor of the section ω_t. In order to show this, it is only necessary to prove that the meromorphic 1-form

$$\Omega_t = \frac{\omega_t}{f_{1t} \cdot f_{2t}}$$

is closed on a nonempty open subset of M.

By Theorem 2.3, the transversal type of the Kupka set is constant through the deformation, thus, on a neighborhood of any point of the Kupka set, there exists a never vanishing holomorphic function $\rho_{\alpha t} \in$

$\mathcal{O}^*(U_\alpha)$, such that the meromorphic 1-form Ω_t has the local expression:

$$\rho_{\alpha t} \cdot \Omega_t|_{U_\alpha} = p\,\frac{dx_{\alpha t}}{x_{\alpha t}} - \frac{dy_{\alpha t}}{y_{\alpha t}} = \eta_{\alpha t},$$

and this equality holds, because $\{x_{\alpha t} = 0\} = \{f_{1t} = 0\} \cap U_\alpha$ and $\{y_{\alpha t} = 0\} = \{f_{2t} = 0\} \cap U_\alpha$, and both forms have the same pole (with multiplicity). Now, in **[C-L]** it is shown that, when $U_\alpha \cap U_\beta \neq \varnothing$, the meromorphic 1-forms $\eta_{\alpha t}$ glue to a meromorphic 1-form η_t defined on the open set $U = \bigcup_\alpha U_\alpha$, and this implies that the family of functions $\rho_{\alpha t}$ define a never vanishing holomorphic function on $\rho_t \in \mathcal{O}(U)$.

On the other hand, the Kupka set K_t, of the foliation represented by the section ω_t, is the transversal intersection of the positive divisors

$$K_t = \{f_{1t} = 0\} \cap \{f_{2t} = 0\},$$

this implies, as was pointed in **[C-L]**, that the function ρ_t may be extended to all M, hence it is a constant, and then, the meromorphic 1-form Ω_t is closed on the neighborhood of the Kupka set $K_1 \subset U = \bigcup_\alpha U_\alpha$, this implies that the meromorphic form Ω_t is closed, hence, $f_{1t} \cdot f_{2t}$ is an integrating factor for the foliation ω_t.

(2) Assume that $p = 1 < q$. In this case, $\{f_2 = 0\}$ is a priori the unique stable compact leaf, thus there exists an analytic family $f_{2t} \in H^0(M, \mathcal{O}(L_2))$ such that $\{f_{2t} = 0\}$ is a leaf of the foliation represented by ω_t.

We claim that f_{2t}^{q+1} is an integrating factor for ω_t, as in the above case, we will show that

$$\Omega_t = \frac{\omega_t}{f_{2t}^{q+1}}$$

is a meromorphic 1-form and it is closed on an open subset of M.

Now, in **[C-S]**, it is shown that there exists a rank two holomorphic vector bundle E, with a holomorphic section σ, vanishing precisely on the Kupka set, and $\wedge^2 E = L$.

The main point that we will use here, is that K is the transversal intersection of two positive divisors. This holds if and only if, the vector bundle E splits in a direct sum of positive line bundles.

The total Chern class of the bundle E, is computed in [**C-S**], and it is given by the formulae:

$$c(E) = 1 + c_1(L) + [K],$$

where $[K]$ denotes the fundamental class of K in M.

Now, in our case, we have that $K_t \subset \{f_{2t} = 0\}$, and $\{f_{2t} = 0\}$ is a positive divisor, this implies that there exists the following exact sequence of holomorphic vector bundles:

$$0 \to L_2 \to E \to L_1 \to 0$$

$$L_2 = [\{f_{2t} = 0\}] \; L_1 = E/L_2$$

such exact sequences are classified by the cohomology group

$$H^1(M, \mathcal{O}(L_2 \otimes L_1^{-1})).$$

Now, we have the following relations on the Chern classes:

$$c_1(L) = c_1(L_2) + c_1(L_1) = (q+1) \cdot c_1(L_2) \Rightarrow c_1(L_1) = q \cdot c_1(L_2),$$

hence the holomorphic line bundle $L_2 \otimes L_1^{-1}$ has negative Chern class, and by Kodaira vanishing theorem, we have that

$$H^1(M, \mathcal{O}(L_2 \otimes L_1^{-1})) = 0,$$

hence, the vector bundle E splits in a direct sum of positive line bundles, and K_t is a complete intersection of positive divisors.

As in the first case, on a neighborhood of any point of the Kupka set K_t of ω_t we have

$$\rho_{\alpha t} \cdot \Omega_t|_{U_\alpha} = \frac{\omega_t}{f_{2,t}^{q+1}}|_{U_\alpha} = \frac{1}{y_{\alpha t}^{q+1}} \left(y_{\alpha t} dx_{\alpha t} - q x_{\alpha t} dy_{\alpha t}\right) = d\left(\frac{x_{\alpha t}}{y_{\alpha t}^q}\right),$$

but in this case, we have that

$$\frac{1}{y_{\alpha t}^{q+1}} \left(y_{\alpha t} dx_{\alpha t} - q x_{\alpha t} dy_{\alpha t}\right) = c_{\alpha\beta} \frac{1}{y_{\beta t}^{q+1}} \left(y_{\beta t} dx_{\beta t} - q x_{\beta t} dy_{\beta t}\right) \quad c_{\alpha\beta} \in \mathbb{C}^*,$$

hence, the functions $\rho_{\alpha t}$ defines a holomorphic section on a neighborhood of K_t of a line bundle with Chern class zero, which is the trivial line bundle, by the assumption on $H^1(M, \mathbb{C}) = 0$.

By the same argument as above, this section may be extended to M, thus it defines a non-zero holomorphic function, and the meromorphic

1-form $\omega_t / f_{2,t}^{q+1}$ is closed, this implies that $f_{2,t}^{q+1}$ is an integrating factor as claimed. □

Now, we are in a position to complete the proof of Theorem A:

Proof of theorem A. We are going to consider two cases:

(1) $\omega = p f_2 df_1 - q f_1 df_2 \; 1 < q < q$:

By Lemma 3.2, $f_{1t} \cdot f_{2t}$ is an integrating factor for ω_t, and by Theorem 1.3, we have shown that:

$$\frac{\omega_t}{f_{1t} f_{2t}} = p\frac{df_{1t}}{f_{1t}} - q\frac{df_{2t}}{f_{2t}}.$$

This implies that

$$\omega_t = p f_{2t} df_{1t} - q f_{1t} df_{2t},$$

and ω_t has the meromorphic first integral $\phi_t = f_{1t}^p / f_{2t}^q$.

(2) $\omega = f_2 df_1 - q f_1 df_2$:

By Lemma 3.2, f_{2t}^{q+1} is an integrating factor for ω_t, and again by theorem 1.3, we have:

$$\frac{\omega_t}{f_{2t}^{q+1}} = \lambda \frac{df_{2t}}{f_{2t}} + d\left(\frac{f_{1t}}{f_{2t}^q}\right)$$

where

$$f_{1t} \in H^0(M, \mathcal{O}(L_2^q)) = H^0(M, \mathcal{O}(L_1)).$$

Since the divisor $\{f_2 = 0\}$ is positive, we have that $\lambda = 0$, thus we get:

$$\omega_t = f_{2t}^{q+1} d\left(\frac{f_{1t}}{f_{2t}^q}\right) = f_{2t} df_{1t} - q f_{1t} df_{2t}.$$

This finish the proof. □

Remarks.

(1) Cerveau and Lins have shown in [**C-L**], that a codimension one foliation whose singular set has a compact connected component of the Kupka set, which is a complete intersection (i.e. the transversal intersection of two positive divisors), has a meromorphic first integral.

The stability of the leaves of with non-trivial holonomy, implies that after a deformation, the Kupka set is a complete intersection.

(2) If we begin with a unbranched rational function (that is, $L_1 = L_2$) and we consider deformations keeping one leaf stable, then it is possible to find an integrating factor.

(3) If $H^1(M, \mathbb{C}) \neq 0$, then theorem A is not true as the following example shows:

Let θ_t be an analytic curve of closed holomorphic 1-forms with $\theta_0 = 0$. Consider the family of foliations

$$\omega_t = f_{1t} f_{2t} \left(p \frac{df_{1t}}{f_{1t}} - q \frac{df_{2t}}{f_{2t}} + \theta_t \right);$$

Where $f_{it} \in H^0(M, \mathcal{O}(L_i))$ $i = 1, 2$. The foliation represented by the section $\omega_t t \neq 0$ has only two compact leaves.

(4) Assume that $H^1(M, \mathbb{C}) \neq 0$ and $\omega = f_2 df_1 - q f_1 df_2$ where $q > 1$ and the foliation ω satisfying the hypotheses of theorem 3.1.

Let as above, θ_t an analytic curve of closed holomorphic 1-forms, and consider the following family of foliations:

$$\omega_t := f_2^{q+1} \left(d \left(\frac{f_1}{f_2^q} \right) + \theta_t \right).$$

In this case, the foliation ω_t has only one compact leaf, given by $\{f_2 = 0\}$.

4. Universal families

In this section, we will describe irreducible components of the universal families of foliations of codimension 1.

Let M be a projective manifold with $H^1(M, \mathbb{C}) = 0$, we have seen that every holomorphic line bundle on M is determined by its Chern class $c_1(L) \in H^2(M, \mathbb{Z})$. It may be shown that $\cup_c Fol(M, c)$ parameterizes the universal family of foliations of codimension 1 in M [**GM**].

4.1 Definition

The fibers $\{\varphi^{-1}(c)\}$ of a rational map $\varphi: M \to \mathbb{P}^1$ defined on a connected projective manifold M form a *Branched Lefschetz Pencil* if there are global sections f_i of positive line bundles L_i, $i = 1, 2$ with $L_1^p = L_2^q, p, q > 0$ such that:

(1) The subvarieties $\{f_i = 0\}_{i=1,2}$ are smooth and meets transversely in a smooth manifold K called the *center of the Pencil.*

(2) The subvarieties defined by $\lambda f_1^p + \mu f_2^q = 0$ with $\lambda\mu \neq 0$ are smooth on $M - K$ except for a finite set of points $\{(\lambda_i : \mu_i)\}_{i=1,\dots,k}$ where it has just a Morse type singularity over the critical value in $M - K$.

Note that branched Lefschetz pencils are dense in the set of meromorphic maps satisfying only condition (1), thus as a consequence of theorem A we have:

Theorem B. *Let M be a projective manifold whose complex dimension is at least 3 and with $H^1(M, \mathbb{C}) = 0$. Let $\mathcal{F}$ be a codimension one holomorphic foliation with singularities arising from the fibers of a Lefschetz or a Branched Lefschetz Pencil. Then $\mathcal{F}$ is a $\mathcal{C}^0$-structurally stable foliation.*

Proof. We have two cases:

(1) If the meromorphic first integral is a Lefschetz Pencil, it is just part (a) of the Gómez-Mont Lins theorem [**GM-L**].

(2) If the meromorphic first integral is a branched Lefschetz Pencil, by Theorem A, we have that any deformation of a branched Lefschetz pencil has a meromorphic first integral, this implies in particular, that the Kupka set is locally structurally stable, and we can repeat the proof of part (2) of the Gómez-Mont Lins theorem given in [**GM-L**]. □

Theorem C. *Let M be a projective manifold of complex dimension at least 3 and with $H^1(M, \mathbb{C}) = 0$. Let $\mathcal{B}(c)$ be an irreducible component of $Fol(M, c)$ that contains a branched Lefschetz Pencil; then there exists a Zariski dense open subset of $\mathcal{B}(c)$ parameterizing $\mathcal{C}^0$-structurally stable foliations, all of them topologically equivalent and branched Lefschetz pencils.*

Proof. Let L_i be positive line bundless with chern classes $c(L_i) = c_i$ such that $L_1 \otimes L_2 = L$ where $c(L) = c = c_1 + c_2$.

Consider the map

$$\Phi: \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to Fol(M, c_1 + c_2)$$
$$([f_1], [f_2]) \mapsto p f_1 df_2 - q f_2 df_1$$

where $n_i = \dim_{\mathbb{C}} H^0(M, \mathcal{O}(L_i)) - 1$, $i = 1, 2$.

This is a well defined algebraic map. Let $\mathcal{W}$ be the Zariski closure of the image of Φ. We claim that $\mathcal{W}$ is an irreducible component of $Fol(M, c)$, where $c = c_1 + c_2$. We know that any deformation of a branched Lefschetz pencil is again a branched pencil and then, it is in the image of Φ. This show that $\mathcal{W}$ and $Fol(M, c)$ coincide in a neighborhood of a foliation $\mathcal{F}_0$. Since $\mathcal{W}$ is irreducible, then it is an irreducible component of $Fol(M, c)$. Hence $\mathcal{W} = \mathcal{B}(c)$. This proves the theorem. □

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