Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 26, N. 1, 85-116

A Poincaré-Bendixson Theorem for Analytic Families of Vector Fields

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Abstract. We provide a characterization of the limit periodic sets for analytic families of vector fields under the hypothesis that the first jet is non-vanishing at any singular point. Also, applying the family desingularization method, we reduce the complexity of some of these sets.

1. Introduction

Let S be an open subset of $\mathbf{S}^2$ such that $\bar{S}$ (i.e. the closure of S) is a surface with possible corners. Here, when we write that X_λ is an *analytic family in* S, it will mean that X_λ is a family of vector fields defined in some open neighborhood of $\bar{S}$ in $\mathbf{S}^2$ and analytic on it. Also, we suppose that X_λ depends analytically on the parameter $\lambda \in P$, where P is an analytic manifold of finite dimension Λ.

Particularly interesting examples are generic local families, where S is taken to be a neighborhood of the origin in $\mathbb{R}^2$ and P, a neighborhood of the origin in $\mathbb{R}^\Lambda$. Also, in relation to the 16^{th} Hilbert's problem, for every natural number n we consider the analytic family X_λ^n, defined for $S = \mathbf{S}^2$ and $P = \mathbf{S}^{n^2+3n+1}$, which is obtained by compactification of the family of all the polynomial vector fields in $\mathbb{R}^2$ of degree less than or equal to n. Note that this family is non-generic.

If $K \subset S$ is some compact subset invariant by the flow of X_{λ_0}, for some $\lambda_0 \in P$, we will denote by (X_λ, K) the germ of X_λ along $K \times \{\lambda_0\} \subset S \times P$. We will also say that (X_λ, K) is an *analytic unfolding of* K.

Received 24 October 1994.

One of the most interesting questions about analytic families concern the configuration (i.e. the number and spatial disposition) of the limit cycles when the parameter varies. To study this, we have to look primarily to the degenerate dynamical structures inside the family, where possibly new limit cycles are originated when the parameter is slightly varied. There is a close analogy between this question and the analysis of the possible ω-limit sets for a single vector field, made in the Poincaré-Bendixson Theorem. First, we have to define an analogue to the notion of ω-set, in the context of analytic families:

Definition 1. *A compact subset $\Omega \subset S$, is a limit periodic set (simply a l.p.s.) of the family X_λ at λ_0 if it is invariant by X_{λ_0} and there exists a sequence $\{\lambda_k\} \to \lambda_0$, each X_{λ_k} having a limit cycle $\gamma_k \subset S$, such that $\{\gamma_k\} \to \Omega$ by the usual Hausdorff metric on compact sets.*

If an l.p.s. Ω contains only isolated singular points of X_{λ_0}, it is known that Ω is a singular point, a periodic orbit or a *graphic* (i.e. a finite union of singular points $p_1, p_2, \ldots, p_{m+1} = p_1$ (not necessarily distinct) and regular orbits $s_1, s_2, \ldots, s_m$ connecting them). The proof of this fact, which uses the same idea as the proof of the Poincaré-Bendixson Theorem (see [**El.Mor - R**]) will be recalled in section 3 of this work.

Of course, the cases above are the only possible l.p.s. inside generic families. But we may also be interested by non generic families, which exhibit, for example, *non-isolated singularities* for some values of the parameter. For instance, this is the case for the family X_λ^n defined above.

Also, any *singular perturbation equation* can be transformed in an analytic family eventually having non isolated singular points. In fact, if the problem has the form

$$X_\varepsilon = \begin{cases} \dot{x} &= f(x,y,\varepsilon) \\ \varepsilon\dot{y} &= g(x,y,\varepsilon) \end{cases}$$

then, after a rescaling of time $\tau = \varepsilon t$ we obtain the new family

$$\hat{X}_\varepsilon = \begin{cases} \dot{x} &= \varepsilon f(x,y,\varepsilon) \\ \dot{y} &= g(x,y,\varepsilon). \end{cases}$$

in which the vector field $\hat{X}_0$ has a whole analytic set of singularities at $\{g(x,y,0) = 0\}$.

Another situation in which we eventually have to deal with non isolated singularities is when we perform a family rescaling, in the sense of [**T**], [**B**] (or the more general blow-up operation for families, in the sense of [**D-R**]) in some generic family. In these cases, we obtain new families which may degenerate on the exceptional divisor.

To proceed our study, we will impose a restrictive condition on the analytic vector fields to be studied. Given an analytic vector field X defined in S, we establish the following hypothesis:

Hypothesis (H). *For any $x \in S$ such that $X(x) = 0$ we have that* $j^1X(x) \neq 0$.

We will say that an analytic family obeys hypothesis (H) when it is verified for every value of the parameter.

As we will see in section 2, when a analytic vector field X satisfies (H) we can decompose the singularity set $\mathcal{Z}(X)$ in three disjoint finite subsets:

- The set of isolated singularities: which can be either hyperbolic, semi-hyperbolic, degenerate focus, centers or nilpotent singularities (these last ones occur when $j^1X(x_0)$ is linearly conjugated to $y\frac{\partial}{\partial x}$).
- Hyperbolic-nilpotent set: which is constituted of a finite number of connected 1-dimensional manifolds, each one of which contains a finite number isolated nilpotent singularities, whose complement are open arcs of semi-hyperbolic singularities.
- Nilpotent set: which is constituted of a finite number connected of 1-dimensional manifolds, each of which is composed entirely of nilpotent singularities.

Definition 2. *In the following, we will say shortly that a 1-manifold of singularities is an $H.N$-*curve *or an N-*curve *accordingly to the fact that it belongs to the second or to the third set above, respectively.*

We can now generalize the notion of graphic:

Definition 3. *A* degenerate graphic *is an invariant set formed by a finite union of isolated singular points, arcs of $H.N$-curves, and regular orbits* *connecting those elements.*

The section 3 is devoted to the study of the structure of l.p.s. appearing in analytic families under the hypothesis (H). We will prove the following:

Theorem 1. *Let Ω be a limit periodic set at λ_0 of an analytic family X_λ obeying hypothesis (H). Then, if X_{λ_0} has a curve of singularities Γ such that $\Omega \cap \Gamma \neq \varnothing$, there are two possible cases:*

i) *If Γ is a N-curve then $\Omega \subset \Gamma$,*

ii) *If Γ is a $H.N$-curve, then $\Omega \cap \Gamma$ is composed of a finite number of closed sub-arcs $\Omega_1, \Omega_2, \ldots, \Omega_k$ (not necessarily disjoint). Eventually, we can have $\Omega_j = \{p\}$, and in this case $p \in \Gamma$ is necessarily a nilpotent singularity.*

In the second case, Ω is a degenerate graphic.

Given a l.p.s. Ω, we can ask how many limit cycles it origins by bifurcation. This is precised by the following notion:

Definition 4. *We will say that a limit periodic set Ω has* finite cyclicity *if there is a neighborhood V of λ_0, and there exists $\varepsilon > 0$, $n \in N$ such that for each $\lambda \in V$ the number of limit cycles of the field X_λ at a Hausdorff distance smaller than ε of Ω, is less than n. The* cyclicity *of Ω is the infimum of such n as the diameter of V and ε go to 0.*

A general conjecture, stated in [**R1**], is

Finiteness conjecture. *For any analytic family X_λ with a compact manifold as parameter space, there exists a uniform bound for the number of limit cycles of any vector field in the family.*

Applied to the family X^n_λ, this conjecture would imply a positive answer to the 16^{th} Hilbert Problem. As it was noticed in that same article, this conjecture can be reduced by a compactness argument to the following:

Finite cyclicity conjecture. *In an analytic family X_λ, any l.p.s. has finite cyclicity.*

This conjecture is intuitively easier than the preceding, because of its local character.

In [**D-R**], it has been introduced a general desingularization method, aiming to reduce the question of finite cyclicity of general l.p.s. to the same question restricted to *elementary* l.p.s. (i.e. l.p.s. in which every point is either a regular point of the vector field or a singularity with at least a non-zero eigenvalue in the Jacobian). The method consists of a sequence of desingularization operations, the most important of which is the blowing-up along sub-manifolds of singularities.

Here, we will apply these ideas to the particular case of l.p.s. contained in N-curves. In fact, given a family X_λ with a N-curve Γ for X_{λ_0}, we will see that we can locally embed X_λ in a larger family $X_{\varphi,\psi,\lambda}$, defined in a neighborhood $V \subset S$ of Γ, in such a way that, after only a single blow-up along Γ (in the sense of [**D-R**]), we obtain analytic families with at most $H.N$-curves. This is the content of the second main result:

Theorem 2. *Any analytic unfolding (X_λ, Γ) of a N-curve can be desingularized (in the sense of [**D-R**]) in a collection of unfoldings $(\hat{X}_{\hat{\lambda}}, \hat{\Gamma})$, where $\hat{\Gamma}$ is an isolated singularity, a periodic orbit, a graphic or a degenerate graphic.*

Since the finite cyclicity is preserved by the operations of desingularizations, we have the following corollary

Corollary. *In order to prove the finite cyclicity conjecture for analytic families under condition (H), it suffices to prove the conjecture for families not containing N-curves.*

2. Structure of the singularity set

We will denote by $\mathcal{Z}(X)$ the singularity set of an analytic vector field X defined in some open neighborhood of $\bar{S}$. In this part, we will discuss the structure of the set $\mathcal{Z}(X)$ for vector fields satisfying hypothesis (H).

Let us write the field X in the form

$$X = \begin{cases} \dot{x} & = F(x,y) \\ \dot{y} & = G(x,y) \end{cases} \tag{1}$$

The first fact is the following:

Proposition 2.1. *Let X be as above. Then $\mathcal{Z}(X)$ is formed by a finite* *union of isolated singular points and connected 1-manifolds of singularities.*

Proof. We have that $\mathcal{Z}(X) = \{F = 0\} \cap \{G = 0\}$. The hypothesis (H) says that, in a neighborhood of each point of intersection, at least one of these two sets can be locally expressed as an analytic 1-manifold C (for this, it suffices to use the implicit function theorem).

If the intersection is an isolated point then we are done. Otherwise, we can see immediately that $\mathcal{Z}(X)$ coincides locally with C. So, at any non-isolated singularity of S, we can express $\mathcal{Z}(X)$ locally as an analytic 1-manifold.

In order to show that $\mathcal{Z}(X)$ is formed by a finite number of connected components we use the fact that X is analytic in $\bar{S}$. By this, the set of isolated singular points cannot accumulate at the boundary of S. Also, the connected 1-manifolds of singularities are topologically isolated in S. So, by the compactness of $\bar{S}$, $\mathcal{Z}(X)$ can have only a finite number of elements. □

Let us consider now one of the connected 1-manifolds of singularities, which we will denote by Γ. There are two possible cases: either Γ is entirely contained in S, and so is diffeomorphic to $\mathbf{S}^1$, or else it has its both ends on the boundary of S, and so it is diffeomorphic to an interval. In both cases, we can find an analytic parameterization

$$\xi : U \subset \mathbf{S}^1 \longrightarrow S$$

where U is a connected subset such that $\xi(U) = \Gamma$ and $\dot{\xi}(t) \neq 0$ for any $t \in U$. Now, we perform the following local change of coordinates

$$(x, y) = \Psi(t, s) = \xi(t) + s \cdot \dot{\xi}(t)^{\perp}$$

where $t \in U$ and $s \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. In fact, since Ψ is a local diffeomorphism when restricted to $U \times \{0\}$, and it is also bijective in this set, it is a global diffeomorphism of $W = U \times (-\varepsilon, \varepsilon)$ on its image, for a sufficiently small positive ε.

In these new coordinates, the field X has the form

$$X = \begin{cases} \dot{t} &= \bar{F}(t, s) \\ \dot{s} &= \bar{G}(t, s). \end{cases} \tag{2}$$

where $\bar{F}(t, 0) \equiv 0$ and $\bar{G}(t, 0) \equiv 0$. Using again the x and y to name the variables, we write finally

$$X = \begin{cases} \dot{x} &= y\, \tilde{F}(x, y) \\ \dot{y} &= y\, \tilde{G}(x, y). \end{cases} \tag{3}$$

where $\tilde{F}$ and $\tilde{G}$ are analytic functions at W. The line of singularities corresponds exactly to the set $\{y = 0\}$, and we can prove now that there are only two possible types of singularity curves.

Proposition 2.2. *If Γ is a connected 1-manifold of singularities then it is either a H.N-curve or a N-curve, according to definition 2.*

Proof. Using the form (3) we obtain

$$j^1 X(x, 0) = \begin{bmatrix} 0 & \tilde{F}(x, 0) \\ 0 & \tilde{G}(x, 0) \end{bmatrix} \tag{4}$$

so we have two possible situations:

- $\tilde{G}(x, 0) \equiv 0$: Then, by hypothesis (H), $\tilde{F}(x, 0) \neq 0$ for any $x \in U$, and Γ is a N-curve.
- $\tilde{G}(x, 0) \not\equiv 0$: So it has a finite number of isolated zeros in Γ. This points correspond exactly to the isolated nilpotent singularities inside the line of semi-hyperbolic singularities. So, Γ is a $H.N$-curve. □

Remark. It is possible to understand more geometrically the classification above. For this, we define a new vector field on W given by

$$\hat{X} = \begin{cases} \dot{x} &= \tilde{F}(x, y) \\ \dot{y} &= \tilde{G}(x, y). \end{cases} \tag{5}$$

with $\tilde{F}$ and $\tilde{G}$ as in (3). By hypothesis (H), $\hat{X}(x, y) \neq 0$ for $\forall (x, y) \in \Gamma$, and so we can identify Γ as a $H.N$-curve or a N-curve according to the *contact* of this curve with the field $\hat{X}$. Indeed, if $p \in S$ is a point where an orbit of $\hat{X}$ crosses Γ transversally, then p is a semi-hyperbolic singularity of X; if there is a higher order contact, p is a nilpotent singularity. If the intersection is non-transversal at every point of Γ, then, by analycity, the curve Γ coincides with a regular orbit of $\hat{X}$; and, in this case, it is a N-curve of X.

Still using the vector field $\hat{X}$ defined above, we can demonstrate the following transversality results:

Proposition 2.3. *Suppose that Γ is a N-curve of singularities of X, and Σ is an analytic curve in S which cuts Γ transversally at a point p. Then, there exists a neighborhood $W \subset S$ of p such that any regular orbit in this neighborhood cuts Σ transversally.*

Proof. Since Γ is a N-curve, the remark above gives us that Γ corresponds to a regular orbit of the field $\hat{X}$. So, by the continuity of the flow in a neighborhood of a regular point, the regular orbits of $\hat{X}$ are all transversal to Σ for a sufficiently small neighborhood of p. Now, we observe that the field X defines exactly the same foliation as the field $\hat{X}$ outside Γ. □

Proposition 2.4. *Suppose that Γ is a connected curve of semi-hyperbolic singularities. Then, there exist two disjoint analytic connected curves Σ^0 and Σ^1 contained in S such that:*

i) *The field X is transversal to Σ^0 and Σ^1;*

ii) *If γ is a regular orbit of X which accumulates at Γ, then it necessarily cuts Σ^0 or Σ^1.*

Proof. It suffices to take both the positive and the negative images of the curve Γ by the flow transformation $\hat{\phi}_t$ generated by $\hat{X}$. We take a time t_0 sufficiently small to have $\Sigma^0 = \hat{\phi}_{t_0}$ and $\Sigma^1 = \hat{\phi}_{-t_0}$ disjoint. Since the flow transformation $\hat{\phi}_{\pm t_0}$ is a diffeomorphism, and the field $\hat{X}$ is transversal to Γ, it is also transversal to $\hat{\phi}_{\pm t_0}(\Gamma)$. To finish, we use the fact that X defines the same foliation as $\hat{X}$ outside Γ. □

3. Structure of the limit periodic set

Consider an analytic family X_λ with a l.p.s. $\Omega \subset S$ at λ_0. In this part, we will study the dynamical structure of Ω.

In order to start, we will prove a basic lemma:

Lemma 3.1. *Let X_λ and Ω be as above. If Σ is a connected curve transversal to the field X_{λ_0}, then Ω cuts Σ in at most one point.*

Proof. Let us suppose, by contradiction, that Ω cuts Σ in more than one

point. We consider then a sub-arc $\Sigma_1 \subset \Sigma$ containing at least two points of intersection, and such that the X_{λ_0} does not vanish in $\bar{\Sigma}_1$. Then, by continuity, there exists a neighborhood $V \subset P$ of λ_0 such that for every $\lambda \in V$, X_λ is transversal to Σ_1. Since Ω is approximated by limit cycles, there exists a $\lambda_1 \in V$ such that X_{λ_1} has a limit cycle γ_1 which cuts Σ_1 more than one time.

Now, there is a natural ordering of the points in the intersection set $\Sigma_1 \cap \gamma_1$, which is given by the orientation of the limit cycle. We consider any two consecutive points p_1 and p_2 of this ordering. The union of the arc of orbit between p_1 and p_2 and the sub-arc of Σ_1 between these two points form a topological closed curve which separates S in two connected components. By the transversality at Σ_1, one of this components is invariant by the flow (either forward invariant or backward invariant). This obviously prevents the orbit which passes by p_2 to be connected with the orbit which passes by p_1, and contradicts the fact that both these points belong to a closed orbit. □

We proceed now to prove a general result about the structure of closed invariant sets of $C^r (r \geq 1)$ vector fields with isolated singularities. In the following, we will denote by $\omega(U)$ (resp. $\alpha(U)$) the union of the ω-limits (resp. α-limits) of all the elements of U.

Proposition 3.1. *Let X be a $C^r (r \geq 1)$ vector field defined in S, and having a finite number of singular points. Let $K \subset S$ be a compact, connected subset which is invariant by the flow, and which cuts every connected curve transversal to the flow in at most one point. Then, K is either:*

i) *an isolated singularity;*

ii) *a periodic orbit;*

iii) *a union of a finite number of singular points and regular orbits connecting them.*

Proof. Since K is a closed and invariant set, clearly $\omega(K) \subset K$. So, if K is a single point, it must be a singular point. Otherwise, by connectedness K contains regular points.

Let us suppose first that there exists a regular point $p \in K$ such that

$\omega(p)$ or $\alpha(p)$ *is not* a single point. Eventually by reversing the sense of the flow, we can suppose that $\omega(p)$ is not a single point.

Then, since $\omega(p)$ is connected, it contains a regular point $q \in S$. Since $\omega(K) \subset K$, we have also that $q \in K$. If we consider a small curve transversal to the orbit passing by q, we have that $\omega(p)$ cuts this curve exactly once, and so it must coincide with the regular orbit passing by q. This implies that p and q belongs to a periodic orbit γ. Now, by the same argument, K coincides with γ in a neighborhood of q. So, by connectedness, $K = \gamma$.

So, if K is not a periodic orbit nor a single point, every regular point in K has *a single point* as ω-limit and as α-limit. Since these points are invariant, they must be a singular points of the flow. As X have only a finite number of singular points, we obtain case (iii). □

Remark. If $K = \omega(x)$ for an arbitrary point $x \in S$, then we obtain the classical Poincaré-Bendixson Theorem. If K is a l.p.s., the conclusion of the proposition also holds due to Lemma 3.1.

Corollary 3.1. *If we suppose that X is an analytic vector field, then K forms a* graphic *in case (iii) (i.e. K contains only a finite number of regular orbits). Moreover, each regular orbit in K tends to a singular point in a well-defined direction.*

Proof. It is a consequence of the Seidenberg's desingularisation theorem that in a sufficiently small neighborhood of each singular point of X we have two possible behaviors:

i) We can decompose this neighborhood in a finite number of hyperbolic, parabolic and elliptic sectors.

ii) There exists a small closed curve containing the singular point which is everywhere transversal to the field. This is the case when there are no separatrices.

Let us suppose by contradiction that K contains an infinite number of regular orbits. Then, there exists a singular point $p \in K$ which is accumulated by an infinite number of regular orbits in K. By the local decomposition above, we can exclude the case (ii), since that would imply that K cuts the closed curve transversal to the field an infinite

number of times.

In case (i), we know that a regular orbit accumulating in p is either a separatrix of a hyperbolic sector; or it is a member of an elliptic sector or a parabolic sector. Since there exists only a finite number of hyperbolic separatrices and a finite number of sectors of each type, there exists an elliptic or a parabolic sector containing an infinite number of regular orbits of K. Now, it is easy to construct in each case a small curve transversal to the field which cuts K an infinite number of times. This contradicts the hypothesis. □

It is not difficult to provide examples of analytic families containing l.p.s. of each one of the types above. When Ω is a periodic orbit, it is well-known that any analytic unfolding of Ω has finite cyclicity (see, e.g. [F-P]). When Ω is a graphic, there are some partial results of finite cyclicity of the unfolding for elementary graphics (i.e. graphic in which all the singularities are hyperbolic or semi-hyperbolic) and nilpotent loops, under generic conditions (see, e.g. [**Mo**], [**R2**], [**I-Y**]).

Now, we will deal with the degenerate case. First, we will show that if Ω intersects a N-curve then necessarily Ω is contained in it:

Proposition 3.2. *Let us consider X_λ and Ω as defined above. Then, if Γ is a N-curve of X_{λ_0} such that $\Gamma \cap \Omega \neq \varnothing$, necessarily Ω is contained in Γ. More precisely, Ω is a closed connected sub-arc of Γ.*

Proof. The proof is by contradiction. Suppose that there is a decomposition $\Omega = (\Omega \cap \Gamma) \cup \Omega_1$, where $\Omega_1 \neq \varnothing$. Since Ω is connected, $\bar{\Omega}_1$ must intersect Γ. But we have seen that Γ is topologically isolated as a singular set, that is, there exists a neighborhood $W \subset S$ of Γ such that X_{λ_0} does not contains singularities in $W - \Gamma$. So, the set $\Omega_2 = \Omega_1 \cap W$ must be constituted entirely of regular orbits.

Let us consider a point $p \in \bar{\Omega}_2 \cap \Gamma$. By proposition 2.3, if we take a curve Σ transversal to Γ at p, every regular orbit sufficiently near p must also cut this curve transversally. So, as Ω_2 approaches p, it must cut Σ an infinite number of times. Since we have supposed that Ω_2 belongs to a limit periodic set, this contradicts lemma 3.1. □

Let us see an example of analytic family with a l.p.s. which is an entire closed sub-arc of nilpotent singularities inside a N-curve.

Example 3.1. Consider the family X_λ given by

$$X_\lambda = \begin{cases} \dot{x} = y \\ \dot{y} = \lambda^2(a+x^2) + \lambda y(b+x) \end{cases} \quad (2)$$

where $a, b \in \mathbb{R}$ are fixed values for which X_1 (i.e. the field X_λ for $\lambda = 1$) presents a limit cycle. It is easy to see that, as $\lambda \to 0$, this limit cycle vertically shrinks and accumulates to a closed segment of the N-curve $\Gamma = \{y = 0\}$ of X_0.

If Γ is a H.N-curve of X_{λ_0}, then there are several possibilities. The next result characterizes the possible intersections of Γ with a l.p.s. of the family X_λ.

Proposition 3.3. *Let X_λ and Ω be as above. Suppose that Γ is a H.N-curve of X_{λ_0} and $\Omega \cap \Gamma \neq \varnothing$. Then, there are two possible cases:*

i) $\Omega \not\subset \Gamma$: *In this case, $\Omega_\Gamma = \Omega \cap \Gamma$ is composed of a finite number of closed connected components, each one of them containing at least one nilpotent singularity.*

ii) $\Omega \subset \Gamma$: *Then, either*

ii.1) $\Omega = \{p\}$ *and p is nilpotent singularity, or*

ii.2) $\Omega = \Gamma$ *and Γ is diffeomorphic to $\mathbf{S}^1$;*

Proof. We will prove each item separately:

(i) First, let us suppose that $\Omega \not\subset \Gamma$. Then, if we take an isolated connected component $\tilde{\Omega} \subset \Omega_\Gamma$, necessarily there exists a neighborhood $V \subset S$ of $\tilde{\Omega}$ such that $R = (\Omega \cap V) - \tilde{\Omega}$ is a union of regular orbits which accumulates at $\tilde{\Omega}$. We assume, by contradiction, that $\tilde{\Omega}$ is composed entirely of semi-hyperbolic singularities, say, of the attractive type. So, every regular orbit of R has as ω-limit some point of $\tilde{\Omega}$.

Now, we use the general fact that, if a regular orbit belongs to a l.p.s., then all the limit cycles bifurcating from this l.p.s. must have the same orientation of this regular orbit (this is simply a consequence of the continuity of the family X_λ). In the case above, it is easy to see that the sense of the regular orbits in R prevents $\tilde{\Omega}$ from being accumulated by a limit cycle. In fact, since only a portion of the limit cycles approaches $\tilde{\Omega}$, the part of the limit cycles which goes out of a neighborhood of $\tilde{\Omega}$ should

follow the orientation of some regular orbit in R; and this contradicts the fact that every orbit in R accumulates in $\tilde{\Omega}$. So, $\tilde{\Omega}$ must contain at least one nilpotent singularity.

We proceed to show that Ω_Γ contains a finite number of components. For this, we will use the proposition 2.4. We consider the set of sub-arcs $\{\Gamma_1, \ldots, \Gamma_k\}$ of Γ which form the complement of the finite set of nilpotent singularities. For each Γ_i, we take the transversal segments Σ_i^0 and Σ_i^1 as in proposition 2.4. Now, if a connected component Ω_1 of Ω_Γ has one of its extremities in Γ_i, necessarily there exists a regular orbit $\gamma \subset \Omega$, accumulating at Ω_1, and which cuts transversally either Σ_i^0 or Σ_i^1. Using lemma 3.1, this implies that each Γ_i can contain the extremities of at most *two* connected components of Ω_Γ (because there exists the possibility that one regular orbit of Ω cuts Σ_i^0 and another orbit cuts Σ_i^1). Now, since there are a finite number of arcs of semi-hyperbolic singularities, and a finite number of nilpotent singularities, Ω_Γ can have only a finite number of connected components.

(ii) Let us suppose now that $\Omega \subset \Gamma$. Then, by connectedness, either Ω is a point, a closed sub-arc or the entire curve Γ. We will show that: in the first case, the point is always a nilpotent singularity; that the second case cannot happen; and that in the third case Ω is diffeomorphic to $\mathbf{S}^1$.

We will make use of a result in **[P-T]** which states that, if $p \in S$ is a semi-hyperbolic singularity of X_{λ_0}, there exists a neighborhood $W \subset S \times P$ of $\{p\} \times \{\lambda_0\}$ in which the family is C^0-equivalent to the family

$$Y_\tau = \begin{cases} \dot{x} = f(x,\tau) \\ \dot{y} = \pm y \end{cases} \quad (3)$$

We will denote by $\Psi(x,y,\lambda) = (\psi_1(x,y,\lambda), \psi_2(\lambda))$ the equivalence function between the two families. Now, we analyze each case separately:

(ii.1) Suppose that Ω is a single point p. By absurd, let us assume that it is a semi-hyperbolic singularity. Since $\{p\}$ is a l.p.s. for X_λ at λ_0, there is a λ near λ_0 such that X_λ has a limit cycle γ arbitrarily near p. And so, $\{\gamma\} \times \{\lambda\}$ is eventually contained in W. But this would imply that the family (3) has also limit cycles arbitrarily near the origin, which is an absurd (because this would imply that, for some value of τ, the field

Y_τ would have an orbit which is horizontal at a point outside the line $\{y = 0\}$).

(ii.2) Suppose now that the l.p.s. Ω is a closed sub-arc of Γ. Since Γ is a $H.N$-curve, there exists a semi-hyperbolic singularity q of X_{λ_0} inside Ω. So, we can apply the result above and consider a neighborhood $W = U \times V \subset S \times P$ of $\{q\} \times \{\lambda_0\}$ in which the family is equivalent to the family (3). Now, we choose any two points p_1 and p_2 in $U \cap \Omega$ such that q is contained in the sub-arc of Γ between these two points. The image by $\psi_1(\cdot, \lambda_0)$ of this sub-arc is a closed segment $\tilde{\Omega}$ of the x-axis.

We consider a sequence $\lambda_k \to \lambda_0$ such that X_{λ_k} has a limit cycle γ_k, and $\gamma_k \to \Omega$ as $k \to \infty$. If $\gamma_k \times \{\lambda_k\}$ is sufficiently near $\Omega \times \{\lambda_0\}$, we can consider the image of $\gamma_k \cap U$ by $\psi_1(\cdot, \lambda_k)$. Since $\gamma_k \to \Omega$, what we will obtain are two regular orbits $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ approaching the segment $\tilde{\Omega}$ and *going in opposite horizontal directions*. This implies that, in (3), the function $f(x, \tau_k)$ does not vanish, for any x in some sub-interval $I \subset (\psi_1(p_1, \lambda_k), \psi_1(p_2, \lambda_k))$, and $\tau_k = \psi_2(\lambda_k)$ sufficiently near $\psi_2(\lambda_0)$. But now, we have obtained a contradiction to the expression (3), because since $f(x, \tau_k)$ does not vanish in I, there can exist only one horizontal sense for the regular orbits near $\tilde{\Omega}$.

(ii.3) Now we assume that $\Omega = \Gamma$. If Ω is a l.p.s. entirely contained in a $H.N$-curve and different from a single point, the limit cycles bifurcating from Ω cannot approach a same segment of Γ going in both senses. In other words, using the same construction as above, we see that each limit cycle γ_k sufficiently near Ω must have one definite sense with respect to Ω. So, the only possibility is that the limit cycles turns all around Ω before closing. That means that all the curve Γ originates a limit cycle, and Γ is diffeomorphic to $\mathbf{S}^1$. □

Finally, we have all the elements to prove the result stated in the introduction:

Theorem 1. *Let Ω be a limit periodic set at λ_0 of an analytic family X_λ obeying hypothesis (H). Then, if X_{λ_0} has a curve of singularities Γ such that $\Omega \cap \Gamma \neq \varnothing$, there are two possible cases:*

i) *If Γ is a N-curve then $\Omega \subset \Gamma$,*

ii) *If Γ is a $H.N$-curve, then $\Omega \cap \Gamma$ is composed of a finite number of closed sub-arcs $\Omega_1, \Omega_2, \dots, \Omega_k$. Possibly, we can have $\Omega_j = \{p\}$, and in this case $p \in \Gamma$ is necessarily a nilpotent singularity.*

In the second case, Ω is a degenerate graphic.

Proof. It follows directly from propositions 3.2 and 3.3. □

4. Desingularization of limit periodic sets of nilpotent type

In this part, we will utilize the family desingularization method (see [**D-R**]) to reduce the question of finite cyclicity of l.p.s. contained in N-curves to the problem of finite cyclicity of graphics and degenerate graphics (see definitions in the introduction).

So, suppose that we have an analytic family X_λ defined in $S \times P$ and that for some parameter value $\lambda_0 \in P$, X_{λ_0} has a N-curve Γ. We will proceed in the following fashion:

i) Define a new analytic family $X_{\varphi,\psi,\lambda}$ (with $(\varphi, \psi, \lambda) \in \tilde{P}$), which induces the family X_λ in a sufficiently small neighborhood of $\Gamma \times \{\lambda_0\}$. This new family will present an analytical expression convenient to the blowing-up operation.

ii) Effectuate the blow-up along $\Gamma \times P$ inside the family $X_{\varphi,\psi,\lambda}$. In this particular case, we will need only a part of the general theory developed in [**D-R**].

As we will see, after the blow-up operation we obtain a new object which is no more an analytic family of vector fields, but for which the definitions of limit cycle, limit periodic set, cyclicity etc, can easily be extended. Further, this new object can be partially seen as an analytic family $\tilde{X}_{\tilde{\lambda}}$ in some special chart. We will prove that if all the l.p.s. of $\tilde{X}_{\tilde{\lambda}}$ have finite cyclicity, then all the possible l.p.s. of the original family X_λ contained in Γ also have finite cyclicity. This represents a real reduction of the complexity of the problem, since the new family $\tilde{X}_{\tilde{\lambda}}$ will not contain N-curves of singularities.

4.1 The new family $X_{\varphi,\psi,\lambda}$

Let X_λ and Γ be as defined above. As we have seen in section 2, we can

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4.1 The new family $X_{\varphi,\psi,\lambda}$

Let X_λ and Γ be as defined above. As we have seen in section 2, we can find a system of coordinates in a neighborhood $M \subset S$ of Γ in which the family has the expression

$$X_\lambda = \begin{cases} \dot{x} & = F(x,y,\lambda) \\ \dot{y} & = G(x,y,\lambda) \end{cases} \tag{1}$$

where $\lambda \in P$, $x \in U \subset \mathbf{S}^1$, $y \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$; and the N-curve Γ is simply given by $\{y = 0\}$. In this coordinate system, we have,

$$j^1 X_{\lambda_0}(x,0) = \begin{bmatrix} 0 & \frac{\partial F(x,0,\lambda_0)}{\partial y} \\ 0 & 0 \end{bmatrix} \tag{2}$$

and, by hypothesis (H),

$$\frac{\partial F(x,0,\lambda_0)}{\partial y} \neq 0 \tag{3}$$

for any $x \in U$. So, we can define the following change of coordinates

$$\begin{aligned} \hat{x} &= x \\ \hat{y} &= F(x,y,\lambda) \\ \hat{\lambda} &= \lambda \end{aligned} \tag{4}$$

In fact, this map is bijective when restricted to the set $\Gamma \times \{\lambda_0\}$, and, by (3), it is also a local diffeomorphism in this set. So, there exists a neighborhood $N \subset U \times (-\varepsilon, \varepsilon) \times P$ of $\Gamma \times \{\lambda_0\}$ in which (4) is a diffeomorphism.

Writing the family in this new coordinates (and dropping the hats), we obtain

$$X_\lambda = \begin{cases} \dot{x} & = y \\ \dot{y} & = H(x,y,\lambda) \end{cases} \tag{5}$$

where H is an analytic function in N. If we develop H in power of y up to the second order

$$H(x,y,z) = g(x,\lambda) + yf(x,\lambda) + y^2 Q(x,y,\lambda), \tag{6}$$

the condition (3) for a N-curve can be exactly expressed as

$$f(x,\lambda_0) \equiv 0 \text{ and } g(x,\lambda_0) \equiv 0\,. \tag{7}$$

Now, we will put the functions f and g in a form which will be more convenient to effectuate the blow-up in the next section. For this, we will introduce the concept of *ideal of coefficients*, first considered in [**R3**].

We consider an analytic function $h(x,\lambda)$ with domain $U \times V$, where U is a compact connected subset of $\mathbf{S}^1$ and V is an open neighborhood of a given point λ_0 in $\mathbb{R}^\Lambda$. We suppose that $h(x,\lambda_0) \equiv 0$. Given a point $x_0 \in U$, there exists an open interval $I \subset U$ centered at x_0 in which we can write the expansion:

$$h(x,\lambda) = \sum_{i=0}^{\infty} a_i(\lambda, x_0)(x - x_0)^i\,. \tag{8}$$

Since U is compact, we can suppose, by eventually reducing V, that this expansion is valid for any $\lambda \in V$. Now, we define J_{x_0} as the ideal generated by $(a_0, a_1, \ldots)$ in the ring of analytic functions in V. One important result of the theory is that this ideal is independent of the point x_0. This is a direct consequence of the following result:

Proposition. (see [**R3**]). *Let $h(x,\lambda)$ be as above. Then, there exists a constant $R > 0$ such that, for any point x_0 of U, we can find a compact neighborhood $V_{x_0} \subset V$ of λ_0 and analytic functions $h_0(x,\lambda,x_0), \ldots, h_k(x,\lambda,x_0)$, defined in $[x_0 - R, x_0 + R] \times V_{\lambda_0}$, such that in this domain,*

$$h(x,\lambda) = \sum_{i=0}^{k} a_i(\lambda, x_0)\; h_i(x,\lambda,x_0)\,. \tag{9}$$

Moreover, each function has the form $h_i(x,\lambda,x_0) = (x - x_0)^i[1 + O(x - x_0, \lambda)]$.

If we consider the ideal J_{x_1} for a point $x_1 \in (x_0 - R, x_0 + R)$, it is easy to see by (9) that $J_{x_1} \subset J_{x_0}$. By symmetry, also $J_{x_0} \subset J_{x_1}$. So, as we have claimed, the ideal is independent of the base point x_0 that we choose.

The ideal $\mathbb{J} = J_{x_0}$ is called the *ideal of coefficients* of the analytic function $h(x,\lambda)$. It is an ideal in the ring of germs of analytic functions at λ_0. Since this ring is Noetherian, $\mathbb{J}$ is finitely generated.

We consider now any minimal set of generators $(\varphi_0, \varphi_1, \ldots, \varphi_r)$ of the ideal J (for us, this will mean that $([\varphi_0], \ldots, [\varphi_r])$ is a basis of $\mathbb{J}/\mathcal{M}\mathbb{J}$, where $\mathcal{M}$ is the maximal ideal of the ring of analytic germs at λ_0).

Our goal is to show the following result:

Lemma 4.1. *There exist a neighborhood $V_0 \subset V$ of λ_0 and functions*

$h_0, \dots, h_r$ analytic in $U \times V_0$ such that we can write the expansion

$$h(x,\lambda) = \sum_{i=0}^{r} \varphi_i(\lambda)\ h_i(x,\lambda)\ ,\ \text{for}\ x \in U,\ \lambda \in V_0\ . \tag{10}$$

Moreover, the functions $h_0, \dots, h_r$ are $\mathbb{R}$-independent (i.e. for any fixed $\lambda_1 \in V_0$, any non-trivial linear combination of $h_0(\cdot,\lambda_1), \dots, h_r(\cdot,\lambda_1)$ does not vanish identically).

Remark. This is not a direct consequence of the previous proposition, because we do not have uniqueness in the local expansion (9).

Proof. Our first step is to complexify the problem, in order to apply results of the theory of holomorphic functions in several complex variables.

Since $h, \varphi_0, \dots, \varphi_r$ are analytic functions in $U \times V$, there exists a neighborhood $\tilde{W} \subset \mathbb{C}^{\Lambda+1}$ of $U \times \{\lambda_0\}$ such that $\{h, \varphi_0, \dots, \varphi_r\} \subset H(\tilde{W})$, where $H(\tilde{W})$ is the set of holomorphic functions in $\tilde{W}$.

Now, it is easy to see that we can find a neighborhood $W \subset \tilde{W}$ of $U \times \{\lambda_0\}$ which is a *domain of holomorphy* (by definition, this means that it is the maximal domain of definition of some holomorphic function). In fact, it suffices to take some W which can be written as a Cartesian product of $\Lambda + 1$ domains of $\mathbb{C}$.

One of the most important properties of domains of holomorphy is the following (see e.g. [**G-R**]):

(The Division Property). *Let $f, f_1, \dots, f_k \in H(D)$, for some domain of holomorphy D of $\mathbb{C}^n$. Suppose that the zero set of f contains the intersection of the zero sets of $f_1, \dots, f_k$ (counted with multiplicities) or, equivalently, that $\mathbf{f}_z$ (the germ of f in the point z) is contained in the ideal $(\mathbf{f}_{1,z}, \dots, \mathbf{f}_{k,z})$, for every point $z \in D$. Then, there exists functions $h_1, \dots, h_k \in H(D)$ such that $f = \sum h_i f_i$.*

By our construction, it is clear that the hypothesis are fulfilled, and so we obtain a complex expansion:

$$h(x,\lambda) = \sum_{i=0}^{r} \varphi_i(\lambda)\ h_i(x,\lambda) \quad \text{for} \quad (x,\lambda) \in W.$$

Finally, it suffices to take the real parts of the functions h_i to get the

desired real analytic expansion. The domain V_0 of the enunciate is obtained by taking the projection of W into the original real space of parameters.

It remains to show that $h_0, \dots, h_r$ are $\mathbb{R}$-independent. For this, we take any point $x_0 \in U$ and consider the power series expansion (8) around x_0. It is easy to see that we can choose a minimal set of generators of $\mathbb{J}$ inside the set $\{a_0, a_1, \dots\}$. Suppose that the set of generators is given by

$$\varphi_0' = a_{i_0}, \dots, \varphi_r' = a_{i_r}\ \ ,\ \text{for}\ 0 \le i_0 < \dots < i_r.$$

The previous proposition give us a local expansion of h in $[x_0 - R, x_0 + R]$,

$$h(x,\lambda) = \sum_{i=0}^{r} \varphi_i'(\lambda, x_0)\ h_i'(x,\lambda,x_0)\ ,$$

from which it is immediate to see that the h_i' are $\mathbb{R}$-independent. Indeed, the demonstration given in [**R3**], consists basically of showing the validity of the following operation:

$$\begin{aligned} h(x,\lambda) &= \sum_{j=0}^{\infty} a_j(\lambda,x_0)\ (x-x_0)^j \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{r} b_k^j(\lambda)\ a_{i_k}(\lambda,x_0) \right) (x-x_0)^j \\ &= \sum_{k=0}^{r} a_{i_k}(\lambda,x_0) \left(\sum_{j=0}^{\infty} b_k^j(\lambda)\ (x-x_0)^j \right) \\ &= \sum_{k=0}^{r} \varphi_k'(\lambda,x_0)\ h_k'(x,\lambda,x_0)\ , \end{aligned}$$

where we have written $a_j = \sum_{k=0}^{r} b_k^j\ a_{i_k}$. From this, it is clear that the particular term $(x-x_0)^{i_k}$ appears in the asymptotic expansion of h_k' and of *no other* $h^l{}'$. So, there can be no non-trivial linear combination of the functions $h_0', \dots, h_r'$ which vanishes identically.

Now, in order to show that the h_i are also $\mathbb{R}$-independent, it suffices to observe that, in a neighborhood of the point $x_0 \in U$, there exists a linear isomorphism which takes $\{h_i'\}$ into $\{h_i\}$, i.e. a matrix $U(\lambda, x_0)$

such that:

$$h_i(x,\lambda) = \sum_{j=0}^{r} u_{ij}(\lambda, x_0)\ h'_j(x,\lambda,x_0)\ .$$

Indeed, $\{[\varphi'_j]\}$ and $\{[\varphi_i]\}$ are two bases of the vector space $\mathbb{J}/\mathcal{M}\mathbb{J}$ (where $\mathcal{M}$ is the maximal ideal of the ring of germs of analytic functions at λ_0), and the isomorphism above is the change of base transformation. So, since the matrix U_{ij} has evidently rank $r+1$ (over the reals), and the set $\{h'_0, \dots, h'_r\}$ is $\mathbb{R}$-independent, the same is valid for the set $\{h_0, \dots, h_r\}$. □

Finally, applying the lemma 4.1, we can find a neighborhood K of λ_0 such that the expansions

$$f(x,\lambda) = \sum_{i=0}^{r} \varphi_i(\lambda) f_i(x,\lambda) \ \text{ and } \ g(x,\lambda) = \sum_{j=0}^{s} \psi_j(\lambda) g_j(x,\lambda)\ , \tag{11}$$

are valid for all $(x,\lambda) \in \Gamma \times K$. We remember the important fact that the functions $\{f_i\}$ and $\{g_j\}$ form $\mathbb{R}$-independent sets.

Our last step will consist of embedding the family X_λ in a larger family $X_{\varphi,\psi,\lambda}$, which is constructed simply by taking the functions $\varphi_0, \dots, \varphi_r, \psi_0, \dots, \psi_s$ as independent parameters, taking values in $\mathbb{R}^{r+s+2}$. This operation of embedding usually receives the name of *family induction.* It is immediate to see that if Γ contains a l.p.s. Ω of X_λ at λ_0, then Ω is also a limit periodic set of $X_{\varphi,\psi,\lambda}$ at $(0,0,\lambda_0)$. Also, the cyclicity of Ω with respect to the latter family is never smaller than its cyclicity with respect to the former one.

The final family takes the form

$$X_{\varphi,\psi,\lambda} = \begin{cases} \dot{x} &= y \\ \dot{y} &= \sum\limits_{j=0}^{s} \psi_j\ g_j(x,\lambda) + y\ (\sum\limits_{i=0}^{r} \varphi_i\ f_i(x,\lambda)) + y^2 Q(x,y,\lambda) \end{cases}$$

By the terminology of the introduction, this family is an analytical unfolding of Γ.

For shortness, we will again denote the new phase space by S, and

the parameter space $K \times R^{r+s+2}$ by P.

4.2 The blow-up operation

The basic idea of the family blow-up is to consider the analytic family $X_{\varphi,\psi,\lambda}$ as a global vector field $\mathcal{X}$ defined in the total space $T = S \times P$ (using the same notation as in the previous section). The important characteristic of $\mathcal{X}$ is that it is a *foliated vector field*; that is, it is everywhere tangent to a foliation $\mathcal{F}$ given by $\{d\varphi = 0, d\psi = 0, d\lambda = 0\}$. To every analytic family we can always associate a foliated vector field in the Cartesian product of the phase space and the parameter space; and the foliation will evidently be always regular. Conversely, under certain hypothesis, we can see a foliated vector field with a regular foliation as an analytic family: the phase space will be diffeomorphic to the leaves and the parameter space to the normal space.

The sub-manifold $M = \{(x,y,\varphi,\psi,\lambda) \in T \colon (y,\varphi,\psi) = (0,0,0)\}$ contains the curve of singularities Γ, and is evidently contained in $\mathcal{Z}(\mathcal{X})$, the set of singularities of $\mathcal{X}$. Our desingularization operation will consist of blowing-up all the manifold M. For this, we express the normal space at M, parameterized by (y,φ,ψ), in quasi-homogeneous coordinates as follows:

$$\begin{aligned} y &= \tau^a \bar{y} \\ \varphi_i &= \tau^{b_i} \bar{\varphi}_i \\ \psi_j &= \tau^{c_j} \bar{\psi}_j \end{aligned} \tag{12}$$

where $\tau \in \mathbb{R}^+$, and $(\bar{y}, \bar{\varphi}_0, \dots, \bar{\varphi}_r, \bar{\psi}_0, \dots, \bar{\psi}_s) \in \mathbf{S}^{r+s+2}$.

The values of $a, b_0, \dots, b_r, c_0, \dots, c_s$ will be precised later. The transformation (12) is a diffeomorphism for $\tau \neq 0$.

Let us denote $\tilde{T}$ the new analytic manifold given by $(x, (\bar{y}, \bar{\varphi}, \bar{\psi}), \lambda, \tau)$. According to i the observation above, there is a surjective projection map $\pi \colon \tilde{T} \to T$ which is bijective in $T - M$. Let us denote the set $\pi^{-1}(M)$ by $\tilde{M}$, usually called the *exceptional divisor.* Clearly, we can pull-back both the vector field $\mathcal{X}$ and the foliation $\mathcal{F}$ defined in T to obtain new analytic objects in $\tilde{T} - \tilde{M}$.

It can be proved (see lemma II.11 at [**D-R**]) that there exists an integer $s > 0$ such that the vector field $\tilde{\mathcal{X}} = \tau^{-s} \cdot (\pi^* \mathcal{X})$ can be analytically extended to all $\tilde{T}$.

On the other hand, the foliation $\tilde{\mathcal{F}}' = \pi^* \mathcal{F}$ cannot be *regularly* extended to $\tilde{M}$. As we will see below, there is a *singular* maximal foliation $\tilde{\mathcal{F}}$ which coincides with $\tilde{\mathcal{F}}'$ in $\tilde{T} - \tilde{M}$. The vector field $\tilde{\mathcal{X}}$ will be everywhere tangent to $\tilde{\mathcal{F}}$.

Explicitly writing, the foliation $\tilde{\mathcal{F}}'$ is given by

$$\tilde{\mathcal{F}}' = \begin{cases} d\lambda = 0 \\ d(\tau^{b_i}\bar{\varphi}_i) = 0 \ ; \ i \in \{0, \ldots, r\} \\ d(\tau^{c_j}\bar{\psi}_j) = 0 \ ; \ j \in \{0, \ldots, s\} \end{cases} , \tag{13}$$

where $\tau \neq 0$. To better understand (13), we will decompose $\tilde{T}$ in two different domains:

$$\begin{aligned} G &= \{(x, (\bar{y}, \bar{\varphi}, \bar{\psi}), \lambda, \tau) \in \tilde{T} \colon |\bar{y}| > \rho_2\} \\ F &= \{(x, (\bar{y}, \bar{\varphi}, \bar{\psi}), \lambda, \tau) \in \tilde{T} \colon |\bar{y}| < \rho_1\} , \end{aligned} \tag{14}$$

for some $0 < \rho_2 < \rho_1 < 1$. In the following subsections, we will analyze the foliation in these two domains separately.

4.2.1 The family chart

We can define another quasi-homogeneous change of coordinates by:

$$\begin{cases} y = \tilde{\tau}^a \tilde{y} \\ \varphi_i = \tilde{\tau}^{b_i}\tilde{\varphi}_i \ ; \ i \in \{0, \ldots, r\} \\ \psi_j = \tilde{\tau}^{c_j}\tilde{\psi}_j \ ; \ j \in \{0, \ldots, s\} \end{cases} \tag{15}$$

where $(\tilde{y}, (\tilde{\varphi}, \tilde{\psi}), \tilde{\tau}) \in \mathbb{R} \times \mathbf{S}^{r+s+1} \times \mathbb{R}^+$.

It is clear that, restricted to the domain F, there exists a diffeomorphism Ψ_F which maps the blow-up (12) into the blow-up (15); i.e. which maps the domain $((\bar{y}, \bar{\varphi}, \bar{\psi}), \tau) \in (\mathbf{S}^{r+s+2} \times \mathbb{R}^+) \cap F$ into $(\tilde{y}, (\tilde{\varphi}, \tilde{\psi}), \tilde{\tau}) \in [-L, L] \times \mathbf{S}^{r+s+1} \times \mathbb{R}^+$ (where L depends of ρ_1), and commutes with the blow-up transformations. The pair (Ψ_F, F) is known as the *family chart*, by a reason that we will see below.

The image $\tilde{\mathcal{F}}_F$ of the foliation $\tilde{\mathcal{F}}'|_F$ by Ψ_F is given by:

$$\tilde{\mathcal{F}}_F = \begin{cases} d\lambda = 0 \\ d(\tilde{\tau}^{b_i}\tilde{\varphi}_i) = 0 \ ; \ i \in \{0, \ldots, r\} \\ d(\tilde{\tau}^{c_j}\tilde{\psi}_j) = 0 \ ; \ j \in \{0, \ldots, s\} \end{cases} . \tag{16}$$

Each leaf can be described as $\{\tilde{\tau} = \text{const}, \tilde{\varphi}_i = \text{const}, \tilde{\psi}_j = \text{const}, \lambda = \text{const}\}$. Indeed, from (16) we obtain the following differential system:

$$\begin{aligned} b_i\tilde{\tau}^{b_i-1}\tilde{\varphi}_i \, d\tilde{\tau} + \tilde{\tau}^{b_i} \, d\tilde{\varphi}_i = 0 \ \text{ for } \ i \in \{0, \ldots, r\} \\ c_j\tilde{\tau}^{c_j-1}\tilde{\psi}_j \, d\tilde{\tau} + \tilde{\tau}^{c_j} \, d\tilde{\psi}_j = 0 \ \text{ for } \ j \in \{0, \ldots, s\} \end{aligned}$$

Multiplying the lines of the first group by $\tilde{\tau}^{S-b_i}\tilde{\varphi}_i$ and the lines of the second group by $\tilde{\tau}^{S-c_j}\tilde{\psi}_j$ (where $S = \sum_{i=0}^r b_i + \sum_{j=0}^s c_j$), we can sum up all the lines. The second members will annihilate since they represent the differential of a constant (i.e. the modulus of the vectors contained in a sphere). So, we finish with the equation:

$$\tilde{\tau}^{S-1}\Big(\sum_{i=0}^{r} b_i\varphi_i^2 + \sum_{j=0}^{s} c_j\psi_j^2\Big)d\tilde{\tau} = 0.$$

From this, we get immediately that $d\tau = 0$. Also, $d\tilde{\varphi}_i = 0$ and $d\tilde{\psi}_j = 0$.

The exceptional divisor has the following expression in this chart:

$$\tilde{M}_F = \tilde{M} \cap F = \{\tilde{\tau} = 0\}.$$

By this, we can extend the foliation (16) in a regular way to $\tilde{M}_F$. In fact, the leaves in the exceptional divisor will simply have the form $\{\tilde{\tau} = 0, \tilde{\varphi}_i = \text{const}, \tilde{\psi}_j = \text{const}, \lambda = \text{const}\}$. The vector field $\tilde{\mathcal{X}}_F = (\Psi_F)_* \tilde{\mathcal{X}}$ will be everywhere tangent to this regular foliation. So, using the duality between vector fields tangent to a regular foliations and an analytic families (as we have observed in the beginning of this sub-section), we can consider $\tilde{\mathcal{X}}_F$ as an analytic family, say $\tilde{X}_{\tilde{\lambda}}$, where the phase space $\tilde{S}$ can be realized as an open set of $\mathbf{S}^2$ and the parameter space $\tilde{P}$ is given by $\mathbb{R}^+ \times \mathbf{S}^{r+s+1} \times K$ (see the previous sub-section for the definition of K). The new parameter $\tilde{\lambda}$ is equal to $(\tau, \tilde{\varphi}, \tilde{\psi}, \lambda)$.

It is not hard to write the analytical expression of the family $\tilde{X}_{\tilde{\lambda}}$ in the family chart. All we have to remember is that in this chart the parameter space is tangent to the field, which means that the vector field $\tilde{\mathcal{X}}_F$ obeys $\dot{\tilde{\lambda}} = 0$. If we apply the change of coordinates given by (15) in the original family $X_{\varphi,\psi,\lambda}$, we obtain:

$$\tilde{X}'_{\tilde{\lambda}} = \begin{cases} \dot{x} = \tilde{\tau}^a \tilde{y} \; ; \\ \tilde{\tau}^a \dot{\tilde{y}} = \sum_{j=0}^{s} \tilde{\tau}^{c_j} \tilde{\psi}_j \; g_j(x,\lambda) + \tilde{\tau}^a \tilde{y} \; (\sum_{i=0}^{r} \tilde{\tau}^{b_i} \tilde{\varphi}_i \; f_i(x,\lambda)) \\ \quad + \tilde{\tau}^{2a} \tilde{y}^2 Q(x,\tilde{y},\lambda,\tilde{\tau}). \end{cases}$$

This expression permits us to choose conveniently $(a, b_0, \ldots, b_r, c_0, \ldots, c_s)$, the weights for the quasi-homogeneous change of coordinates (12). The choice that will suffice to us is:

$$a = 1, \; b_i = 1 \; (i = 0, \ldots, r), \; c_j = 2 \; (j = 0, \ldots, s).$$

So, we get:

$$\tilde{X}'_{\tilde{\lambda}} = \begin{cases} \dot{x} = \tilde{\tau} \tilde{y} \\ \dot{\tilde{y}} = \tilde{\tau} \sum_{j=0}^{s} \tilde{\psi}_j \; g_j(x,\lambda) + \tilde{\tau} \tilde{y} \; (\sum_{i=0}^{r} \tilde{\varphi}_i \; f_i(x,\lambda)) + \tilde{\tau} \tilde{y}^2 Q(x,\tilde{y},\lambda,\tilde{\tau}) \end{cases}$$

Finally, we can divide this family by $\tilde{\tau}$ (which means that we extend the vector field $\pi^* \mathcal{X}$ to the exceptional divisor by taking $s = 1$ in the transformation $\tilde{\mathcal{X}} = \tau^{-s} \cdot (\pi^* \mathcal{X})$). This important operation permits us to get a non-trivial dynamics at the exceptional divisor $\tilde{M}_F$. The final form of the family is:

$$\tilde{X}_{\tilde{\lambda}} = \begin{cases} \dot{x} = \tilde{y} \\ \dot{\tilde{y}} = \sum_{j=0}^{s} \tilde{\psi}_j \; g_j(x,\lambda) + \tilde{y} \; (\sum_{i=0}^{r} \tilde{\varphi}_i \; f_i(x,\lambda)) + \tilde{y}^2 Q(x,\tilde{y},\lambda,\tau) \end{cases} \tag{17}$$

Now, the crucial step is to observe that the analytic family (17), which is evidently equivalent to the original family $X_{\varphi,\psi,\lambda}$ outside the exceptional divisor, does not present any N-curve of singularities. Indeed, since the vector of parameters $(\tilde{\varphi}_0, \ldots, \tilde{\varphi}_r, \tilde{\psi}_0, \ldots, \tilde{\psi}_s) \in \mathbf{S}^{r+s+1}$ never vanish altogether, we conclude from the $\mathbb{R}$-independence of the $\{f_i\}$ and the $\{g_j\}$ that, for any fixed $\tilde{\lambda}_1 = (\tau_1, \tilde{\varphi}, \tilde{\psi}, \lambda_1) \in \tilde{P}$, either

$$\sum_{i=0}^{r} \tilde{\varphi}_i \; f_i(x,\lambda_1) \not\equiv 0 \;\text{ or }\; \sum_{j=0}^{s} \tilde{\psi}_j \; g_j(x,\lambda_1) \not\equiv 0.$$

So, at the worst case, the vector field $\tilde{X}_{\tilde{\lambda}_1}$ presents a $H.N$-curve.

4.2.2 The germ chart

Now, we proceed to analyze the foliation $\tilde{\mathcal{F}}'$ in the set G. This set is composed of two connected components $G_+ = \{\bar{y} > 0\} \cap G$ and $G_- = \{\bar{y} < 0\} \cap G$. As in the previous case, it will be convenient to define a diffeomorphism which will simplify the geometry of the foliation. We consider the following map:

$$\Psi_G = \begin{cases} \check{y} = \pm 1 \\ \check{\varphi}_i = \bar{\varphi}_i / n^{b_i} \;\; ; \;\; i \in \{0, \ldots, r\} \\ \check{\psi}_j = \bar{\psi}_j / n^{c_j} \;\; ; \;\; j \in \{0, \ldots, s\} \\ \check{\tau} = n \cdot \tau, \end{cases} \tag{18}$$

where $n = (\bar{y})^{\frac{1}{a}}$. This map takes $(\bar{y}, \bar{\varphi}, \bar{\psi}) \in \mathbf{S}^{r+s+2}$ into $(\check{y}, \check{\varphi}_i, \check{\psi}_j) \in \{-1, +1\} \times B(0, Q)$ (where $B(0, Q) \subset \mathbb{R}^{r+s+2}$ is a ball of radius $Q = \sqrt{1 - \rho_2^2/\rho_2}$). The chart (G, Ψ_G) is usually called the *germ chart*, since it contains the blow-up of the single vector field $X_{0,0,\lambda_0}$.

In the germ chart, the foliation $\tilde{\mathcal{F}}_G = (\Psi_G)_*(\tilde{\mathcal{F}}')$ has the expression:

$$\tilde{\mathcal{F}}_G = \begin{cases} d\lambda = 0 \\ d(\check{\tau}\check{\varphi}_i) = 0 \;\; ; \;\; i \in \{0, \ldots, r\} \\ d(\check{\tau}^2 \check{\psi}_j) = 0 \;\; ; \;\; j \in \{0, \ldots, s\} \end{cases} . \tag{19}$$

The regular leaves of $\tilde{\mathcal{F}}_G$ are clearly two dimensional surfaces. Next to the exceptional divisor, the foliation accumulates at the following two sets:

$$\tilde{\mathcal{L}}_\Lambda = \{(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \mid \lambda \in K, \; (\check{\varphi}, \check{\psi}) = (0,0)\}$$

and

$$\tilde{M}_G = \tilde{M} \cap G = \{\check{\tau} = 0\}.$$

These sets intersects transversally, and the intersection is a $(\Lambda + 1)$-dimensional sub-manifold D (where Λ is the dimension of the parameter space of the original family X_λ). By this, it is clear that the foliation (19) cannot be regularly extended to all the space $\Psi_G(G)$. Anyway, it is possible to obtain a singular foliation, defined globally, and which extends $\tilde{\mathcal{F}}_G$.

First, we define foliation in $\tilde{\mathcal{L}}_\Lambda - D$ by taking as leafs the sets:

$$\tilde{L}_{\lambda_1} = \tilde{\mathcal{L}}_\Lambda \cap \{\lambda = \lambda_1\}$$

for each $\lambda \in K$.

To extend the foliation to $\tilde{M}_G - D$, we define the leaves as follows: We choose one parameter, say $\check{\varphi}_0$, and take a typical leaf as

$$\tilde{L} = \left\{ \begin{array}{r} (x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \,|\, \check{\varphi}_1 = c_1 \cdot \check{\varphi}_0, \dots, \check{\varphi}_r = c_r \cdot \check{\varphi}_0, \\ \check{\psi}_0 = d_0 \cdot \check{\varphi}_0^2, \dots, \check{\psi}_s = d_s \cdot \check{\varphi}_0^2, \\ \lambda = \text{ const}, \ \check{\tau} = 0 \end{array} \right\},$$

where $(c_1, \cdots, d_s)$ are constants, and $\check{\varphi}_0 \in [-1, 0) \cup (0, 1]$. This defines the foliation except at the hyper-plane $\{\check{\varphi}_0 = 0\}$. Then, we proceed inductively by choosing another parameters, up to cover all the set $\tilde{M}_G - D$ by two dimensional leaves.

Finally, we must also foliate the set D by including one-dimensional leaves of the form:

$$\tilde{L} = \left\{ \begin{array}{c} (x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \mid \check{\varphi}_0 = 0, \dots, \check{\varphi}_r = 0, \check{\psi}_0 = 0, \dots, \check{\psi}_s = 0, \\ \lambda = \text{ const}, \check{\tau} = 0 \end{array} \right\}$$

It is easy to see that this foliation of $\tilde{M}_G$ is compatible with the foliation of $\tilde{M}_F$ defined in the family chart (i.e. the function $\Psi_G \circ \Psi_F^{-1}$ maps one foliation into the other in the intersection $F \cap G$). By this, we get a globally defined singular foliation in the manifold $\tilde{T}$, which we will denote $\tilde{\mathcal{F}}$.

In the germ chart, the blown-up vector field $\tilde{\mathcal{X}}$ can no more be considered as an analytic family. Anyway, as we will show below, this fact does not represent a difficulty to the desingularization procedure in this particular case.

Let us calculate the vector field $\tilde{\mathcal{X}}$ in the germ chart. Applying (18)

composed with (12) in the family $X_{\varphi,\psi,\lambda}$ we get

$$\tilde{\mathcal{X}}_G{}' = \begin{cases} \dot{x} = \pm\check{\tau} \\ \dot{\check{\tau}} = \pm\check{\tau}^2[\sum_{j=0}^{s} \check{\psi}_j \, g_j(x,\lambda) \pm (\sum_{i=0}^{r} \check{\varphi}_i \, f_i(x,\lambda)) + Q(x, \pm 1, \lambda, \check{\tau})] \\ \dot{\check{\tau}} \, \check{\varphi}_i + \check{\tau} \, \dot{\check{\varphi}}_i = 0 \qquad ; \, i \in \{0, \dots, r\} \\ 2\check{\tau}\dot{\check{\tau}} \, \check{\psi}_j + \check{\tau}^2 \, \dot{\check{\psi}}_j = 0 \ ; \ j \in \{0, \dots, s\} \\ \dot{\lambda} = 0 \end{cases} \tag{20}$$

If we write $H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) = \sum_{j=0}^{s} \check{\psi}_j \, g_j(x,\lambda) \pm (\sum_{i=0}^{r} \check{\varphi}_i \, f_i(x,\lambda)) + Q(x, \pm 1, \lambda, \check{\tau})$, we can substitute $\dot{\check{\tau}}$ in the second and third rows to obtain:

$$\tilde{\mathcal{X}}_G{}' = \begin{cases} \dot{x} = \pm\check{\tau} \\ \dot{\check{\tau}} = \pm\check{\tau}^2 H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \\ \dot{\check{\varphi}}_i = \mp 2\check{\tau} \; H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \cdot \check{\varphi}_i \quad ; \, i \in \{0, \dots, r\} \\ \dot{\check{\psi}}_j = \mp\check{\tau} \; H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \cdot \check{\psi}_j \quad ; \, j \in \{0, \dots, s\} \\ \dot{\lambda} = 0 \end{cases} \tag{21}$$

which we can divide by $\check{\tau}$ to have the new vector field:

$$\tilde{\mathcal{X}}_G = \begin{cases} \dot{x} = \pm 1 \\ \dot{\check{\tau}} = \pm H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \cdot \check{\tau} \\ \dot{\check{\varphi}}_i = \mp 2 H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \cdot \check{\varphi}_i \quad ; \, i \in \{0, \dots, r\} \\ \dot{\check{\psi}}_j = \mp H(x, \pm 1, \check{\varphi}, \check{\psi}, \lambda, \check{\tau}) \cdot \check{\psi}_j \quad ; \, j \in \{0, \dots, s\} \\ \dot{\lambda} = 0 \end{cases} \tag{22}$$

This last system is analytically equivalent to the original system $\mathcal{X}$ restricted to the set $G - M_G$. Also, if we define the following family of leaves:

$$\mathcal{L}_\Lambda = \{L_\lambda = (x, y, \varphi, \psi, \lambda) \in T \mid \lambda \in K, \ (\varphi, \psi) = (0, 0)\},$$

then each regular leaf $L \in \mathcal{F}$ not belonging to $\mathcal{L}_\Lambda$ is mapped into a regular leaf $\tilde{L} \in \tilde{\mathcal{F}}$.

We observe also that $\tilde{\mathcal{X}}_G$ has two regular orbits, which we will denote by Γ_G^+ and Γ_G^-, passing exactly by the intersection $\tilde{L}_{\lambda_0} \cap \tilde{M}_G$.

4.2.3 Study of the cyclicity

As we know, in each leaf $L \in \mathcal{F}$ it is defined an unique element of the family $X_{\varphi,\psi,\lambda}$ (i.e. a vector field acting in the phase space S), which we will denote simply by X_L. We have seen above that, if $L \notin \mathcal{L}_\Lambda$, there is a corresponding regular leaf $\tilde{L} \in \tilde{\mathcal{F}}$ and a vector field $\tilde{X}_{\tilde{L}}$ defined in this leaf which is analytically equivalent to X_L. In this section, we will use the language of foliations to state our results.

In the following, we will use the term *limit cycles in a leaf* L to designate the periodic orbits of the field $\mathcal{X}$ which correspond to limit cycles of the restriction of this field to the leaf $L \in \tilde{\mathcal{F}}$.

First, we show that Γ_G^+ and Γ_G^- have a *finite cyclicity,* in the following precise sense:

. *There exists a neighborhood $V \subset \tilde{T}$ of the orbits Γ_G^+ and Γ_G^-, and a natural number N, such that the number of limit cycles which intersect V, in each leaf $L \in \tilde{\mathcal{F}}$, is bounded by N.*

For this, we will prove a more general result, valid for arbitrary analytic foliated vector fields (we refer to [**D-R**] for the definitions):

Proposition 4.1. *If Γ is a periodic orbit of an analytic foliated vector field $(E, \pi, \Lambda, \tilde{\mathcal{X}})$, then there exists a neighborhood $V \subset E$ of Γ and a natural number N such that the number of limit cycles which intersect V, for each leaf of the foliation, is uniformly bounded by N.*

Proof. We can construct an analytic hyper-surface Σ, transversal to the field $\tilde{\mathcal{X}}$, and which cuts Γ in a single point p_0. For a sufficiently small compact neighborhood $\Sigma_1 \subset \Sigma$ of p_0, there exists an analytic first return map

$$\rho\colon \Sigma_1 \longrightarrow \Sigma .$$

Since the vector field $\tilde{\mathcal{X}}$ is everywhere tangent to the singular foliation $\tilde{\mathcal{F}}$, it follows that this foliation is also transversal to the section Σ. So, we can define an induced analytic foliation $\tilde{\mathcal{G}} = \tilde{\mathcal{F}} \cap \Sigma$ of the transversal section.

The set of periodic orbits of $\tilde{\mathcal{X}}$, which cuts Σ_1, is given by:

$$A = \{p \in \Sigma_1 \mid \rho(p) = p\} .$$

This is obviously an analytic set, since it is defined as the zero set of the analytic function. Now, we consider the following analytic map:

$$\pi : A \longrightarrow \tilde{\mathcal{G}} ,$$

associating to each point of A the leaf of $\tilde{\mathcal{G}}$ to which it belongs. Since the leafs of $\tilde{\mathcal{G}}$ are defined by a finite set of parameters, we can consider the map π as having as image the space $\mathbb{R}^m$, for some natural number m.

Let us now state a well-known result of A.Gabrielov:

Theorem. (see [G]). *If M is a compact analytic set in the real space, and $g : M \to \mathbb{R}^m$ a real analytic map, then the number of the connected components of the pre-images $g^{-1}(a)$ is bounded from above uniformly over $a \in \mathbb{R}^m$.*

By this, we conclude that there exists a uniform bound N for the number of components of A in each leaf of $\tilde{\mathcal{G}}$. Since the limit cycles are, by definition, topologically isolated inside each leaf, we have obtained a bound for the number of limit cycles which cut Σ_1 in each leaf L. Taking the inverse image of the section Σ_1 by the flow generated by $\tilde{\mathcal{X}}$, we obtain the desired neighborhood V of Γ. □

As a direct consequence, if Γ_G^+ and Γ_G^- are periodic orbits, we obtain a neighborhood V in which number of cycles in uniformly bounded.

We suppose now that $\Gamma_G^\pm$ is not a periodic orbit. Then, it is easy to see, by the expression (22), that either $\Gamma_G^\pm$ is transversal to the boundary of $\tilde{T}$; or else it is tangent to the boundary and its closure is a periodic orbit. In the former case, there exists a neighborhood of V in which $\mathcal{X}$ has no periodic orbit (since we have always supposed that a l.p.s. is entirely contained in the original phase space S). In the latter case, we can still apply the argument above to find a transversal section and a bound for the number of limit cycles by leaves in a neighborhood of $\Gamma_G^\pm$.

Finally, we are ready to demonstrate that the blow-up operation reduces the question of finite cyclicity of a l.p.s. $\Omega \subset \Gamma$, of the original family X_λ, to the same question for the new family $\tilde{X}_{\tilde{\lambda}}$. Below, we will use the following result:

Lemma 4.2. (see [**R1**]) *Let Y_τ be an analytic family as defined in the introduction. Suppose that there exists a convergent sequence of parameter values $\tau_k \to \tau_0$, such that the corresponding vector fields Y_{τ_k} have an increasing number of limit cycles. Then, there exists a l.p.s. Ω at τ_0 with infinite cyclicity.*

Now, our result can be stated as follows:

Proposition 4.2. *If the analytic family $\tilde{X}_{\tilde{\lambda}}$ has the finite cyclicity property, then all the l.p.s. of the original family X_λ contained in Γ have finite cyclicity.*

Proof. We are going to suppose that there exists a l.p.s. $\Omega \subset \Gamma$ of X_λ at λ_0 which has an infinite cyclicity. This means there exists a sequence $\lambda_k \to \lambda_0$, each X_{λ_k} having a set $C_k = \{\gamma_k^1, \dots, \gamma_k^{n_k}\}$ of limit cycles such that $C_k \to \Omega$, and $n_k \to \infty$ as $k \to \infty$. Let us denote by $\{L_k\}$ the sequence of leaves in $\mathcal{F}$ corresponding to the sequence of parameter values $\{\lambda_k\}$.

Now, we look at the inverse image of this sequence by the blow-up map $\pi\colon \tilde{T} \to T$. If $L_k \cap M = \varnothing$, then $\pi^{-1}(L_k)$ is a well-defined regular leaf $\tilde{L}_k \in \tilde{\mathcal{F}}$. Otherwise, since $M \subset \mathcal{Z}(\mathcal{X})$, it suffices to take $\pi^{-1}(L_k - M)$, because it is clear that $C_k \cap M = \varnothing$. To simplify the notation, we will also denote the inverse image in this case by $\tilde{L}_k$.

So, we obtain a sequence of leaves $\{\tilde{L}_k\}$ accumulating at the union $\tilde{M} \cup \tilde{L}_{\lambda_0}$, such that in each leaf $\tilde{L}_k$ there is a set of limit cycles $\tilde{C}_k$, with an increasing cardinality, and which approaches the exceptional divisor $\tilde{M}$ as $k \to \infty$.

By proposition 4.1, we can find a $\rho_2 > 0$ in the definition (14) of G such that, for all k sufficiently big, the subset of the limit cycles in $\tilde{C}_k$ which intersects G, has a bounded cardinality. Choosing a ρ_1 in the interval $(\rho_2, 1)$, we consider the subset $\tilde{C}_k^F \subset \tilde{C}_k$ of limit cycles entirely contained in the set F. Obviously, the cardinality of the set $\tilde{C}_k^F$ tends to infinity as $k \to \infty$.

Since the sets $\tilde{C}_k^F$ are entirely contained in the family chart, there exists a sequence $\{\tilde{\lambda}_{j(k)}\}$ in the parameter space $\tilde{P}$ of the family $\tilde{X}_{\tilde{\lambda}}$, such that, for each k, $\tilde{C}_k^F$ is in the set of limit cycles of the vector field $\tilde{X}_{\tilde{\lambda}_{j(k)}}$. The sequence $\{\tilde{\lambda}_{j(k)}\}$ accumulates on the compact subset $(\tau, \tilde{\varphi}, \tilde{\psi}, \lambda) \in \{0\} \times S^{r+s+1} \times \{\lambda_0\}$ of the parameter space $\tilde{P}$. So, there exists a subsequence of $\{\tilde{\lambda}_{j(k)}\}$ (which we will denote simply by $\{\tilde{\lambda}_i\}$) which tends to a definite point $\tilde{\lambda}_0 \in \{0\} \times S^{r+s+1} \times \{\lambda_0\}$. The corresponding vector fields $\tilde{X}_{\tilde{\lambda}_i}$ have each a set of limit cycles $\tilde{C}_i$ whose cardinality tends to infinity as $\tilde{\lambda}_i \to \tilde{\lambda}_0$. This implies, by the lemma 4.2, that the family $\tilde{X}_{\tilde{\lambda}}$ has a l.p.s. $\tilde{\Omega} \subset \tilde{S}$ of infinite cyclicity for the value $\tilde{\lambda}_0$ of the parameter (we observe that we can apply the lemma 4.2 to the family $\tilde{X}_{\tilde{\lambda}}$ because its phase space can be realized as an open set of $\mathbf{S}^2$). □

Finally, we state the second main theorem:

Theorem 2. *Any analytic unfolding (X_λ, Γ) of a N-curve can be desingularized (in the sense of [**D-R**]) in a collection of unfoldings $(\hat{X}_{\hat{\lambda}}, \hat{\Gamma})$, where $\hat{\Gamma}$ is an isolated singularity, a periodic orbit, a graphic or a degenerate graphic.*

Proof. It follows easily from the above results. □

The following corollary is now clear:

Corollary. *In order to prove the finite cyclicity conjecture to analytic families under condition (H), it suffices to prove the conjecture for families not containing N-curves.*

Proof. As we have seen above, the finite cyclicity property is preserved by the operations of desingularization. □

Bibliography

[**B**] Bogdanov R., *Versal deformations of a singular point of a vector field on the plane in the case of zero eigenvalues*, Seminaria Petrovsky 1976, Sel.Math.Sov. 1 no. 4 (1981).

[**D-R**] Denkowska Z., Roussarie R., *A method of desingularization for analytic two-dimensional vector fields families*, Bol. Soc. Bras. Mat. 22 no. 1 (1991), 93-126.

[**El.Mor-R**] El Morsalani M., Roussarie R., *Techniques in the Theory of Local Bifurcations: Cyclicity and Desingularization*, in Bifurcations and Periodic Orbits of Vector Fields, D.Scholomiuk ed., NATO ASI Series, Series C, Vol.408, 1993.

[F-P] Françoise J. P., Pugh C. C., *Keeping Track of Limit Cycles*, Journal of Differential Equations 65 (1986), 139-157.

[G] Gabrielov A. M., *On projections of semianalytic sets*, Funct. Anal. Appl. 2 no. 4 (1968), 18-30.

[G-R] Gunning R. C., Rossi H., *Analytic Functions of Several Complex Variables*, Prentice-Hall, 1965.

[I-Y] I'llyashenko Y., Yakovenko S., *Finite cyclicity of elementary polycycles in generic families*, to appear in Advances in Soviet Mathematics.

[M] Mourtada A., *Ciclicité finie des polycycles hyperboliques des champs de vec-teurs du plan, Algorithme de finitude*, Annales de l'Institut Fourier, tome 41 fascicule 3 (1991), 719-753.

[P-T] Palis J., Takens F., *Topological equivalence of normally hyperbolic dynamical systems*, Topology 16 (1977), 335-345.

[R1] Roussarie R., *Note on the Finite cyclicity property and Hilbert's 16th Problem*, Lecture Notes in Math. no.1331, Dynamical Systems – Valparaiso 1986, R.Bamon, R.Labarca, J.Palis ed. (1988).

[R2] Roussarie R., *On the number of limit cycles which appear by perturbation of separatrix loops of planar vector fields*, Bol. Soc. Bras. Mat. 17 (1986), 67-101.

[R3] Roussarie R., *Cyclicité finie des lacets et des points cuspidaux*, Nonlinearity 2 (1989), 73-117.

[T] Takens F., *Forced oscillations and bifurcations*, in Applications of Global Analysis I, Communications of Math. Inst. Rijksuniversitat, Ultrecht 3, 1974.

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