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Centralizers of Expanding Maps on Tori

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Abstract. We prove here that the elements of an open and dense subset of expanding maps on tori have trivial centralizers; i.e., the maps commute only with their own powers.

1. Introduction

Let $Imm(T^n)$ be the space of C^∞ immersions of the torus T^n (i.e. mappings $f: T^n \to T^n$ for which $D_x f$ is invertible for all $x \in T^n$) endowed with the C^∞ topology. For $f \in Imm(T^n)$, it *centralizer* $Z(f)$ in $Imm(T^n)$ is defined as the set of elements that commute with f. We say that f has *trivial centralizer* if $Z(f)$ is reduced to the iterates $\{f^n, n \in \mathbb{N}\}$ of f.

An immersion $f: T^n \to T^n$ is *expanding* if there exists $\lambda > 1$ such that $\|D_x f(v)\| > \lambda \ \|v\|$ for all $x \in T^n$ and $v \in T_x(T^n)$. The expanding maps form an open subset $\text{Exp}(T^n)$ of $Imm(T^n)$ and by a result of Shub [**8**], they are structurally stable.

The objective of this paper is to prove the following result:

Theorem. *For an open and dense subset of* $\text{Exp}(T^n)$, *the centralizer is trivial.*

This result is a version to expanding maps of a theorem of Palis-Yoccoz [**6**], who showed the triviality of the centralizer for an open and dense subset of Anosov diffeomorphisms of T^n.

In the proof of the theorem, we adapt to the context of expanding maps, the techniques of [**6**] and [**7**]. Besides some combinatorial reason-

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ing, perhaps the novelty here are the arguments in Section 5. We use Markov partitions to prove a version to expanding maps of Proposition 3 of [6], which is fundamental in the proof of the theorem.

When T^n is the circle S^1, the theorem was proved in a previous work of the author [3]. In that paper we also apply a result of [2] to show the triviality of the centralizer for an open and dense subset of C^∞ immersions of S^1. This result is an extension to immersions of a similar theorem of Kopell [5] to diffeomorphisms of S^1.

2. Preliminaries

We collect some basic concepts and recalling the arguments of localization of [7].

Formally we will think of the torus as $\mathbb{R}^n/\mathbb{Z}^n$ and use π to denote the canonical projection. Thus every continuous map f of the torus has a *lift*, i.e. a continuous map $\tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$f \circ \pi = \pi \circ \tilde{f}.$$

Let $\mathcal{U}$ be the set of $f \in \mathrm{Exp}(T^n)$ which verify:

(i) The spectra of Df at the different fixed points are distinct;

(ii) At each fixed point p of f, the eigenvalues of D_pf are simple and D_pf is non-resonant.

Clearly $\mathcal{U}$ is an open and dense subset of $\mathrm{Exp}(T^n)$. Let $\mathcal{U}^*$ be the set of $f \in \mathcal{U}$ such that $f(0) = 0$.

For $f \in \mathcal{U}^*$, let $A(\tilde{f}) = D_0\tilde{f}$. By using a parametric version of Sternberg's, linearization theorem [1], we have that there exists a C^∞ neighbourhood V of f in $\mathcal{U}^*$ such that for any $g \in V$, there is a diffeomorphism $H(g): \mathbb{R}^n \to \mathbb{R}^n$, with the following properties:

(1) $H(g)$ depends continuously on g in the C^∞ topology on compact subsets of $\mathbb{R}^n$;

(2) There exist natural numbers r, s with $r + 2s = n$, coordinates $x_1, \dots, x_{r+s} \in \mathbb{R}^r \times \mathbb{C}^s$ in $\mathbb{R}^n$, and a continuous map

$$\lambda = (\lambda_1, \dots, \lambda_{r+s}): V \to (\mathbb{R}^*)^r \times (\mathbb{C}^* - \mathbb{R}^*)^s$$

such that $A(\tilde{g}) = H(g)^{-1} \circ \tilde{g} \circ H(g)$ and $A(\tilde{g})$ is non-resonant and

diagonalizable in the coordinates (x_i) with eigenvalues $\lambda_i(g)$.

Denote by $Z(A(\tilde{f}))$ the centralizer group of $A(\tilde{f})$ in the space of the linear automorphisms of $\mathbb{R}^n$, $Gl(\mathbb{R}^n)$. Observe that if $h \in Z(f)$, then $h(0) = 0$ [because of property (1)] and so

$$\tilde{h}(\mathbb{Z}^n) \subset \mathbb{Z}^n.$$

Moreover by (ii), $H(f)^{-1} \circ \tilde{h} \circ H(f) \in Z(A(\tilde{f}))$, and the map

$$h \mapsto H(f)^{-1} \circ \tilde{h} \circ H(f)$$

is injective.

By [7], if V is a small C^∞ neighbourhood of f in $\mathcal{U}^*$, then for any $g \in V$, $Z(A(\tilde{g}))$ is identified with the Lie group

$$Z_{r,s} = \mathbb{R}^{r+s} \times (\mathbb{Z}_2)^r \times (S^1)^s,$$

and $Z(A(g))/A(g))$ is identified with a fixed quotient $Z_0 = Z_{r,s}/(\varepsilon)$ of $Z_{r,s}$. Let Z_1 be the maximal compact subgroup of Z_0, and let

$$\theta: Z_{r,s} \to Z_0$$

the canonical projection.

The *compact part* of $Z(f)$ is defined as the set of $h \in Z(f)$ such that

$$\overline{h} = \theta(H^{-1}(f) \circ \tilde{h} \circ H(f)) \in Z_1.$$

For $f \in \mathcal{U}^*$, we define

$$Z_1(f) = \{\overline{h} \in Z_1;\ \theta_1(\overline{h})[H^{-1}(f)(\mathbb{Z}^n)] \subset H^{-1}(f)(\mathbb{Z}^n)\}.$$

It is clear that if $h \in Z(f)$, then $\overline{h} \in Z_1(f)$. Moreover since $H^{-1}(Z^n)$ is discrete, $Z_1(f)$ is a closed subgroup of Z_1. From this and by using the same arguments as in Lemmas 5.2 and 5.3 of [7] it can be shown the following result.

Proposition 2.1. *There is an open and dense subset $\mathcal{U}_1^*$ of $\mathcal{U}^*$ such that for any $f \in \mathcal{U}_1^*$ the centralizer $Z(f)$ has trivial compact part.*

3. Basic Results and Proof of the Theorem

We establishing some basic material on centralizers of expanding maps.

Lemma 3.1. *Let $f \in \text{Exp}(T^n)$ and $g \in Z(f)$. If $g \circ f^k = f^l$ for some $k, l \in \mathbb{N}$, then g is a power of f.*

Proof. We claim that $l \geq k$. Otherwise, if $y \in T^n$ and $x \in f^{-l}(y)$, then

$$g \circ f^{k-l}(y) = g \circ f^k(x) = f^l(x) = y.$$

Hence $g \circ f^{k-l} = f^{k-l} \circ g = id_{T^n}$. This is a contradiction because f^{k-l} is expanding. Therefore $l \geq k$ and by the same argument above, we have that $g = f^{l-k}$ and the lemma is proved.

Keeping the same notation as Section 2, let $f \in \mathcal{U}^*$. Denote by K_i the x_i-axis, $1 \leq i \leq r+s$. Let J be a non trivial subset of $\{1, \ldots, r+s\}$; that is, J is neither the empty set nor the whole set. We then denote by $W_J = W_J(f)$ the image of $\prod_{i \in J} K_i = K_J$ under $H(f)$.

Let $L = L(f) \in Gl_n(\mathbb{Z})$ which is induced by f in homology. By [8], there is a neighbourhood V_f of f in $\mathcal{U}^*$ such that for any $g \in V_f$ there exists an unique homeomorphism h_g of T^n satisfying $h_g \circ g = L(f) \circ h_g$ and $h_g(0) = 0$. Moreover h_f depends continuously on f. We denote by $\tilde{h}_f$ the lift of h_f.

Lemma 3.2. *If $g \in \text{Exp}(T^n) \cap Z(f)$, then $h_f \circ g \circ h_f^{-1} \in Gl_n(Z)$.*

Proof. Let $L(h_f)$, $L(g)$ be the elements of $Gl_n(Z)$ which are induced in homology by h_f and g respectively. Denote

$$h = h_f \circ g \circ h_f^{-1} \quad \text{and} \quad B = L(h_f) \circ L(g) \circ L^{-1}(h_f).$$

Since

$$\sup_{\mathbb{R}^n} \left\| \tilde{h}_f(y) - L(h_f)(y) \right\| < +\infty,$$

we have that

$$\sup_{\mathbb{R}^n} \left\| B^{-1} \circ \tilde{h}(y) - (y) \right\| < +\infty.$$

This property together with the fact that B commutes with $L(f)$ imply that, for any $y \in \mathbb{R}^n$,

$$\sup_n \left\| L^n(f)(B^{-1}\tilde{h}(y) - y) \right\| = \sup_n \left\| B^{-1}\tilde{h}(L^n(f)(y)) - L^n(f)(y) \right\| < \infty.$$

As $L^n(f)$ is expanding, this implies that $B^{-1} \circ \tilde{h}$ is the identity, and so $h \in Gl_n(\mathbb{Z})$.

The following result which is basic to the proof of the theorem relates the conjugacies $H(f)$ and $\tilde{h}_f$.

Proposition 3.3. *There is an open and dense subset $\mathcal{U}_2^*$ of $\mathcal{U}_1^*$ (endowed with the C^∞ topology) such that for any $f \in \mathcal{U}_2^*$ and any non trivial subset J of $\{1, \ldots, r+s\}$, the dimension of the linear subspace of $\mathbb{R}^n$ generated by $\tilde{h}_f(W_J(f))$ is strictly greater than the dimension of K_J.*

Observe that the openess of $\mathcal{U}_2^*$ follows from the continuity of the maps $f \to H(f)$ and $f \to \tilde{h}_f$.

Before proving density, we show how to deduce the theorem from proposition 3.3 and similar arguments of [6].

Proof of Theorem. For $p \in T^n$, we denote by α_p the rotation $\alpha_p(x) = x - p$, $x \in T^n$. Let

$$\mathcal{U}_2 = \{f \in \mathcal{U};\ \alpha_p \circ f \circ \alpha_p^{-1} \in \mathcal{U}_2^* \text{ for some fixed point } p \text{ of } f\}.$$

By Proposition 3.3, $\mathcal{U}_2$ is an open and dense subset of $\mathcal{U}$. Hence, to prove the theorem it is sufficient to show that the elements of $\mathcal{U}_2^*$ have trivial centralizers.

Let $f \in \mathcal{U}_2^*$, $g \in Z(f)$. For some $l \in \mathbb{N}$, $g \circ f^l$ is expanding. Thus by lemma 3.1, we can assume that g is expanding. Denote by $\mu_1, \ldots, \mu_{r+s}$, the eigenvalues of $H^{-1} \circ \tilde{g} \circ H(f)$. Then, as g does not belong to the compact part of $Z(f)$, the numbers

$$\frac{\log|\mu_i|}{\log|\lambda_i(f)|}$$

are not equal. Then there exist $k, l \in \mathbb{Z}$ such that $D_0(\tilde{g}^k \circ \tilde{f}^l)$ is hyperbolic. Since by lemma 3.2,

$$B = \tilde{h}_f \circ \tilde{g} \circ \tilde{h}_f^{-1} \in Gl_n(\mathbb{Z}),$$

we conclude that

$$B^k \circ L^l(f) = \tilde{h}_f \circ \tilde{g}^k \circ \tilde{f}^l \circ \tilde{h}_f^{-1}$$

is hyperbolic. This implies that there exists a non trivial subset J of $\{1, \ldots, r+s\}$ such that $W_J(f)$ and $\tilde{h}_f(W_J(f))$ are the unstable manifolds of $\tilde{g}^k \circ \tilde{f}^l$ and $B^k L^l$ respectively. Thus the dimension of $\tilde{h}_f(W_J(f))$ is equal to the dimension of K_J. This contradicts the definition of $\mathcal{U}_2^*$ and shows that any $f \in \mathcal{U}_2^*$ has trivial centralizer.

4. Proof of Proposition 3.3

To prove that $\mathcal{U}_2^*$ is dense in $\mathcal{U}_1^*$, we consider some $f \in \mathcal{U}_1^*$ and non-trivial subset J of $\{1, \dots, r+s\}$, such that $\tilde{h}_f(W_J(f))$ is equal to a non-trivial L-invariant linear subspace of $\mathbb{R}^n$. We will show that there exist arbitrarily small perturbations f' of f in $\mathcal{U}_1^*$ such that the linear subspace of $\mathbb{R}^n$ generated by $\tilde{h}_{f'}(W_J(f'))$ is equal to $\mathbb{R}^n$.

We choose a minimal non-zero L-invariant subspace E of $\tilde{h}_f(W_J(f))$; the dimension of E is 1 or 2.

The proof of Proposition 3.3 is based on the following result, which is an adaptation of Proposition 3 of [**6**] to pre-images of 0 under L.

Proposition 4.1. *Let K be a neighbourhood of 0 in $\mathbb{R}^n$ such that*

$$L_K = L(f)/\pi(K)$$

is injective. Then, there exist $\delta_0 > 0$, and, for any $\epsilon_0 > 0$, points $z_1, \dots, z_n$ in E, $y_1, \dots, y_n \in \mathbb{R}^n$ satisfying the following properties:

(i) *$z_i \in K$ and $B(\pi(z_i), \delta_0) \subset \pi(K)$ for $1 \le i \le n$, where $B(x, r)$ denotes the closed ball in T^n with center x and radius r for the euclidean norm $\| \ \|$.*

(ii) *$\|z_i - y_i\| < \epsilon_0$ for $1 \le i \le n$;*

(iii) *$y_1, \dots, y_n$ is a basis of $\mathbb{R}^n$;*

(iv) *There exists $m \in \mathbb{N}$ such that for every $1 \le i \le n$, $L^m(\pi(y_i)) = 0$ in T^n;*

(v) *$L^m(\pi(y_i)) \notin B(\pi(z_j), \delta_0) = B_j$ for $m > 0$, $\quad 1 \le i, j \le n$;*

(vi) *$B_i \cap L_K^m(B_j) = \varnothing$ for $m < 0$, $\quad 1 \le i, j \le n$;*

(vii) *$B_i \cap B_j = \varnothing$ for $i \neq j$.*

Before proving this proposition, we finish the proof of Proposition 3.3, by adapting the arguments of [[**6**], Proposition 2].

We choose $r > 0$ such that for the set

$$K_1 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n; \quad |x_i| < r \quad \text{for} \quad 1 \le i \le n\},$$

we have that $K_1 \subset K$ and $\tilde{\pi} = \pi/K_1$, $f_{K_1} = f/\pi(K_1)$ are injective.

Let

$$K_2 = \{x \in \mathbb{R}^n; |x_i| \le \frac{r}{2}, \quad \text{for} \quad 1 \le i \le n\}$$
$$K = \tilde{h}_f(K_2)$$

and let $\delta_0 = \delta_0(K)$ be as in Proposition 4.1. Let $\delta \in (0, \frac{r}{2})$ be such that $\|y - z\| \le 2\delta$ for $y, z \in K_1$, implies that $h_f \circ \tilde{\pi}(y) \in B(h_f \tilde{\pi}(z), \delta_0)$.

Let $\mathcal{W}$ be a neighbourhood of f in $\mathcal{U}_1^*$. We fix some C^∞ function η: $[0, +\infty] \to [0, 1]$, satisfying $\eta(t) = 0$ for $t \ge 1$ and $\eta(t) = 1$ for $t \le \frac{1}{2}$. By [**6**], Section 6, there exists $\epsilon = \epsilon(\delta, \mathcal{W}) < \frac{\delta}{2}$ such that if $q_1, \dots, q_n \in K_2$, $q'_1, \dots, q'_n \in K_1$ satisfy

- $\|q_i - q_j\| \ge 2\delta$ for $i \neq j$;
- $\|q'_i - q_i\| \le \epsilon$ for $1 \le i \le n$.

Then the C^∞ vector field X in $\mathbb{R}^n$ defined by

$$X(y) = \sum_{j=1}^{n} \eta(\delta^{-1} \ \|y - q_j\|)(q'_j - q_j)$$

generates a flow $(F_t)_{t \in \mathbb{R}}$ such that $f_t = \tilde{\pi} \circ F_t \circ \tilde{\pi}^{-1} \circ f$ belongs to $\mathcal{W}$ for $0 \le t \le 1$. Observe that X has support contained in the interior of K_1, and $F_1(q_i) = q'_i$ for $1 \le i \le n$.

By uniform continuity, there exists $\epsilon_0 > 0$ such that if $y, z \in \mathbb{R}^n$ satisfy $\tilde{h}_f^{-1}(z) \in K_2, d(y, z) < \epsilon_0$, then

$$\tilde{h}_f^{-1}(y) \in K_1 \quad \text{and} \quad \left\|\tilde{h}_f^{-1}(y) - \tilde{h}_f^{-1}(z)\right\| < \epsilon.$$

For this ϵ_0 we apply Proposition 4.1 and get points $y_i, z_i \in \mathbb{R}^n$, $1 \le i \le n$, satisfying properties (i)-(vii). We now consider the flow $(F_t)_{t \in \mathbb{R}}$ defined above with data $q_i = \tilde{h}_f^{-1}(z_i)$ and $q'_i = \tilde{h}_f^{-1}(y_i)$, we may assume that for $0 \le t \le 1$, f_t induced L in homology, and $h_t \circ h_f^{-1}$ is sufficiently near of the identity, where h_t denotes the global conjugacy of f_t and L with $h_t(0) = 0$ (so that $h_0 = h_f$).

We will prove that $f_1 \in \mathcal{U}_2^*$, and more precisely that the linear subspace of R^n generated by $\tilde{h}_1(W_J(f_1))$ is $\mathbb{R}^n$.

With the notation of Proposition 4.1, we have that for $0 \le t \le 1$, the image under h_f of the support of $\tilde{\pi} \circ F_t \circ \tilde{\pi}^{-1}$ is contained in $\bigcup_{i=1}^{n} B_i$ (by definition of δ); by (vi), 0 does not belong to the support of $\tilde{\pi} \circ F_t \circ \tilde{\pi}^{-1}$ and the manifolds $W_J(f)$, $W_J(f_1)$ coincide in a neighbourhood of 0 in $\mathbb{R}^n$. By (v), we have; for $1 \leq i \leq n$, $0 \leq t \leq 1$ and $m \geq 0$:

$$f_t^m(h_f^{-1}\pi(y_i)) = f^m(h_f^{-1}\pi(y_i)) = h_f^{-1}L^m(\pi(y_i)),$$

hence, by (iv), $f_t^m(h_f^{-1}\pi(y)i)) = 0$ for some $m \geq 0$. This implies that $L^m(h_t(h_f^{-1}(\pi(y_i)) = 0$. Since the set $L^{-m}(0)$ is finite and h_t depends continuously on t,

$$h_1 \circ h_f^{-1}(\pi(y_i)) = h_0 \circ h_f^{-1}(\pi(y_i)) = \pi(y_i).$$

Using the fact that $h_1 \circ h_f^{-1}$ is near of the identity, we obtain that

$$\tilde{h}_1(q_i') = y_i. \qquad (*)$$

On the other hand, by (vi), we have that

$$(f_t/\pi(K_1))^m(h_f^{-1}(\pi(y_i)) = f_{K_1}^m \tilde{\pi} F_t^{-1} \tilde{\pi}^{-1} h_f^{-1}(\pi(y_i))$$

for $1 \leq i \leq n$, $0 \leq t \leq 1$, and $m < 0$.

By construction of F_1,

$$\tilde{\pi} F_1 \tilde{\pi}_{-1} h_f^{-1} h_f^{-1}(\pi(z_i)) = h_f^{-1}(\pi(y_i)),$$

hence

$$(f_1/\pi(K_1))^m(h_f^{-1}(\pi(y_i))) = f_{K_1}^m(h_f^{-1}\pi(z_i)),$$

and thus

$$\tilde{f}_1^m(q_i') = \tilde{f}_1^m \circ \tilde{h}_f^{-1}(y_i) = \tilde{f}^m(\tilde{h}_f^{-1}(z_i)) = \tilde{f}^m(q_i)$$

for $1 \leq i \leq n$ and $m < 0$.

Since $\tilde{f}^m(q_i) \in W_J(f)$ and $W_J(f)$, $W_J(f_1)$ coincide near 0 in $\mathbb{R}^n$, we have that $q_i' \in W_J(f_1)$. This together with $(*)$ and property (iii) imply that $f_1 \in \mathcal{U}_2^*$. This concludes the proof of Proposition 3.3.

5. Proof of Proposition 4.1

We first recall the definition of Markov partitions for expanding maps. For $f \in \mathrm{Exp}(T^n)$, we say that a collection $\mathcal{R} = \{R_1, \ldots, R_k\}$ of open subsets of T^n form a *Markov partition* for f with diameter r if

(a) $R_i \cap R_j = \varnothing$ for $i \neq j$ and $T^n = \bigcup_{i=1}^n \overline{R_i}$, and

(b) diameter $R_i < r$ for $1 \leq i \leq k$, and if $f(R_j) \cap R_i \neq \varnothing$ for some $1 \leq j \leq k$, then $f(R_j) \supset R_i$.

We have that there exist arbitrarily small Markov partitions for f (see [4]).

Now we prove the Proposition 4.1. Let K be a neighbourhood of 0 in $\mathbb{R}^n$ such that π/K and $L_K = L/\pi(K)$ are injective. We may assume that K is a ball in $\mathbb{R}^n$ with center 0. We choose a 1-dimensional linear subspace E_1 of E. Denote $\tilde{E}_1 = E_1 \cap K$ and $D = K - L^{-1}(K)$.

We consider a Markov partition $\mathcal{R}$ for L in T^n satisfying the following conditions:

(1) diameter $\mathcal{R} < d(\partial K, \partial(L_K^{-1}(K)))$, where ∂X denotes the boundary of X in T^n,

(2) There exist $V_0, V_1, \ldots, V_n \in \mathcal{R}$ such that $0 \in V_0$, and for $1 \leq i \leq n$, $V_i \subset D$, $V_i \cap \tilde{E}_1$ is a proper line segment of $\tilde{E}_1$ and, $d(\overline{V}_i, \overline{V}_j) > 0$ for $i \neq j$.

Let $s < 0$ be an integer such that for all $1 \leq i \leq n$, $L^s(V_i) \subset V_0$. By (2) we can choose a line segment $J_i = J_i(V_i)$ of $L^{s-1}(\tilde{E}_1)$ such that $J_i \subset L^{s-1}(D)$, $V_i \cap \tilde{E}_1$ is a proper line segment of $L^{-s+1}(J_i)$, and $d(\overline{J}_i, \overline{J}_j) > 0$ for $i \neq j$. This implies that there exists $\delta_0 > 0$ sufficiently small so that if V_i' denotes the δ_0-neighbourhood of J_i in T^n then:

(3) $V_i' \subset L_K^{s-1}(D)$ for $1 \leq i \leq n$;

(4) $\overline{V_i'} \cap \overline{V_j'} = \varnothing$ for $i \neq j$,

(5) If $J_i' \subset V_i'$ is a line segment parallel and with the same length to J_i, then $L^{-s+1}(J_i')$ intersects ∂V_i in two points for $1 \leq i \leq n$.

Let ϵ_0 be given, and let $1 \leq i \leq n$. Let S_{ϵ_0} be the ball in the orthogonal subspace of $L^{s-1}(E_1)$ with center 0 and radius ϵ_0. We may assume that $S_{\epsilon_0} \times J_i \subset V_i'$. Let $J_i' \subset S_{\epsilon_0} \times J_i$ be a line segment satisfying (5), and let $C_i \subset S_{\epsilon_0} \times J_i$ be an arbitrarily cylinder in $\mathbb{R}^n$ with generator J_i'. We claim that there exist a point y in C_i such that in T^n, $L^m(y) = 0$ for some $m \in \mathbb{N}$, and $L^j(y) \notin V_i'$ for $1 \leq i \leq n$, $1 \leq j \leq m$. In fact, let $j \in \mathbb{N}$ such that in T^n, $L^m(C_i) \cap V_l' = \varnothing$ for $1 \leq m < j$,

$1 \leq l \leq n$, and $L^j(C_i) \cap V'_l \neq \varnothing$ for some $1 \leq l \leq n$. Then by (5) and the definition of Markov partition, $L^j(C_i)$ intersects $L_K^{s-1}(\partial K)$ and $\partial L_K^s(K)$, so $L^{j-s}(C_i)$ intersects ∂K and $\partial(L_K^{-1}(K)$ in T^n. Hence there exists an open subset C'_i of C_i such that $L^{j-s}(C'_i)$ is a cylinder in $\mathbb{R}^n$, and in T^n, $L^m(C'_i) \cap V'_l = \varnothing$ for all $1 \leq m < j-s$, $1 \leq l \leq n$, and $L^{j-s}(C'_i)$ intersects ∂K and $\partial(L_K^{-1}(K))$. This fact together with properties (b), (1), and the fact that L is expanding imply that there exist $m \in \mathbb{N}$ and open subset C''_i of C'_i such that in T^n, $L^p(C''_i) \cap V'_l = \varnothing$ for all $1 \leq p \leq m$, $1 \leq l \leq n$, $L^m(C''_i) = W$, where $W \subset D$ and $W \in \mathcal{R}$. Now if q is the first natural number such that $L^q(W)$ intersects some V'_l, $1 \leq l \leq n$, then by definition of Markov partition; $0 \in V_0 \subset L^{m+q}(C''_i)$, so the claim is proved.

Now we choose $y_q \in S_{\epsilon_0} \times J_1$ satisfying the claim. By induction, suppose that there exist $y_i \in S_{\epsilon_0} \times J_i$, $1 \leq i < l < n$, such that y_i satisfying the claim above and $y_1, \ldots, y_{l-1}$ are linearly independent in $\mathbb{R}^n$. Then there exist a line segment $J'_l \subset S_{\epsilon_0} \times J_l$ satisfying (5), and a cylinder $C_l \times S_{\epsilon_0} \times J_l$ with generator J'_l such that C_l does not intersects the linear subspace of $\mathbb{R}^n$ generated by $y_1, \ldots, y_{l-1}$. Hence we can choose $y_l \in C_l$ satisfying the claim and such that $y_1, \ldots, y_l$ are linearly independent in $\mathbb{R}^n$. Therefore there exist $y_i \in S_{\epsilon_0} \times J_i$, $l \leq i \leq n$ satisfying the claim above and such that $\{y_1, \ldots, y_n\}$ is a basis of $\mathbb{R}^n$. By definition of V'_i, we can choose points $z_i \in J_i$, $1 \leq i \leq n$ such that $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ satisfy the conditions of the Proposition 4.1.

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