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Intersecting Self-Similar Cantor Sets

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Abstract. We define a self-similar set as the (unique) invariant set of an iterated function system of certain contracting affine functions. A topology on them is obtained (essentially) by inducing the C^1-topology of the function space. We prove that the measure function is upper semi-continuous and give examples of discontinuities. We also show that the dimension is not upper semicontinuous. We exhibit a class of examples of self-similar sets of positive measure containing an open set. If C_1 and C_2 are two self-similar sets C_1 and C_2 such that the sum of their dimensions $d(C_1) + d(C_2)$ is greater than one, it is known that the measure of the intersection set $C_2 - C_1$ has positive measure for almost all self-similar sets. We prove that there are open sets of self-similar sets such that $C_2 - C_1$ has arbitrarily small measure.

1. Introduction

This note describes some results for and some examples of self-similar sets (a precise definition will be given in section 2). More particularly, we investigate measure theoretic properties of the intersection of two such sets.

At a recent conference Jacob Palis [**Pa**] and Gustavo Moreira [**Mo**] considered self-similar Cantor sets S_1 and S_2 contained in $\mathbb{R}$, and asked measure theoretic questions about their difference set

$$S_2 - S_1 = \{t \in \mathbb{R} \mid \exists x_i \in S_i \text{ with } t = x_2 - x_1\} \quad .$$

Moreira and Palis were led to these questions by investigating how common hyperbolic behavior is for generic diffeomorphisms of a surface. This train of thought can be found in [**PT**] (see also [**PT87**]). These ideas lead so naturally to questions about Cantor sets, that it is useful to present a survey of them. Because of its brevity, the outline we present is necessarily very sketchy. For details we refer to these two

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sources.

Let $\phi_\mu : M \to M$ be a one parameter family of C^2 diffeomorphisms of a closed surface M. Assume that for $\mu < 0$, the non-wandering set Ω_μ is persistently hyperbolic, that is: the non-wandering set every $\tilde{\phi}$ sufficiently (C^2-)close to ϕ is hyperbolic. It follows that for $\mu < 0$ the maps $\tilde{\phi}_\mu|_{\Omega_\mu}$ are all topologically conjugate.

The stable and unstable manifolds $W^s(x)$ and $W^u(x)$ at $x \in \Omega_\mu$ (for $\mu < 0$) are smooth leaves. The union of these leaves may be a complicated set (in the transversal direction, see figure 1.1(b). Denoting the union of the unstable leaves by $\mathcal{F}^u(\Omega)$ and the stable one by $\mathcal{F}^s(\Omega)$, we have that Ω is the intersection of the two bundles. The leaves $W^u(x)$ and $W^s(x)$ for $x \in \Omega$ foliate $\mathcal{F}^s(\Omega) \cup \mathcal{F}^u(\Omega)$. In fact, we may smoothly extend this foliation to a neighborhood of $\mathcal{F}^s(\Omega) \cup \mathcal{F}^u(\Omega)$.

The way in which one dimensional Cantor sets arise is crucial for the construction of our examples in later sections. Let ℓ be a smooth curve tranversal to the unstable foliation. Then each point of Ω projects (along the unstable foliation) to a point on ℓ. Define

$$C_1(\ell) = \mathcal{F}^u(\Omega) \cap \ell$$

Similarly, for the projection along the stable manifolds we have:

$$C_2(\ell') = \mathcal{F}^s(\Omega) \cap \ell' \quad .$$

Suppose that in fact C_1 (and C_2) are contained in a compact segment K_1 of ℓ (and K_2 of ℓ'). Choose $\ell = W^s(p)$ and note that $\phi^{-1}|_{C_1}$ is expanding and maps C_1 into itself. In fact, at p the derivative of this map is precisely the reciprocal of the 'stable' eigenvalue of $D\phi|_p$. We will model this map by a collection of expanding maps sending disjoint intervals in K_1 onto K_1. The set C_1 is modelled by the largest compact invariant set of this map (called the presentation function, borrowing this from [**Fe**]). The same can be done for C_2. Sets that can be constructed this way are called 'dynamically defined' ([**PT**]). They can be topologized by inducing the $C^{1+\epsilon}$-topology of the space of presentation functions. (This is similar to what was done in [**Su**] and [**Fe**].)

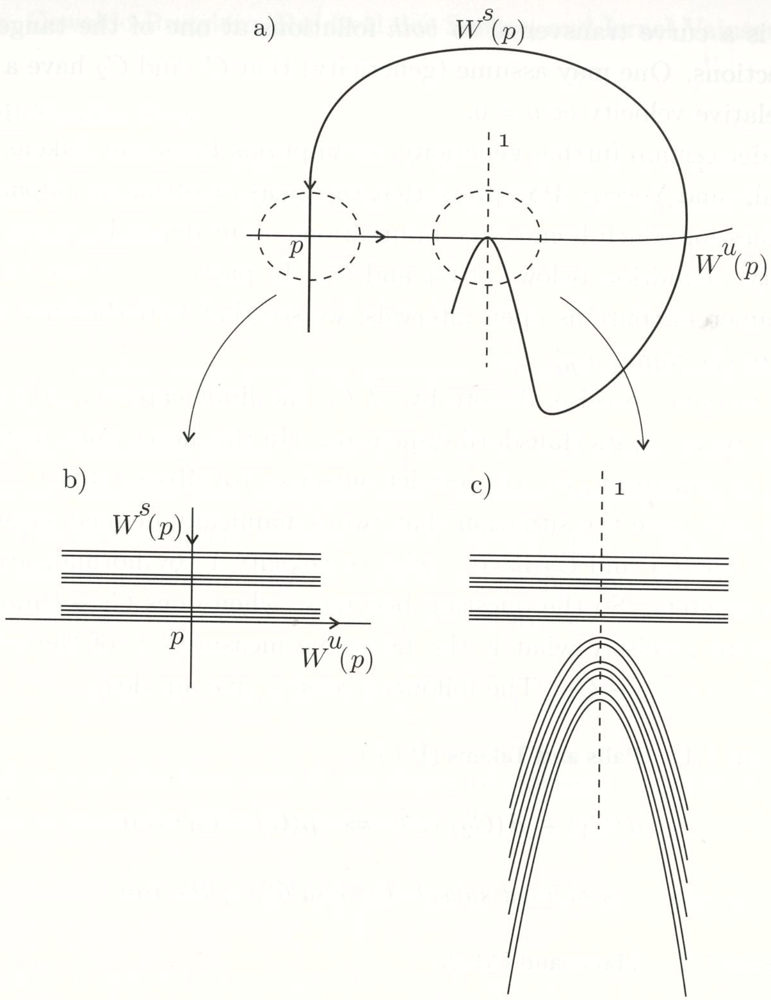

Figure 1.1 – Homoclinic tangency of stable and unstable manifolds emanating from a fixed point p

Assume further that at $\mu = 0$, the unstable and stable manifolds associated with a fixed point $p \in \Omega_0$) intersect tangentially (as in the figure). Now the non-wandering set Ω_0 is the union of a hyperbolic set and the orbit $\mathcal{O}$ of homoclinic tangency. Note that if the foliation is smooth all properties depending on asymptotics (such as dimension) are invariant under transport along the unstable foliation. In particular, we may choose $\ell = W^s(p)$ as in figure 1.1(c) or as in figure 1.1(b), where

$\ell = \ell'$ is a curve transversal to *both* foliations at one of the tangential intersections. One may assume (genericity) that C_1 and C_2 have a non-zero relative velocity et $\mu = 0$.

Under certain further genericity assumptions Palis and Takens [**PT**] and Palis and Yoccoz [**PY**] prove that the measure of the set of parameter values for which homoclinic bifurcations occur depends on the limit capacity (definition below) of C_1 and C_2. In particular, when this set of parameters contains open intervals, we see that hyperbolicity is not dense in the family ϕ_μ!

To be more precise, denote by $d(C)$ the limit capacity of a set C and by $\text{Hdim}(C)$ its Hausdorff dimension. (In this note, when using the notion of dimension, we will restrict ourselves to subsets of $\mathbb{R}$.)

We now have the situation that two dynamically defined subsets of the real line C_1 and C_2 have a relative velocity 1 (by normalization of the parameter). So the question becomes: when does $C_1 + t$ intersect C_2? More precisely what is the Lebesgue measure (μ) of the set of t such that $t \in C_2 - C_1$? The following results give an idea.

Theorem 1.1. **(Palis and Takens [PT].)**

$$d(C_1) + d(C_2) < 1 \quad \Rightarrow \quad \mu(C_2 - C_1) = 0$$

for all dynamically defined subsets C_1 and C_2 of the line.

Theorem 1.2. **(Marstrand [Ma].)**

$$\text{Hdim}(C_1) + \text{Hdim}(C_2) > 1 \quad \Rightarrow \quad \mu(C_2 - C_1) > 0$$

for almost all dynamically defined subsets C_1 and C_2 of the line.

In section 2, we give the general construction of (affinely) self-similar sets and some properties. In section 3, we give examples of difference sets that contain open sets, and also some that have measure zero although the sum of the capacities involved is bigger than that of the ambient space. We will show that affine difference sets of positive measure must contain an open interval.

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2. Self-Similar Sets

Let $\mathcal{T}_n$ be the space of pairs (M, R), where M is an $n \times n$ matrix with eigenvalues of modulus greater than one, and $R \subset \mathbb{R}^n$ a finite set. On $\mathcal{T}_n \times \mathcal{T}_n$ we define

$$Td((M,R),(N,S)) = \|(N^{-1} - M^{-1})\| + \text{ Hd } (R,S) \quad ,$$

where $\|\cdot\|$ is the usual norm on matrices. This is easily seen to be a metric on $\mathcal{T}_n$.

For any complete metric space X, let $H(X)$ be the space of compact and apriori bounded sets equipped with the usual Hausdorff metric (see [**Fa**]). This is a complete compact metric space ([**Hu**]). Moreover, this metric defines a distance in $H(X)$ which we denote by $\text{ Hd } (\cdot,\cdot)$.

Now define a function $\Lambda : \mathcal{T}_n \to H(\mathbb{R}^n)$ as follows:

$$\Lambda(M,R) = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{\infty} M^{-i} r_i \; , \; r_i \in R\} \quad . \tag{2.1}$$

We will call M the base and R the digits of the set $C = \Lambda(M,R)$. We will also employ the following notation.

$$C = \sum_{i=0}^{\infty} M^{-i} R \quad ,$$

where $X + Y$ means all $x + y$ with $x \in X$ and $y \in Y$. Note that C is not necessarily a Cantor set: when $M = 3$ and $R = \{0,1,2\}$, we obtain the usual representation on the base 3 for $C = [0,1]$.

Our discussion will be invariant under affine coordinate transformations. For a fixed vector $s' = \sum_{i=0}^\infty M^{-i} s \in \mathbb{R}^n$ and linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ that commutes with M, we have

$$C + s' = C + \sum_{i=0}^{\infty} M^{-i} s = \sum_{i=0}^{\infty} M^{-i}(R + s)$$

$$AC = \sum_{i=0}^{\infty} (A^{-1} M A)^{-i} AR = \sum_{i=0}^{\infty} M^{-i} AR \quad .$$

So we can always add vectors to R or multiply R by a scalar. In particular, we will assume most of the time that $0 \in R$.

The sets constructed here can also be described as the (unique) invariant set of an iterated function system. Define the following map from $H(\mathbb{R}^n)$ to itself.

$$\tau(A) = \bigcup_{r \in R} M^{-1}(A + r) = M^{-1}(A + R) \quad .$$

It is easy to see that τ is a contraction and that its unique fixed point is equal to the set C just described:

$$\tau(C) = C \quad .$$

It is thus justified to call C self-similar, since

$$C = M^{-1}C + M^{-1}R \quad . \tag{2.3}$$

(Note, though, that these different 'copies' of C may overlap.)

There is yet another useful description of self-similar sets. Since M^{-1} is a contraction it is easy to find a closed ball B such that $\tau(B) \subset B$. It is then easy to see that $\tau^n(B) \subset \tau^{n-1}(B)$ and

$$C = \bigcap_{i=0}^{\infty} \tau^i(B) \quad . \tag{2.4}$$

Lemma 2.1. *Λ is a continuous function.*

Proof. Denote $\Lambda(N, S)$ by C_1 and $\Lambda(M, R)$ by C_2. We will prove that for given $\epsilon > 0$ there is an $\delta > 0$ such that if $Td((M, R), (N, S)) < \delta$, then Hd $(C_1, C_2) < \epsilon$.

Consider all eigenvalues of M and N and pick the one whose modulus is smallest. Call this modulus λ_-. Then pick λ such that $1 < \lambda < \lambda_-$. There is a constant C such that $\|M^{-i}\|$ and $\|N^{-i}\|$ are smaller than $C\lambda^{-i}$.

Now, for a given point $x_1 = \sum_{i=1}^{\infty} M^{-i} r_i$ in C_1 its distance to C_2

satisfies (using equation (2.1)):

$$\begin{aligned} \min_{x_2 \in C_2} |x_2 - x_1| &\leq \sum_{i=1}^{\infty} \min_{s_i \in S} |M^{-i} r_i - N^{-i} s_i| \\ &\leq \sum_{i=1}^{\infty} \min_{s_i \in S} |M^{-i} r_i - M^{-i} s_i| + \\ &\quad + \sum_{i=1}^{\infty} \min_{s_i \in S} |N^{-i} s_i - M^{-i} s_i| \\ &\leq \sum_{i=1}^{\infty} C\lambda^{-i}\delta + \sum_{i=1}^{\infty} \|M^{-i} - N^{-i}\| \max_{s \in S} |s| \quad . \end{aligned}$$

To estimate this last term, observe that

$$\begin{aligned} \|M^{-i} - N^{-i}\| &= \|\sum_{j=0}^{i-1} M^{j-i+1}(M^{-1} - N^{-1})N^{-j}\| \\ &\leq \sum_{j=0}^{i-1} C^2 \lambda^{-(i-1)} \delta \leq i C^2 \lambda^{-(i-1)} \delta \quad . \end{aligned}$$

Putting together these estimates proves that for all $\epsilon > 0$ there is an δ small enough so that C_1 is contained in a ϵ-neighborhood of C_2. By symmetry, the reverse is also true which proves the lemma. □

In dimension one, this result is a special case of a 'folklore' result that asserts continuity even if $\mathcal{T}_n$ is a space of $C^{1+\epsilon}$ expansions.

We need some more notation. Denote by $\Gamma_k \in \mathbb{R}^n$ the set of points:

$$\Gamma_k = \{x \in \mathbb{R}^n | \; x = \sum_{i=0}^{k-1} M^{-i} r_i \; , \; r_i \in R\} \quad . \tag{2.5}$$

Note that $\Gamma_0 = R$. Denote the cardinality of a finite set X by $|X|$.

Lemma 2.2. *If for some $k \in \mathbb{N}$, $|\Gamma_k| < |\det M|^k$, then $\mu(C(M, R)) = 0$.*

Proof. By iterating formula (2.3), we have:

$$C = M^{-k}C + \Gamma_k \;\Rightarrow \mu(C) \leq \mu(C) \cdot \frac{|\Gamma_k|}{|\det M|^k} \quad . \quad \square$$

The converse of this lemma is not true as we will see in the next section. However, in the case of M having integer entries and $R \in \mathbb{Z}^n$ we do have a converse.

Theorem 2.3. *Let $M : \mathbb{Z}^n \to \mathbb{Z}^n$ and $R \in \mathbb{Z}^n$. Then $\mu(C) > 0$ if and only if $\lim_{k\to\infty} \frac{|\Gamma_k|}{|\det M|^k} > 0$.*

Proof. By the set theoretical continuity of the Lebesgue measure (see for example [**KF**], section 25) and (2.4), we have

$$\mu(C) = \lim_{k\to\infty} \mu(\tau^k(B)) \quad .$$

Now let I be the standard (unit) cube and choose B big enough so that $I \subseteq B$. Observe that

$$\tau^n(I) = M^{-k}I + \Gamma_k$$

whose intersections have measure zero. Since we can cover B by, say, K standard cubes, we obtain

$$|\Gamma_k| \cdot \mu(M^{-k}I) \leq \mu(C) \leq K \cdot |\Gamma_k| \cdot \mu(M^{-k}I) \quad . \quad \square$$

Note that by the previous lemma, for the measure to be positive, the limit must actually be greater than or equal to one.

The measure function $\mu : \mathcal{T}_n \to \mathbb{R}^+$ is given by

$$\mu(M, R) = \mu(\Lambda(M, R)) \quad ,$$

that is: the (Lebesgue) measure of the invariant set generated by the digits R on the base M. It is not a continuous function as we will see in the next section, but we do have the following weaker result:

Theorem 2.4. *The measure function $\mu : \mathcal{T}_n \to \mathbb{R}^+$ is upper semi-continuous.*

Proof. The measure function is the composition of the continuous function Λ and the usual measure function from $H(\mathbb{R}^n)$ to $\mathbb{R}^+$ that assigns to a set its (Lebesgue) measure. It is sufficient to prove that the latter one is semi-continuous.

We are given a set $C_0 \in H(\mathbb{R}^n)$ and a number $\epsilon > 0$. Let U_δ be a closed δ-neighborhood of C_0. Now, note that $U_{1/(n+1)} \subset U_{1/n}$ and so the (Hausdorff) limit of this sequence of sets is the compact set $\cap_{n\in\mathbb{N}} U_{1/n}$. Since Hd $(\cap_{n\in\mathbb{N}} U_{1/n}, C_0) = 0$, the two sets must be equal (Hd is a metric). Then, using the set theoretical continuity of the measure,

$$\lim_{n\to\infty} \mu(U_{1/n}) = \mu(C_0) \quad .$$

On the other hand, any C such that Hd $(C, C_0) < \delta$ is contained in U_δ and so $\mu(C) < \mu(U_\delta)$. Thus, for δ small enough,

$$\text{Hd } (C, C_0) < \delta \Rightarrow \mu(C) \leq \mu(C_0) + \epsilon \quad . \quad \square$$

Note that the semi-continuity is not uniform.

Theorem 2.5. *Let $M : \mathbb{Z}^n \to \mathbb{Z}^n$ and $R \in \mathbb{Z}^n$. If $\mu(C) > 0$, then C contains an open set.*

Proof. Since C has positive measure, almost all points in C are density points (Lebesgue's theorem). Let x_0 be such a point. For each $\epsilon > 0$, there is a r_ϵ such that for all $r \leq r_\epsilon$

$$\mu(C \cap B(r)) \geq (1-\epsilon)\mu(B(r) \quad ,$$

where $B(r)$ is a ball with radius r and center x_0.

Since M is a contraction, we can now find a number k_ϵ such that the parallellograms of $M^{-k_\epsilon}(I + \mathbb{Z}^n)$ inscribed in $B(r_\epsilon)$ cover more than half the volume of $B(r_\epsilon)$. Then there must be at least one of these parallellograms $M^{-k_\epsilon}(I + z_{k_\epsilon})$ ($z_{k_\epsilon} \in \mathbb{Z}^n$) that is well covered by copies of C, that is:

$$\mu((M^{-k_\epsilon}C + \Gamma_{k_\epsilon}) \cap M^{-k_\epsilon}(I + z_{k_\epsilon})) \geq (1-2\epsilon)\mu(M^{-k_\epsilon}(I + z_{k_\epsilon})) \quad .$$

The number of distinct copies of C contained in $M^{-k_\epsilon}C + \Gamma_{k_\epsilon}$ intersecting $M^{-k_\epsilon}(I + z_{k_\epsilon})$ is bounded. Denote them by $M^{-k_\epsilon}C + Z_{k_\epsilon}$, where Z_{k_ϵ} is a subset of Γ_{k_ϵ}. Now after an affine transformation (preserves relative measure), we have

$$\mu((C + M^{k_\epsilon}(Z_{k_\epsilon}) - z_{k_\epsilon}) \cap I) \geq (1-2\epsilon)\mu(I) \quad .$$

Recall that this is valid for all ϵ. So clearly there exists a finite subset Z of $\mathbb{Z}^n$ such that the translates $C + Z$ cover the unit cube. Then—by Baire's theorem—C contains an open set. $\square$

We remark that a statements similar to theorems 2.3 and 2.5 were proved by different methods in [**HSV**] if R contains a complete set of residues modulo $M\mathbb{Z}^n$. Similar statements also appeared in [**Ke**].

We now turn our attention to the dimension of self-similar sets. Recall that we only consider the dimension of subsets of the line. First, we define the limit capacity. let $N(\epsilon)$ be the minimum number of intervals of length ϵ needed to cover a set $C \in \mathbb{R}^n$. Then the limit capacity of C is given by

$$d(C) = \limsup_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log \epsilon} \quad .$$

For the notion of Hausdorff dimension, we refer to [**Fa**]. We can now define the limit capacity function $d : \mathcal{T}_n \to \mathbb{R}^+$ and the Hausdorff dimension function Hdim $: \mathcal{T}_n \to \mathbb{R}^+$ in the same way we defined the measure function.

Note that we always have

$$\text{Hdim}\,(C) \leq d\,(C) \leq \frac{\ln|R|}{\ln|\det M|} \quad , \tag{2.6}$$

(see, for example [**Ta**]). However, when C is dynamically defined, the first two of thse notions coincide (see [**Ta**], [**PT**]). Cantor sets arising from surface diffeomorphism as described in section 1 are dynamically defined. The difference set of two such sets may *not* be dynamically defined. Thus equality does not hold for difference sets.

A stronger condition than 'dynamically defined' (it implies dynamically defined), but similar and easier to state is the 'open set' condition (section 8.3 of [**Fa**]). We say that (M, R) satisfies the open set condition (section 8.3 of [**Fa**]) if there is an open set $V \subset \mathbb{R}$ such that

1. $M^{-1}(V + R) \subset V$
2. $M^{-1}(V + r_1) \cap M^{-1}(V + r_2) = \varnothing, \quad r_1 \neq r_2, \quad r_1, r_2 \in R$.

For pairs (M, R) satisfying the 'open set' condition, the dimension of its invariant set is easy to calculate:

$$\text{Hdim}(C) = \frac{\ln|R|}{\ln|\det M|} \quad . \tag{2.7}$$

In fact, more generally, for dynamically defined sets, the dimension depends continuously on the presentation function ([**MM**], [**Ta**]). In the next section, we will see that this is not the case for difference sets.

3. Examples of Difference Sets

Suppose we have two self-similar sets C_1 and C_2, generated by the same

base M but using different digit sets, namely R_1 and R_2, respectively. Define the intersection set $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$ associated with M, R_1, and R_2 as follows

$$\Delta = \{(t, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in (C_1 + t) \cap C_2\} \quad . \tag{3.1}$$

The meaning of this set is that its projection $p(\Delta)$ onto the t-axis gives those values of t for which $C_1 + t$ intersects C_2. Thus

$$p(\Delta) = C_2 - C_1 \quad .$$

This is the set we will investigate in this section. In the fiber we find the set $(C_1 + t) \cap C_2$:

$$p^{-1}(t) \cap \Delta = (C_1 + t) \cap C_2 \quad .$$

Meanwhile note that Δ is contained in $(C_1 - C_2) \times C_2$.

We restrict the discussion now to Cantor-sets C_1 and C_2, such that

$$\frac{|R_1|}{|\det M|}, \frac{|R_2|}{|\det M|} < 1 < \frac{|R_1||R_2|}{|\det M|} \quad .$$

We will first give examples in which $C_2 - C_1$ contains an open set. Then we will give examples such that the measure of $C_2 - C_1$ is zero. Note that $C_2 - C_1$ is the set generated by base M and digit set $D = R_2 - R_1$.

Theorem 3.1. *Suppose that $M : \mathbb{Z}^n \to \mathbb{Z}^n$ and D contains a complete set of coset representatives of $M\mathbb{Z}^n/\mathbb{Z}^n$ in $\mathbb{Z}^n$. Then $C_2 - C_1$ contains an open set.*

Proof. An independent proof of this appears in [**HSV**]. But it also follows almost immediately from theorem 2.5.

Let R be the complete set of representatives and notice that $C(M, R) \subset C_2 - C_1$. So, by theorem 2.3, it is sufficient to show that Γ_k as defined in equation (2.5) has more than $|\det M|^k$ distinct points. But this is easy, because

$$\sum_{i=0}^{k-1} M^{-i}\tilde{r}_i = \sum_{i=0}^{k-1} M^{-i} r_i$$

implies that

$$\tilde{r}_i = r_i \mod M\mathbb{Z}^n \;\Rightarrow\; \tilde{r}_i = r_i \quad .$$

(Recall that we may assume that $0 \in R_i$.) Thus all $|\det M|^k$ expressions in the definition of Γ_k give rise to distinct points. □

In fact, we could have stated a slightly more general result, because of the affine invariance discussed at the beginning of section 2. In dimension one, for the case where $M = 3$ and $|R_i| = 2$, the theorem would read:

Corollary 3.2. *In dimension one, let $M = \pm 3$ and $R_i = \{0, t_i\}$, $i \in \{1,2\}$. Then $C_2 - C_1$ contains an interval if there is a real α such that $\alpha t_i \in \mathbb{Z} \setminus 3\mathbb{Z}$.*

Proof. Just check that $\alpha(R_2 - R_1) + t_1$ contains a complete set of residues modulo 3 if and only if the condition holds. □

We turn to the examples. The first one shows that μ is not lower semi-continuous.

Example 3.3. *Let $M = 3$ and $R_t = \{0, t, 2\}$. Then, $\mu(\Lambda(M, R_t)) = \frac{2}{q}$ if and only if $t = 2\frac{p}{q}$ and $pq \mod 3 = 2$, where p and q are relative prime.*

Proof. The 'if' part follows from the fact that $\frac{q}{2} \cdot R_t$ is a complete set of residues with greatest common divisor equal to 1. The measure of the set associated with such a digit set is one (see for example [**HSV**]). Its converse is a corollary of results of Lagarias and Wang [**LW**]. □

We remark that there is older result [**Od**] that implies that $\mu(\Lambda(M, R_t))$ is zero for almost all t.

The dimension is not upper semi-continuous:

Example 3.4. *Suppose that $M = 3$ and $R_k = \{0, 3^{-k}, 1 - 3^{-k}\}$. Then, $\lim_{k\to\infty} \text{Hdim}(C_k) = 1$, but $\text{Hdim}(\lim_{k\to\infty} C_k) = \frac{\ln 2}{\ln 3}$.*

Proof. The first statement follows from the previous example. The second follows from lemma 2.1 and the fact that $\lim_{k\to\infty} R_k = \{0, 1\}$. So we obtain the middle third Cantor set. □

It is easy to find such examples for difference sets.

The lower semi-continuity for dimensions is still open. However, recall that for dynamically defined Cantor sets the dimension function is continuous ([MM], [Ta]).

It is by no means clear that a difference set with more than $|\det M|$ digits should have positive measure. We have not succeeded in manufacturing a one dimensional counter-example on the base 3. When we drop the requirement that it has to be a difference set, things become quite easy:

Example 3.5. *Let $C \subset \mathbb{R}$ have base 3 and digits $\{0, 1, 3, 9\}$. Then $\mu(C) = 0$.*

Proof. It is sufficient to prove that for some k, $|\Gamma_k| < 3^k$. This can in principle be accomplished by simple counting. One finds that when $n = 10$, $|\Gamma_{10}| = 56563$ and $3^{10} = 59049$. □

If we pick a slightly larger base (still in one dimension), counter-examples are easier to come by.

Example 3.6. *In one dimension, let $C_1 = C(4, \{0, 4\})$ and $C_2 = C(4, \{0, 1, 4\})$. Then $\mu(C_2 - C_1) = 0$.*

However, this example is not satisfying either, because C_2 does not satisfy the open set condition. Upon closer inspection one verifies in fact that

$$d(C_2) \leq \frac{\ln(7 + 3\sqrt{5}) - \ln 2}{\ln 16}$$

(we leave this bit of digit-counting to the reader). Thus, in this case, the sum of the limit capacities is smaller than one!

Here is our 'good' counter-example:

Example 3.7. *In one dimension, let $C_1 = C(5, \{0, 5\})$ and $C_2 = C(5, \{0, 2, 5\})$. Then the sum of their (Hausdorff) dimensions is bigger than one, but their difference has measure zero.*

Proof. Both C_1 and C_2 satisfy the open set condition at the end of section 2 (take $V = (0, 5/4)$). Thus

$$\text{Hdim}(C_1) = \frac{\ln 2}{\ln 5} \quad \text{and} \quad \text{Hdim}(C_2) = \frac{\ln 3}{\ln 5} \quad .$$

Now observe that Γ_1 associated with the difference set contains only 23 distinct points. Thus in fact the difference set has dimension smaller than one (see equation (2.6)). □

Corollary 3.8. *Take C_1 and C_2 as in example 3.7. For any $\epsilon > 0$ there are open δ-neighborhoods in $H(\mathbb{R})$: $N_{1,\delta}$ of C_1 and $N_{2,\delta}$ of C_2, such that if $X_1 \in N_{1,\delta}$ and $X_2 \in N_{2,\delta}$ then $\mu(X_2 - X_1) < \epsilon$.*

Proof. Follows directly from upper semi-continuity of the measure function and the continuity of the difference operation in $H(\mathbb{R})$. □

Apparently, it is an open question whether there exists an affinely self-similar set of positive measure but which is nowhere dense. Theorem 2.5 implies that there is no *integrally* affinely self-similar nowhere dense set of positive measure. In the setting of C^∞ presentation functions Sannami has constructed a nowhere dense invariant set of positive measure [**Sa**].

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