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The Derivatives of Equilibrium States

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— To the memory of Ricardo Mañé.

Abstract. We find some estimates for the derivatives of equilibrium states of subshifts of finite type. We prove the differentiability (with respect to the potential) of integrals of certain discontinuous functions for the equilibrium state of a potential.

Introduction

In this paper we are interested in certain singular integrals with respect to equilibrium states of a subshift of finite type.

Let $\sigma : \Sigma \hookleftarrow$ be a topologically transitive subshift of finite type endowed with one usual metric (e.g. diameter(n-cylinder) $= 2^{-n}$), let $\phi : \Sigma \to \mathbb{R}$ be a Hölder continuous function and let μ_ϕ be its equilibrium state.

Let $K \subseteq \Sigma$ be a compact subset such that $\mu_\phi(\epsilon$-neighbourhood of $K)$ $\leq A\,\epsilon^\alpha$, A, $\alpha > 0$. Consider a measurable function $L : \Sigma \to \mathbb{R}$ having a singular set K of order $|L(x)| \leq B\,|\log d(x,K)|$, where $B > 0$ is a constant and $d(\cdot,\cdot)$ is the distance in Σ. Then, using the condition on μ_ϕ and K, one can prove that L is μ_ϕ-integrable. On the other hand, it is known that the map $C^\beta(\Sigma,\mathbb{R}) \ni \phi \mapsto \mu_\phi \in \left(C^\beta(\Sigma,\mathbb{R})\right)^*$ is real analytic. We will prove (theorem B) that, if moreover, L has local Hölder constants $D(x)$, $|L(x) - L(y)| \leq D(x)\,d(x,y)^\gamma$, such that $D(x) \leq C\,d(x,K)^{-\delta}$, C, γ, $\delta > 0$, then the map $C^\beta(\Sigma,\mathbb{R}) \ni \phi \mapsto \int L\,d\mu_\phi \in \mathbb{R}$ is C^∞.

In order to prove theorem B, one has to show that $D_\phi(\mu_\phi(L))$ exists.

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For this we need estimates on the derivative $D_\phi(\mu_\phi(\mathbf{1}_{C_n}))$, where $\mathbf{1}_{C_n}$ is the characteristic function of an n-cylinder. Observe that one can not expect to have a bound

$$|(D_\phi\,\mu_\phi(C_n))(\varphi)| = |(D_\phi\,\mu_\phi)\,(\varphi)\cdot\mathbf{1}_{C_n}| \le A\;\|\varphi\|\;|\mu_\phi(C_n)|$$

(where the first equality is proven in (6)), because for $\phi_t = \phi_0 + t\,\varphi$, we would have an estimate

$$\left|\mu_{\phi_t}(C_n)\right| \le \exp(A\,\|\varphi\|\,t)\,\mu_{\phi_0}(C_n),$$

implying the absolute continuity $\mu_{\phi_t} \ll \mu_{\phi_0}$, which is false. The Theorem A below is the following estimate for n-cylinders C_n:

$$\left|\left(D_\phi^k\,\mu_\phi(C_n)\right)(\varphi,\overset{k}{\ldots},\varphi)\right| \le n^k\;\|\varphi\|^k\;A(\phi,k)\;\mu_\phi(C_n)\,.$$

where $\|\quad\|$ is the α-Hölder norm. We show here two applications of theorem B.

a. One dimensional dynamics.

The first application is in 1-dimensional dynamics. Consider the map $f:[-1,1]\hookleftarrow;\ f(x)=1-2x^2$. This map is conjugated to the Tent map $g:[-1,1]\hookleftarrow,\ g(y)=1-2\,|y|$. The conjugacy $h:[-1,1]\to[-1,1]$, $f\circ h = h\circ g$ is given by $h(y) = sin\left(\frac{\pi}{2}\,y\right)$. The full 2-shift is semiconjugated to the Tent map by an α-Hölder semiconjugacy $k:\Sigma_2\to[-1,1]$, for some $0<\alpha<1$. The function $F:[-1,1]\to\mathbb{R}$, $F(p)=\log|f'(p)|$ has at $p=0$ a singularity of order $|F(x)|\sim-\log|x|$, and has local Lipschitz constants of order $\sim F'(x)\sim 1/|x|$. Since h is Lipschitz, if we choose $K=k^{-1}\{0\}$, $L:\Sigma_2\to\mathbb{R}$, $L(x)=F(h\circ k(x))$, then L has a singularity at K of order $L(x)\sim-\log d(x,K)^\alpha\sim-\log d(x,K)$ and local α-Hölder constants of order $\sim 1/d(x,K)^\alpha$. In particular the pair (K,L) satisfies the hypothesis of theorem B. If $\phi:[-1,1]\to\mathbb{R}$ is α-Hölder, then $\psi=\phi\circ h\circ k:\Sigma_2\to[-1,1]$, is β-Hölder for some $\beta=\beta(\alpha)$, and the map $C^\alpha(\Sigma_2,\mathbb{R})\to C^\beta(\Sigma_2,\mathbb{R}):\phi\mapsto\psi$ is real analytic. We get that the Lyapunov exponents of equilibrium states of f depend smoothly with respect to the potential ϕ, i.e.

$$C^\alpha([-1,1],\mathbb{R})\ni\phi\mapsto\lambda_\phi := \int\log|f'(p)|\;d\mu_\phi(p)$$

is C^∞, where μ_ϕ is the equilibrium state for (ϕ,f).

b. Average linking numbers.

The second application is to average linking numbers for hyperbolic flows. Let $\Lambda\subseteq S^3$ be a hyperbolic basic set of a flow φ_t in S^3. Let $PO(t)$ be the set of periodic orbits of $\varphi_t|_\Lambda$ with period $\le t$. Denote by $\ell(\gamma,\eta)$ the linking number of the knots γ and η in S^3. Define

$$\mathcal{L}(s,t) := \sum\{\ell(\gamma,\eta)\,|\,\gamma\in PO(s)\,,\,\eta\in PO(t)\,,\,\gamma\ne\eta\}$$

In [**3**] we proved that identifying $S^3\approx\mathbb{R}^3\cup\{\infty\}$, with $\infty\notin\Lambda$, we have that

$$\mathbf{L}(\Lambda)\overset{\text{def}}{=}\lim_{s,t\to\infty}\frac{1}{st}\mathcal{L}(s,t)=\int_{\Lambda\times\Lambda}\mathbb{L}(x,y)\;d(\mu\times\mu)(x,y)$$

where μ is the measure of maximal entropy of $\varphi_t|_\Lambda$, and

$$\mathbb{L}(x,y) = -\frac{1}{\mathrm{vol}(S^3)}\;\frac{F(x)\times F(y)}{\|x-y\|}\cdot\frac{(x-y)}{\|x-y\|^2}$$

where $\times$ is the vector product in $\mathbb{R}^3\approx S^3-\{\infty\}$, $\cdot$ is the inner product in $\mathbb{R}^3$ and F is the vectorfield of φ_t. We show now that there exists a neighbourhood $\mathcal{U}$ of F with the C^3 metric such that the map $\mathcal{U}\ni G\mapsto\mathbf{L}(\Lambda_G)\in\mathbb{R}$ is C^∞, where Λ_G is the continuation of Λ for the vectorfield G.

Using the Taylor expansion of F one can see that the factor

$$\frac{F(x)\times F(y)}{\|x-y\|}$$

is bounded in a neighbourhood of the diagonal $x=y$, therefore $\mathbb{L}(x,y)$ has order $1/\|x-y\|$ near the diagonal. By dimension arguments, it is seen in [**3**] that $\mathbb{L}(x,y)\in\mathcal{L}^1(\mu\times\mu)$.

Using a Markov partition, it is shown in [**2**] that there exists a semi-conjugacy π of a suspension $S(\Sigma,\tau_F)$ of a subshift of finite type Σ onto Λ, where $\tau_F:\Sigma\to]0,+\infty[$ is Hölder continuous and $S(\Sigma,\tau_F)$ is the quotient space

$$S(\Sigma,\tau) := \{(x,t)\,|\,x\in\Sigma\,,\,0\le t\le\tau(x)\,\}/_{\equiv}$$

with the equivalence relation $(x,\tau(x))\equiv(\sigma(x),0)$, where $\sigma:\Sigma\hookleftarrow$ is the shift map. The lift of the measure of maximal entropy μ to the suspension is $(\pi^*\mu) = \left(\int \tau_F \, d\nu\right)^{-1} \nu \times \lambda$ where λ is the Lebesgue measure on $[0, +\infty[$ and ν is the equilibrium state of the function $\phi_F(x) = h_{\rm top}(F)\, \tau_F(x)$, where $h_{\rm top}(F)$ is the topological entropy of $\varphi|_\Lambda$.

Define $\ell : \Sigma \times \Sigma \to \mathbb{R}$, by

$$\ell(x,y) = \int_0^{\tau_F(x)} \int_0^{\tau_F(y)} \mathbb{L}\left(\varphi_s(\pi\, x), \varphi_t(\pi\, y)\right) \, ds\, dt$$

then

$$\mathbf{L}(\Lambda) = \frac{1}{\left(\int \tau_F \, d\nu\right)^2} \int_{\Sigma\times\Sigma} \ell(x,y) \; d(\nu_F \times \nu_F)$$

It can be proved (cf. [**3b**]) that the function $\ell(x,y)$ has order $-\log d(x,y)$ near the diagonal $x = y$ on $\Sigma \times \Sigma$ and has local Lipschitz constants of order $1/d(x,y)$. The structural stability for hyperbolic basic sets and the topological invariance of the linking number yields that when we change the vectorfield on a neighbourhood of F, the new average linking can be calculated using the same shift and *the same function* ℓ. The corresponding orbits under the topological equivalence will have the same linking numbers but their periods will be different. In general the new measures ν_G will be singular with respect to ν_F.

In [**4**] it is proven that if $\mathcal{U}$ is a small neighbourhood of the vectorfield F_0 endowed with the C^2-metric, then the maps $\mathcal{U} \ni F \mapsto \phi_F \in C^\alpha(\Sigma, \mathbb{R})$, and $\mathcal{U} \ni F \mapsto \int \tau_F \, d\nu_F$, are real analytic. So we need to see that

$$\mathcal{U} \ni F \mapsto \int_{\Sigma\times\Sigma} \ell \, d(\nu_F \times \nu_F)$$

is C^∞. Here we apply theorem B to $K = \{(x,y) \in \Sigma\times\Sigma | x = y\}$ and the function ℓ. For this we observe that the product map $\sigma \times \sigma : \Sigma \times \Sigma \hookleftarrow$, $(\sigma\times\sigma)(x,y) = (\sigma(x), \sigma(y))$ is also a subshift of finite type, the measure $\nu_F \times \nu_F$ is the equilibrium state of $\psi_F(x,y) := \phi_F(x) + \phi_F(y)$ for the product shift. Moreover since the measure ν_F satisfies a uniform estimate $\nu_F(\epsilon\text{-neighbourhood of } K) {\sim} \epsilon^\beta$ with $\beta > 0$, one gets that $(\nu_F \times \nu_F)(\epsilon\text{-neighbourhood of } K) {\sim} \epsilon^\beta$. Finally, if one constructs the Markov partition using small embedded differentiable transverse sections to the flows, the projection of ℓ, $J(p,q) := \ell(x,y)$, where $\pi x = p$, $\pi y = q$, $p, q \in$(smooth transversal), is differentiable on the transversal sections

and $|J| \sim -\log \|p - q\|$, $\|D\, J\| \sim \frac{1}{\|p-q\|}$. This gives the required conditions on theorem B.

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Statements of the Results.

Given a matrix $A \in \{0,1\}^{\ell\times\ell}$ such that for all $0 \le i, j \le \ell$ there exists $M = M(i,j) > 0$ such that $A^M_{ij} > 0$, consider the (topologically transitive) subshift of finite type $\sigma\colon \Sigma \hookleftarrow$ (resp. $\sigma_+\colon \Sigma^+ \hookleftarrow$) where $(\sigma(x))_i = x_{i+1}$,

$$\Sigma\colon= \{x = (x_i)_{i=-\infty}^{+\infty} \in \{1, \ldots, \ell\}^{\mathbb{Z}} \,|\, A(x_i, x_{i+1}) = 1, \quad \forall i \in \mathbb{Z}\}$$

$$\Sigma^+\colon= \{x = (x_i)_{i=0}^{+\infty} \in \{1, \ldots, \ell\}^{\mathbb{N}} \,|\, A(x_i, x_{i+1}) = 1, \quad \forall i \in \mathbb{N}\}$$

Endow Σ (resp. Σ^+) with the metric $d(x,y) = b^{\langle x,y\rangle}$ for some fixed $0 < b < 1$ and where

$$\langle x,y\rangle\colon= \max(\{0\} \cup \{k \,|\, x_i = y_i, \forall |i| \le k\}).$$

For $x \in \Sigma$ (resp. Σ^+) let $C_n(x)$ be the $n-$cylinder containing x:

$$C_n(x)\colon= \{y \in \Sigma \,|\, x_i = y_i, \forall |i| \le n\}, \qquad (\text{resp. } 0 \le i \le n).$$

For $0 < \alpha < 1$ let $C^\alpha(\Sigma, \mathbb{R})$ (resp. $C^\alpha(\Sigma^+, \mathbb{R})$) be the Banach space of $\alpha-$Hölder continuous functions on Σ (resp. Σ^+).

$$C^\alpha(\Sigma, \mathbb{R})\colon= \{\phi\colon \Sigma \to \mathbb{R} \,|\, \exists K > 0, |\phi(x) - \phi(y)| < K d(x,y)^\alpha\}$$

with the norm $\|\phi\|\colon= |\phi|_0 + |\phi|_\alpha$, where

$$|\phi|_0\colon= \sup_{x\in\Sigma} |\phi(x)|, \qquad |\phi|_\alpha\colon= \sup_{d(x,y)\neq 0} \frac{|\phi(x) - \phi(y)|}{d(x,y)^\alpha}.$$

It is known (cf. [6]) that the map $\phi \mapsto \mu_\phi$ from $C^\alpha(\Sigma, \mathbb{R})$ into the dual space $(C^\alpha(\Sigma, \mathbb{R}))^*$ (with the dual norm) is real analytic. For $p \in \Sigma$, $n \ge 0$, let $\mathbf{1}_{C_n(p)}$ be the characteristic function of $C_n(p)$. Since $\mathbf{1}_{C_n(p)}$ is $\alpha-$Hölder continuous, it follows that the map $C^\alpha(\Sigma) \to \mathbb{R}\colon \phi \mapsto \mu_\phi(C_n(p))$ is real analytic.

Theorem A.

(a) *Let Σ be a topologically transitive two-sided subshift of finite type, and let $\alpha > 0$, $\phi_0 \in C^\alpha(\Sigma,\mathbb{R})$. Then there exists a neighbourhood $\mathcal{U}$ of ϕ_0 and $D = D(\mathcal{U}) > 0$ such that*

$$\left|(D_\phi^{(k)}(\mu_\phi)(\varphi_1,\dots,\varphi_k)(\mathbf{1}_{C_n(p)})\right| \le n^k\, k!\; D(\mathcal{U})^k\; \|\varphi_1\|\cdots$$
$$\cdots\|\varphi_k\|\;\mu_\phi(C_n(p))\;,$$

for all $n \ge 0$, $k \ge 0$, $p \in \Sigma$, $\phi \in \mathcal{U}$ and $\varphi_1,\dots,\varphi_k \in C^\alpha(\Sigma,\mathbb{R})$, where $\|\varphi\|$ is the β–Hölder norm of φ.

(b) *The same estimate holds for one-sided topologically transitive subshifts of finite type.*

Theorem A will be proven at the end of the paper. In the following theorem we can think on Q_n as $Q_n := \bigcup_{x\in K} C_n(x)$, where $K \in \Sigma$ is a subset of measure 0.

Theorem B. *Let Σ be a topologically transitive two sided subshift of finite type. Let $\phi \in C^\alpha(\Sigma,\mathbb{R})$, $\alpha > 0$, and let μ_ϕ be the equilibrium state for ϕ. Let $(Q_n)_{n\ge0}$ be a collection of subsets of Σ such that $\forall n \ge 0$, $Q_{n+1} \subset Q_n$; Q_n is a (disjoint) union of $n-$cylinders $C_n(x)$ and*

(a) $\limsup_{n\to+\infty} \frac{1}{n}\log\mu_\phi(Q_n) < 0$

Let $L\colon (\Sigma\setminus K) \to \mathbb{R}$, $K \subset \bigcap_n Q_n$, be a function such that there exist $A, B > 0$ and $\gamma > 0$ which satisfy

(b) $|L(x)| \le Bn$ *if* $x \notin Q_n$

(c) $|L(x) - L(y)| \le A^n d(x,y)^\gamma$ *if* $x, y \in \Sigma\setminus Q_n$.

Then there exists a neighbourhood $\mathcal{U}$ of ϕ in $C^\alpha(\Sigma,\mathbb{R})$ such that for all $\varphi \in \mathcal{U}$ the integral $\mu_\varphi(L) := \int L\, d\mu_\varphi$ exists and the mapping $\mathcal{U} \to \mathbb{R}\colon \varphi \mapsto \mu_\varphi(L)$ is C^∞.

Proof of Theorem B

Here we prove theorem B using theorem A. We need the following result.

1. Proposition. *Let Σ be a two-sided subshift of finite type. Suppose that we have*

(a) *λ a probability on Σ.*

(b) *$\mu_k \in (C^\alpha(\Sigma,\mathbb{R}))^*$ such that there exists $k \ge 0$ and $D(k) > 0$ such*

that

$$\forall p \in \Sigma,\ \forall n \ge 0\colon \quad \left|\mu_k(\mathbf{1}_{C_n(p)})\right| \le D(k)\; n^k\; \lambda(C_n(p))\,.$$

(c) *A subset $K \subset \Sigma$ such that if $Q_n := \bigcup_{x\in K} C_n(x)$ then*

$$\limsup_{n\to+\infty} \frac{1}{n}\log\lambda(Q_n) \le -c < 0.$$

(d) *$L\colon (\Sigma\setminus K) \to [0,+\infty[$ a non-negative function such that there exist A, $B \ge 1$ such that*

$$L(x) \le B\,n \quad \text{if } x \notin Q_n$$
$$|L(x) - L(y)| \le A^n\, d(x,y)^\gamma \quad \text{if } x,\, y \in \Sigma\setminus Q_n\,.$$

Then there exists a sequence (g_n) of simple non-negative functions which are locally constant on n-cylinders, such that

$$\sum g_n \uparrow L \text{ outside } \bigcap_m Q_m \supset K$$
$$\sum_n |\mu_k(g_n)| \le D(k)\, G(k)\,,$$

with $G(k) > 0$, in particular $\sum_n |\mu_k(g_n)| < \infty$.

Proof. For simplicity we prove the proposition only for $B = 1$. Observe that μ_k is not necessarily positive and that it can only be applied to Hölder continuous functions. Let $D_N = C_N(p)$ be an N-cylinder in $Q_{N-1}\backslash Q_N$. Let

$$R := \left(\frac{\log A}{-\gamma\log b}\right) + 1$$

and choose $n = n(N) \in \mathbb{N}$ such that

$$\left(\frac{\log A}{-\gamma\log b}\right) N < n \le R\,N$$

so that $A^N b^{\gamma n} < 1$. Let

$$g_n^{D_N}(x) := \sum_{E_n}\left(\min_{z\in E_n} L(z)\right)\mathbf{1}_{E_n}(x),$$

where the sum is over all the n-cylinders E_n such that $E_n \subset D_N$. Inductively, for $m \ge 0$ let

$$g_{n+m+1}^{D_N} := \sum_{E_{n+m+1}\subset D_N}\left(\min_{E_{n+m+1}} L\right)\mathbf{1}_{E_{n+m+1}} - \sum_{r=0}^{m} g_{n+r}^{D_N}$$

where the sum in on all the $(n+m-1)-$cylinders $E_{n+m+1} \subset D_N$. The $g_m^{D_N}$'s are simple functions of the form

$$g_m^{D_N} = \sum_{E_m^r \subset D_N} a_r \, \mathbf{1}_{E_m^r}.$$

(where $a_r \geq 0$ and the E_m^r are $m-$cylinders) and hence $\gamma-$Hölder continuous. Therefore

$$\mu_k(g_m^{D_N}) = \sum_{E_m^r \subset D_N} a_r \, \mu_k(\mathbf{1}_{E_m^r}).$$

Since for $x \in D_N$, $L(x) \leq N$, we have that

$$\left|\mu_k(g_{n(N)}^{D_N})\right| \leq D(k) \sum_{E_n \subset D_N} N\, n^k \lambda(E_n) \leq D(k)\, R^k N^{k+1} \lambda(D_N). \tag{1}$$

Let $E_{n+m-1} \subset D_N$ be an $(n+m-1)$-cylinder in D_N. Since

$$\operatorname*{var}_{E_{n+m-1}} L \leq A^N \, b^{\gamma(n+m-1)},$$

we have that

$$g_{n+m}^{D_N}(x) \leq A^N \, b^{\gamma(n+m-1)}, \quad \forall x \in D_N, \quad \forall m \geq 1.$$

and then

$$\begin{aligned}\left|\mu_k(g_{n+m}^{D_N})\right| &\leq D(k)\, A^N \, b^{\gamma(n+m-1)} \sum_{E_{n+m} \subset D_N} (m+n)^k \, \lambda(E_{m+n}) \\ &\leq D(k)\, b^{-\gamma}\, b^{\gamma m}\, (m+n)^k \, \lambda(D_N).\end{aligned}$$

because $A^N b^{\gamma n} < 1$. Since

$$\begin{aligned}\sum_{m=1}^{\infty} b^{\gamma m}\,(m+n)^k &= \sum_{m=1}^{n} b^{\gamma m}\,(m+n)^k + \sum_{m=n}^{\infty} b^{\gamma m}\,(m+n)^k \\ &\leq 2^k\, n^k \left(\frac{1}{1-b^\gamma}\right) + \sum_{m=1}^{\infty} b^{\gamma m} \cdot 2^k\, m^k ,\end{aligned}$$

writing $S(k){:} = \sum_{m=1}^{\infty} b^{\gamma m}\, m^k$, we have that

$$\sum_{m=1}^{\infty} \left|\mu_k(g_{n(N)+m}^{D_N})\right| \leq D(k) \left(\frac{b^{-\gamma}}{1-b^\gamma}\right) 2^k \,(R^k N^k + S(k))\, \lambda(D_N).$$

From this and (1) we get

$$\sum_{m=0}^{\infty} \left|\mu_k(g_{n(N)+m}^{D_N})\right| \leq D(k) \left(\frac{b^{-\gamma}}{1-b^\gamma}\right) (2R)^k \,(S(k) + 2N^{k+1})\, \lambda(D_N).$$

For each $N \geq 1$ consider a partition $\mathcal{P}$ of $Q_{N-1} \setminus Q_N$ into N-cylinders D_N and the corresponding $\left(g_{n(N)+m}^{D_N}\right)_{m \geq 0}$, we get

$$\begin{aligned}&\sum_{D_N \in \mathcal{P}} \sum_{i \geq n(N)} \left|\mu_k(g_i^{D_N})\right| \leq \\ &\leq D(k) \left(\frac{b^{-\gamma}}{1-b^\gamma}\right) (2R)^k \,(S(k) + 2\, N^{k+1})\, \lambda(Q_{N-1} \setminus Q_N)\end{aligned}$$

Define

$$g_i := \sum_{N \text{ with } n(N) \leq i} g_i^{D_N}.$$

Since $\sum_N \lambda(Q_{N-1} \setminus Q_N) \leq \lambda(\Sigma) = 1$ and $\lambda(Q_{N-1} \setminus Q_N) \leq \lambda(Q_{N-1})$, we have that

$$\sum_i |\mu_k(g_i)| \leq D(k) \left(\frac{b^{-\gamma}}{1-b^\gamma}\right) (2R)^k \left(S(k) + 2 \sum_N N^{k+1} \lambda(Q_{N-1})\right); \tag{2}$$

where the series

$$F(k) := 2 \sum_N N^{k+1} \lambda(Q_{N-1})$$

is convergent by hypothesis (c).□

Given $\phi \in C^\alpha(\Sigma, \mathbb{R})$, let $P(\phi)$ be its topological pressure (cf. [7]).

Proof of theorem B

It is enough to prove theorem B for $B = 1$ and L non-negative. There are constants $A_0, B_0 > 0$, uniform on a neighbourhood of $\mathcal{U}_0$ of ϕ (cf. [**1**]) such that for any n-cylinder $C_n(p)$ in Σ and all $\psi \in \mathcal{U}_0$, we have

$$\mu_\psi(C_n(p)) \in [A_0, B_0] \exp\big(-2n\, P(\psi) + S_n\psi(p)\big)\,, \tag{3}$$

where

$$S_n\psi(x) := \sum_{k=-n}^{n} \psi(\sigma^k(x)).$$

It is easy to see from the definition of pressure that $|P(\phi+\varphi)-P(\phi)|\le |\varphi|_0$. For $\psi=\phi+\varphi\in\mathcal{U}_0$, we have

$$0\le \mu_{\phi+\varphi}(C_n(p))\le B_0\,\exp\big(-2n\,P(\phi)+S_n\phi(p)\big)\,\exp(4\,|\varphi|_0\,n)$$
$$\le A_0^{-1}\,B_0\,\exp(4\,|\varphi|_0\,n)\,\mu_\phi(C_n(p)).$$

Since Q_n is a disjoint union of n-cylinders, we have

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mu_{\phi+\varphi}(Q_n)\le 2\,|\varphi|_0+\limsup_{n\to+\infty}\frac{1}{n}\log\mu_\phi(Q_n)<0$$

if $|\varphi|_0$ is sufficiently small. Choose a neighbourhood $\mathcal{U}_1$ of ϕ and $c>0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mu_\psi(Q_n)<-c<0 \tag{4}$$

for all $\psi\in\mathcal{U}_1$.

Using Theorem A(a) and (4), we can apply Proposition 1 to L, $\lambda=\mu_\phi$, and $\mu_k=(D^{(k)}\mu_\phi)(\varphi_1,\dots,\varphi_k)$ (with $D(k)=k!\,D(\mathcal{U}_2)^k\ \|\varphi_1\|\cdots\cdots\|\varphi_k\|$), and a neighbourhood $\mathcal{U}_2\subset\mathcal{U}_1$. It follows that there exists a decomposition $L=\sum_n g_n$ and for any $r\ge 0$ a number $H(r)>0$, constant on $\mathcal{U}_2$ and so that

$$\sum_n\left|(D^{(r)}_{\overline{\psi}}\mu_\psi)(\varphi_1,\dots,\varphi_r)\cdot g_n\right|\le H(r)\ \|\varphi_1\|\cdots\|\varphi_r\| \tag{5}$$

for all $\varphi_1,\dots,\varphi_r$ in $C^\alpha(\Sigma,\mathbb{R})$ and $\overline{\psi}\in\mathcal{U}_2$.

Now we prove that for all $k\ge 0$, one has

$$(D^{(k)}_\phi\mu_\varphi)(\varphi_1,\dots,\varphi_k)\cdot g_n=\left(D^{(k)}_\phi\mu_\phi(g_n)\right)(\varphi_1,\dots,\varphi_k). \tag{6}$$

Observe that if $(\mu_t)_{t\ge0}\subset(C^\alpha(\Sigma,\mathbb{R}))^*$ and $\omega=\lim_{t\to0}\mu_t$ in the dual $\alpha-$Hölder norm on $(C^\alpha(\Sigma,\mathbb{R}))^*$, then for all $g\in C^\alpha(\Sigma,\mathbb{R})$ we have that $\omega(g)=\lim_{t\to0}\mu_t(g)$. For $k=0$, (6) is trivial. Suppose by induction that (6) is true for $k\ge0$, then

$$\left[(D^{(k+1)}_\phi\mu_\phi)(\varphi_1,\dots,\varphi_{k+1})\right](g_n)=$$
$$=\lim_{t\to0}\frac{1}{t}\left[(D^{(k)}_{\phi+t\varphi_{k+1}}\mu_{\phi+t\varphi_{k+1}})(\varphi_1,\dots,\varphi_k)-(D^{(k)}_\phi\mu_\phi)(\varphi_1,\dots,\varphi_k)\right](g_n)$$

but since g_n is $\alpha-$Hölder, we have

$$=\lim_{t\to0}\frac{1}{t}\Big[(D^{(k)}_{\phi+t\varphi}\mu_{\phi+t\varphi_{k+1}})(\varphi_1,\dots,\varphi_k)(g_n)-(D^{(k)}_\phi\mu_\phi)(\varphi_1,\dots,\varphi_k)(g_n)\Big]$$

and by the induction hypothesis

$$=\lim_{t\to0}\frac{1}{t}\Big[\left(D^{(k)}_{\phi+t\varphi}\mu_{\phi+t\varphi}(g_n)\right)(\varphi_1,\dots,\varphi_k)-\left(D^{(k)}_\phi\mu_\phi(g_n)\right)(\varphi_1,\dots,\varphi_k)\Big]$$
$$=\left(D^{(k+1)}_\phi\mu_\phi(g_n)\right)(\varphi_1,\dots,\varphi_{k+1}).$$

Moreover, for $k\ge0$,we can write

$$\begin{aligned}(D^{(k)}_{\psi+f}\mu_\psi(g_n))(\varphi_1,\dots,\varphi_k)&=(D^{(k)}_\psi\mu_\psi(g_n))(\varphi_1,\dots,\varphi_k)\\&\quad+(D^{(k+1)}_\psi\mu_\psi(g_n))(\varphi_1,\dots,\varphi_k,f)\\&\quad+\int_0^1 s\left(D^{(k+2)}_{\psi+sf}\mu_\psi(g_n)\right)(\varphi_1,\dots,\varphi_k,f,f)ds\end{aligned} \tag{7}$$

From proposition 1 we get that $\mu_\psi(L)=\sum_n\mu_\psi(g_n)$ is finite. Equation (7) for $k=0$ and equations (5) and (6) for $r=0,1,2$, prove the continuity and differentiability of $\psi\mapsto\mu_\psi(L)$ in $\mathcal{U}_2$. By induction we assume that $\psi\mapsto\mu_\psi(L)$ is k-times differentiable, $k\ge1$, and that

$$(D^{(k)}_\psi(\mu_\psi(L)))(\varphi_1,\dots,\varphi_k)=\sum_n(D^{(k)}_\psi\mu_\psi(g_n))(\varphi_1,\dots,\varphi_k).$$

Then (7), (5) and (6) for $r=k,k+1,k+2$, prove that $\psi\mapsto\mu_\psi(L)$ is $(k+1)$-times differentiable in $\psi\in\mathcal{U}_2$.□

Observe that since $S(k)$ and $F(k)$ in proposition 1 have order $k!$, the estimate (2) in proposition 1 is not enough to prove the analyticity of $\phi\mapsto\int L\,d\mu_\phi$.

Proof of Theorem A.

In order to prove Theorem A we need some estimates. Let Σ^+ be a topologically transitive one-sided subshift of finite type. From now on fix $\beta=\alpha/2>0$, $\phi_0\in C^\beta(\Sigma^+,\mathbb{R})$, $p\in\Sigma^+$ and $n\ge0$. Consider the $\beta-$Hölder norm, $\|\ \ \|$, in $C^\beta(\Sigma^+,\mathbb{R})$, we have that

$$|f\cdot g|_\beta\le|f|_0\,|g|_\beta+|f|_\beta\,|g|_0\,;$$
$$\|f\cdot g\|\le\|f\|\,\|g\|\,.$$

Given $\phi \in C^\beta(\Sigma^+, \mathbb{R})$ let $P(\phi)$ be its topological pressure (cf. [7]).

2. Lemma. *For all $\phi \in C^\beta(\Sigma^+, \mathbb{R})$ there exists $A(\phi) > 0$, constant on a neighbourhood of ϕ_0, such that for all $\varphi \in C^\beta(\Sigma^+, \mathbb{R})$ and all $n \geq 0$, $k \geq 0$ we have*

$$\left|D_\phi^{(k)} \exp(-P(\phi)n)(\varphi_1, \dots, \varphi_k)\right| \leq$$
$$\leq n^k\, k!\, A(\phi)^k \exp(-P(\phi)n)\ \|\varphi_1\| \cdots \|\varphi_k\| .$$

Proof. We know that (cf. [6] 5.27) the map $C^\beta(\Sigma^+, \mathbb{R}) \to \mathbb{R}$; $\phi \mapsto P(\phi)$ is analytic. Hence the coefficients of its Taylor series can grow at most exponentially. Therefore there exists $B(\phi) > 0$ such that, for all $k \geq 0$,

$$\frac{1}{k!}\left\|D_\phi^{(k)} P(\phi)\right\| \leq B(\phi)^k .$$

Moreover, $B(\phi)$ can be chosen constant on a neighbourhood of ϕ. Since the map $\phi \mapsto \exp(-P(\phi)\, n)$ is also analytic, we have that

$$D_\phi^{(k)}(\exp(-P(\phi)n))(\varphi_1, \dots, \varphi_k) =$$
$$= \sum_{s=1}^{k} \sum_{\substack{A_1+\dots+A_s=\mathbb{N}_k \\ \#A_i \geq 1}} \exp(-P(\phi)n)(-1)^s n^s \left(D_\phi^{(\#A_1)} P\right)(\varphi_i)_{i\in A_1} \cdots$$
$$\cdots \left(D_\phi^{(\#A_s)} P\right)(\varphi_j)_{j\in A_s} ,$$

where $\mathbb{N}_k := \{1, 2, \dots, k\}$ and $+$ denotes the disjoint union of sets. Therefore

$$|D_\phi^{(k)}(\exp(-P(\phi)n))(\varphi_1, \dots, \varphi_k)| \leq$$
$$\leq \sum_{s=1}^{k} \sum_{A_1+\dots+A_s=\mathbb{N}_k} \exp(-P(\phi)n)\, n^k\, (\#A_1)! \cdots$$
$$\cdots (\#A_s)!\, B(\phi)^k\ \|\varphi_1\| \cdots \|\varphi_k\|$$
$$\leq \sum_{s=1}^{k} \exp(-P(\phi)n)\, n^k\, k!\, B(\phi)^k\ \|\varphi_1\| \cdots \|\varphi_k\|$$
$$\leq n^k\, k\, k!\, B(\phi)^k \exp(-P(\phi)n)\ \|\varphi_1\| \cdots \|\varphi_k\| .$$

Now let $A(\phi) = 2\, B(\phi)$. □

Let f_n be the branch of the inverse of σ_+^n which sends $f_n\colon C_0(\sigma_+^n(p)) \to C_n(p)$. For $\varphi \in C^\beta(\Sigma^+, \mathbb{R})$, $n \geq 0$, $p \in \Sigma^+$, write $\tilde{S}_n\varphi := S_n\varphi \circ f_n$, $\tilde{S}_n\varphi\colon C_0(\sigma_+^n p) \to \mathbb{R}$, where $S_n\varphi(z) := \sum_{k=0}^{n-1} \varphi(\sigma_+^k(z))$. Let $|||\ \ |||$ be the β-Hölder norm in $C^\beta(C_0(\sigma_+^n(p)))$, i.e.

$$|||\psi||| := [\psi]_0 + [\psi]_\beta \quad , \quad [\psi]_0 := \sup_{x\in C_0(\sigma_+^n(p))} |\psi(x)|$$

$$[\psi]_\beta := \sup\left\{ \frac{|\psi(x) - \psi(y)|}{d(x,y)^\beta} \;\middle|\; x, y \in C_0(\sigma_+^n(p)) \ ,\ d(x,y) \neq 0 \right\}.$$

Since $d(f_n(x), f_n(y)) \leq b^n d(x,y)$, we have that

$$|||\varphi \circ f_n||| \leq \|\varphi\| \qquad \text{for all } \varphi \in C^\beta(\Sigma^+, \mathbb{R}). \tag{8}$$

3. Lemma. *The map $C^\beta(\Sigma^+, \mathbb{R}) \to C^\beta(C_0(\sigma_+^n(p)), \mathbb{R})$: $\phi \mapsto \exp(\tilde{S}_n\phi)$ is analytic.*

Moreover, there exist $B(\phi) > 0$ and $E(\phi) > 0$, constant on a neighbourhood of ϕ_0, such that for all $\phi \in C^\beta(\Sigma^+, \mathbb{R})$ and $k \geq 0$ we have

(a) $\left|\left|\left|(D_\phi^{(k)} \tilde{h}_\phi)(\varphi_1, \dots, \varphi_k)\right|\right|\right| \leq k!\, E(\phi)^k\ \|\varphi_1\| \cdots \|\varphi_k\|$.

(b) $\left|\left|\left|(D_\phi^{(k)} \exp(\tilde{S}_n\phi))(\varphi_1, \dots, \varphi_k)\right|\right|\right| \leq B(\phi)\ n^k\ \|\varphi_1\| \cdots \|\varphi_k\| \exp(S_n\phi(p))$

Proof. The estimate (8) shows that the map $C^\beta(\Sigma^+, \mathbb{R}) \to C^\beta(C_0(\sigma_+^n(p), \mathbb{R})$, $\psi \mapsto \psi \circ f_n$ is a bounded linear map. Since the map $C^\beta(\Sigma^+, \mathbb{R}) \hookleftarrow$: $\phi \mapsto h_\phi$ is analytic, we get that $\phi \mapsto \tilde{h}_\phi = h_\phi \circ f_n$ is analytic. This implies the existence of $E(\phi) > 0$.

We have that

$$\left|\left|\left|\tilde{S}_n\phi\right|\right|\right| \leq \sum_{k=0}^{n-1} \left|\left|\left|(\phi \circ \sigma_+^k) \circ f_n\right|\right|\right| = \sum_{k=0}^{n-1} |||\phi \circ f_{n-k}||| \leq n\ \|\varphi\| .$$

Therefore $\tilde{S}_n$ is a bounded linear operator $\tilde{S}_n : C^\beta(\Sigma^+, \mathbb{R}) \to C^\beta(C_0(\sigma_+^n(p)), \mathbb{R})$ and hence analytic. We have that

$$(D_\phi^{(k)} \exp(\tilde{S}_n\phi))(\varphi_1, \dots, \varphi_k) = (\exp \tilde{S}_n\phi)(S_n\varphi_1) \cdots (S_n\varphi_k) ,$$
$$|||(D_\phi^{(k)} \exp(\tilde{S}_n\phi))(\varphi_1, \dots, \varphi_k)||| \leq |||\exp(\tilde{S}_n\phi)|||\, n^k\ \|\varphi_1\| \cdots \|\varphi_k\|$$

If $z, w \in C_0(\sigma_+^n(p))$ and $x := f_n(z)$, $y := f_n(w)$, we have that

$$
\begin{aligned}
\left|\tilde S_n\phi(z) - \tilde S_n\phi(w)\right| &\le \sum_{k=0}^{n-1} \left|\phi(\sigma_+^k x) - \phi(\sigma_+^k y)\right| \\
&\le |\phi|_\beta \sum_{k=0}^{n-1} d(\sigma_+^k x, \sigma_+^k y)^\beta \le |\phi|_\beta \sum_{k=0}^{n-1} b^{-k\beta} d(x,y)^\beta \\
&\le |\phi|_\beta\, B_0\, d(z,w)^\beta \le B_0\ |\phi|_\beta\ ,
\end{aligned}
$$

where $B_0 := (b^{-\beta} - 1)^{-1}$. Thus, taking $z = \sigma_+^n(p)$ we obtain

$$[\exp \tilde S_n \phi]_0 \le (\exp S_n\phi(p)) \exp(B_0\, |\phi|_\beta)\ ,$$

$$
\begin{aligned}
\left|\exp \tilde S_n\phi(z) - \exp \tilde S_n\phi(w)\right| &= \left|\exp \tilde S_n\phi(z)\right| \left|1 - \exp(\tilde S_n\phi(w) - \tilde S_n\phi(z))\right| \\
&\le (\exp S_n\phi(p))\ \exp(B_0\,|\phi|_\beta)\, A_0\, B_0\, |\phi|_\beta\, d(z,w)^\beta\ ,
\end{aligned}
$$

where

$$A_0 := \exp(B_0\,|\phi|_\beta) \ge \sup\left\{ \left|\frac{d}{dx} e^x|_{x_0}\right| \ :\ |x_0| \le B_0\,|\phi|_\beta \right\}.$$

And then

$$|||\exp \tilde S_n\phi||| \le (A_0 + A_0^2\, B_0\, |\phi|_\beta) \exp(S_n\phi(p)).$$

Therefore

$$|||(D_\phi^{(k)} \exp(\tilde S_n\phi))(\varphi_1, \ldots, \varphi_k)||| \le \tilde B(\phi)\, n^k\ \|\varphi_1\| \cdots \|\varphi_k\|\ \exp(S_n\phi(p))$$

where $\tilde B(\phi) := A_0 + A_0^2 B_0\, |\phi|_\beta$. We can choose $B(\phi) = \tilde B(\phi_0) + 1$ for all ϕ on a small neighbourhood of ϕ_0. □

Let $\mathcal{L}_\phi : C^\beta(\Sigma^+, \mathbb{R}) \hookleftarrow$ be the Perron-Frobenius operator:

$$(\mathcal{L}_\phi \varphi)(x) := \sum_{y \in \sigma_+{}^{-1}(x)} \varphi(y)\ \exp(\phi(y)).$$

We use the following notation from Ruelle's theorem ([**1**] (1.7)):

$$
\begin{aligned}
&\mathcal{L}_\phi(h_\phi) = \lambda_\phi\, h_\phi \quad , \quad \int h_\phi\, d\nu_\phi = 1 \quad , \quad \lambda_\phi = e^{P(\phi)}. \\
&\mathcal{L}_\phi^*(\nu_\phi) = \lambda_\phi\, \nu_\phi \quad , \quad \mu_\phi = h_\phi\, \nu_\phi.
\end{aligned}
$$

A proof that the maps $C^\beta(\Sigma^+, \mathbb{R}) \to \mathbb{R} : \phi \mapsto P(\phi)$, $C^\beta(\Sigma^+, \mathbb{R}) \to (C^\beta(\Sigma^+, \mathbb{R}))^* : \phi \mapsto \mu_\phi$, $\phi \mapsto \nu_\phi$ and $C^\beta(\Sigma^+, \mathbb{R}) \to C^\beta(\Sigma^+, \mathbb{R}) : \phi \mapsto h_\phi$ are real analytic can be found in [5].

Define

$$
\begin{aligned}
J_\phi^n(z) &:= \lambda_\phi^{-n}\, \mathcal{L}_\phi^n(h_\phi \cdot \mathbf{1}_{C_n(p)})(z) \\
&= \begin{cases} \exp(-P(\phi)n + S_n\phi(x))\ h_\phi(x) & \text{if } \sigma_+^n x = z, x \in C_n(p) \\ 0 & \text{if } z \notin C_0(\sigma_+^n p) \end{cases}
\end{aligned}
$$

Thus the measure of a one-sided cylinder is given by

$$\mu_\phi(C_n(p)) = \nu_\phi(h_\phi \cdot \mathbf{1}_{C_n(p)}) = \lambda_\phi^{-n}\ \nu_\phi(\mathcal{L}_\phi^n(h_\phi \cdot \mathbf{1}_{C_n(p)})) = \int_{C_0(\sigma_+^n p)} J_\phi^n\ d\nu_\phi\ .$$

Given $p \in \Sigma^+$, $n \ge 0$, denote by f_n the branch of the inverse of σ_+^n sending $f_n : C_0(\sigma_+^n(p)) \to C_n(p)$. Given $\phi \in C^\beta(\Sigma^+, \mathbb{R})$, let

$$
\begin{aligned}
\tilde h_\phi &:= h_\phi \circ f_n \in C^\beta(C_0(\sigma_+^n(p)), \mathbb{R})\ , \\
\tilde J_\phi^n &:= \exp(-P(\phi)n + \tilde S_n\phi)\ \tilde h_\phi\ .
\end{aligned}
$$

4. Lemma. *The map* $C^\beta(\Sigma^+, \mathbb{R}) \longrightarrow C^\beta(C_0(\sigma_+^n(p)), \mathbb{R}) \ :\ \phi \mapsto \tilde J_\phi^n$ *is analytic and there exists* $D_0(\phi) > 0$*, constant on a neighbourhood of* ϕ_0*, such that for all* $\varphi \in C^\beta(\Sigma^+, \mathbb{R})$ *we have*

$$
\begin{aligned}
&|||(D_\phi^{(k)} \tilde J_\phi^n)(\varphi_1, \ldots, \varphi_k)||| \le \\
&\le k!\, D_0(\phi)^k\, n^k\ \exp(-P(\phi)n + S_n\phi(p))\ \|\varphi_1\| \cdots \|\varphi_k\|\ .
\end{aligned}
$$

Proof. We have that

$$
\begin{aligned}
\left(D_\phi^{(k)} \tilde J_\phi^n\right)(\varphi_1 \ldots \varphi_k) = &\sum_{A+B+C=\mathbb{N}_k} \left((D_\phi^{(\#A)} \exp(-P(\phi)n))(\varphi_a)_{a\in A} \right) \cdot \\
&\cdot \left((D_\phi^{(\#B)} \exp(\tilde S_n\phi))(\varphi_b)_{b\in B} \right) \left((D_\phi^{(\#C)} \tilde h_\phi)(\varphi_c)_{c\in C} \right)
\end{aligned}
$$

By lemmas 2 and 3 we have that

$$
\begin{aligned}
\left|\left|\left|(D_\phi^k \tilde J_\phi^n)(\varphi_1, \ldots \varphi_k)\right|\right|\right| \le &\sum_{A+B+C=\mathbb{N}_k} (\#A)!\ n^{\#A}\ A(\phi)^{\#A}\ \exp(-P(\phi)n) \\
&\cdot \prod_{a\in A} \|\varphi_a\|\, B(\phi)\, n^{\#B}\ \exp(S_n\phi(p)) \prod_{b\in B} \|\varphi_b\| \\
&\cdot (\#C)!\ E(\phi)^{\#C} \prod_{c\in C} \|\varphi_c\|\ .
\end{aligned}
$$

$$
\begin{aligned}
\left|\left|\left|(D_\phi^{(k)} \tilde J_\phi^n)(\varphi_1, \ldots, \varphi_k)\right|\right|\right| \le k!\ n^k\, (D_0(\phi))^k \exp(-P(\phi)n + S_n\phi(p)) \\
\cdot \|\varphi_1\| \cdots \|\varphi_k\|\ ,
\end{aligned}
$$

where $D_0(\phi) := \max\{A(\phi), B(\phi), E(\phi), 2\}$.□

Proof of Theorem A(b). Let E_0 be the 0-cylinder $E_0 := C_0(\sigma_+^n(p))$. Consider the map $R: C^\beta(E_0,\mathbb{R}) \to C^\beta(\Sigma^+,\mathbb{R})$ given by $R(\varphi)(z) := \varphi(z)$ if $z \in E_0$, $R(\varphi)(z) := 0$ if $z \notin E_0$. We claim that for $\varphi \in C^\beta(E_0,\mathbb{R})$, we have

$$\|R(\varphi)\| \le (1 + b^{-\beta})\ |||\varphi||| \ .$$

Indeed, clearly $|R(\varphi)|_0 = [\varphi]_0$. Now, if x, $y \in E_0$, then

$$|R(\varphi)(x) - R(\varphi)(y)| = |\varphi(x) - \varphi(y)| \le [\varphi]_\beta\, d(x,y)^\beta.$$

If x, $y \notin E_0$, then $|R(\varphi)(x) - R(\varphi)(y)| = 0$. If $x \in E_0$, $y \notin E_0$, then $d(x,y) = b$ and

$$|R(\varphi)(x) - R(\varphi)(y)| = |R(\varphi)(x)| \le [\varphi]_0 = [\varphi]_0\, b^{-\beta}\, d(x,y)^\beta.$$

Therefore $|R(\varphi)|_\beta \le b^{-\beta}\,[\varphi]_0$.

We have that $\mu_\phi(\mathbf{1}_{C_n(p)}) = \nu_\phi(J_\phi^n) = \nu_\phi(R(\tilde{J}_\phi^n))$. Since the map $(C^\beta(\Sigma^+,\mathbb{R}))^* \times C^\beta(E_0,\mathbb{R}) \to \mathbb{R}: (\nu, J) \mapsto \nu(R(J))$ is bilinear, we have that

$$\Big(D_\phi^{(k)}\mu_\phi(\mathbf{1}_{C_n(p)})\Big)(\varphi_1,\ldots,\varphi_k) =$$
$$= \sum_{A+B=\mathbb{N}_k} \Big((D_\phi^{(\#A)}\nu_\phi)(\varphi_a)_{a\in A}\Big)\ R\Big((D_\phi^{(\#B)}\tilde{J}_\phi^n)(\varphi_b)_{b\in B}\Big)$$

Since $\phi \mapsto \nu_\phi \in \big(C^\beta(\Sigma^+,\mathbb{R})\big)^*$ is analytic, there exists $C(\phi) > 0$, constant on a neighbourhood of ϕ_0, such that

$$\|(D_\phi^{(r)}\nu_\phi)(\varphi_1,\ldots,\varphi_r)\|_* \le r!\ C(\phi)^r\ \|\varphi_1\|\cdots\|\varphi_r\|,$$

where $\|\ \ \|_*$ is the dual norm in $\big(C^\beta(\Sigma^+,\mathbb{R})\big)^*$. Therefore, by lemma 4, we have that

$$\Big|D_\phi^{(k)}\mu_\phi(\mathbf{1}_{C_n(p)})(\varphi_1,\ldots,\varphi_k)\Big| \le$$
$$\le \sum_{A+B=\mathbb{N}_k} (\#A)!\ C(\phi)^{\#A}\cdot(1+b^{-\beta})(\#B)!\ \left|\left|\left|\frac{1}{(\#B)!}\Big(D_\phi^{(s)}\tilde{J}_\phi^n\Big)(\varphi_b)_{b\in B}\right|\right|\right|$$
$$\le k!\ n^k\ D_1(\phi)^k(1+b^{-\beta})\exp\big(-P(\phi)n + S_n\phi(p)\big)\ \ \|\varphi_1\|\cdots\|\varphi_k\|\ , \qquad (9)$$

where $D_1(\phi) = (1 + b^\beta)\max\{\,C(\phi)\,,\,D_0(\phi)\,\}$. Part (b) of Theorem A follows from (9) and recalling (cf. [1]) that there are constants A, $B > 0$, uniform on a neighbourhood of ϕ_0, such that

$$\mu_\phi(C_n(p)) \in [A,B]\exp(-P(\phi)n + S_n\phi(p))\,. \qquad \square$$

In order to prove part (a) of Theorem A we need some definitions. We say that two functions ϕ, $\varphi \in C^\beta(\Sigma,\mathbb{R})$ are *homologous* if there exists $h \in C^\beta(\Sigma,\mathbb{R})$ such that $\phi = \varphi + h\circ\sigma - h$. Homologous functions have the same pressure and the same equilibrium state. There is a natural embedding $C^\beta(\Sigma^+,\mathbb{R}) \hookrightarrow C^\beta(\Sigma,\mathbb{R}): \phi \mapsto \overline{\phi}$ by considering $\overline{\phi}(x) := \phi(x_0, x_1,\ldots)$. The proof of the following lemma appears in pg. 11 of [1].

5. Lemma. *There exists a continuous linear map $A: C^\alpha(\Sigma,\mathbb{R}) \to C^\beta(\Sigma^+,\mathbb{R})$, $\beta = \alpha/2$, such that $A(\psi)$ is homologous to ψ. In particular $A^*: C^\beta(\Sigma^+,\mathbb{R})^* \to C^\alpha(\Sigma,\mathbb{R})^*$, $(A^*\mu)(\varphi) = \mu(A\varphi)$ is a continuous linear map.*

Proof of Therorem A(a). We first claim that there exists $E(\phi) > 0$, constant on a neighbourhood of ϕ_1 such that for all $n \ge 0$, $k \ge 0$, $p \in \Sigma$, we have

$$|(D_\phi^{(k)}\mu_\phi(\mathbf{1}_{C_n(p)}))(\varphi_1,\ldots,\varphi_k)| \le$$
$$\le k!\ (2n)^k\ E(\phi)\ \exp\big(-2n\,P(\phi) + S_n\varphi(p)\big)\ \|\varphi_1\|\cdots\|\varphi_k\|\,, \qquad (10)$$

where

$$S_n\varphi(p) = \sum_{k=-n}^{n}\varphi(\sigma^k p).$$

The proof of this claim consists on using (9) (for 1-sided subshifts) and observing that $\mu_\phi(C_n(p)) = \mu_{\overline{\phi}}^{+}(\sigma^{-n}C_n(p))$, where $\sigma^{-n}(C_n(p))$ is considered as a cylinder in Σ^+ and $\mu_{\overline{\phi}}^{+}$ is the equilibrium state for $\overline{\phi} := A(\phi) \in C^{\alpha/2}(\Sigma^+,\mathbb{R})$ which is homologous to ϕ on Σ. In particular $P_{\Sigma^+}(\overline{\phi}) = P_\Sigma(\phi)$ and $\Big|S_n\phi(p) - S_n\overline{\phi}(p)\Big| \le 2\,|u|$, where $\overline{\phi} = \phi + u\circ\sigma - \sigma$. This introduces a factor $\exp(2\,|u|)$ in the constant $E(\phi)$.

The corollary follows from (10) and recalling the Gibbs Property, stated in formula (3). □

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