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Some Existence Results for Quasilinear Elliptic Equations in a Finite Cylinder

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Abstract. We study the existence of solutions u in the finite cylinder $S_a = (-a, a) \times \omega$ of the quasilinear elliptic equation

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u + f(x, u, \nabla u) = 0$$

with Dirichlet boundary condition on flat parts of ∂S_a and Neumann condition on the curved parts. In this paper, we focus on the technicality caused by the "corners" of S_a. We prove the existence of such solutions provided that suitable sub and super solutions are known and under the condition that the coefficients $a_{1,i}, i \neq 1$ vanish on the corners. We also prove a more general result in $\mathbb{R}^2$.

1. Introduction – Main Results

In the finite cylinder $S_a = \{x = (x_1, y) \in \mathbb{R}^n; -a < x_1 < +a, y \in \omega\}$, we wish to solve the boundary value problem,

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u + f(x, u, \nabla u) = 0 \quad \text{in } S_a \tag{1.1}$$

$$\partial_\nu u(x_1, y) = 0 \quad \text{for } -a < x_1 < +a, y \in \partial\omega \tag{1.2}$$

$$u(-a, y) = \psi_1(y), \quad u(a, y) = \psi_2(y) \tag{1.3}$$

Here, ω is a bounded regular domain in $\mathbb{R}^{n-1}$ and ν is the exterior unit normal to S_a (or ω).

For $i, j = 1, \ldots, n$, the coefficients $a_{i,j}$ are continuous in $\overline{S}_a$, satisfy $a_{i,j} = a_{j,i}$ for $i, j = 1, 2, \ldots, n$ and the usual condition of uniform

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ellipticity,

$$c_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \le C_0|\xi|^2 \qquad (1.4)$$
$$\forall x \in S_a,\ \forall \xi \in \mathbb{R}^n, \quad c_0, C_0 > 0$$

The functions $\psi_1 \le \psi_2$ are assumed to belong to $W^{2,\infty}(\omega)$ and to satisfy compatibility conditions with (1.2), i.e.

$$\partial_\nu \psi_j = 0 \text{ on } \partial\omega,\ j = 1, 2 \qquad (1.5)$$

The function $f: \overline{S}_a \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is required to satisfy certain assumptions detailed below.

H. Amann and M.G. Crandall [**AC**], considered this type of problems in a bounded and regular domain. Under the hypothesis that $f(x, u, p)$ grows at most quadratically in p, they proved that the existence of an ordered pair of sub and super solutions implies the existence of a solution of this problem.

More general quasilinear elliptic equations of the type

$$\sum_{i,j=1}^{n} a_{i,j}(x, u, \nabla u)\partial_{i,j}u + f(x, u, \nabla u) = 0$$

in a bounded regular domain Ω with a Dirichlet boundary condition, were studied in [**LU**] and [**S**]. Using a method due to Bernstein, J. Serrin [**S**] reduces the solvability of this problem to the setting of a priori estimates for solutions:

i) estimation of $\sup |u|$ on Ω

ii) estimation of $\sup |\nabla u|$ on $\partial\Omega$

iii) estimation of $\sup |\nabla u|$ on Ω

This reduction is achieved through the use of the Schauder fixed point Theorem in appropriate function spaces.

P.L. Lions [**L2**] applied this existence method to solve the Dirichlet problem

$$\sum_{i,j=1} a_{i,j}(x)\partial_{i,j}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u + \varphi(\nabla u) = f$$

in a bounded regular domain with a convex function φ.

Coming back now to the problem in the finite cylinders S_a, we denote by Σ the set of "corners" of S_a, $\Sigma = \{(\pm a, y), y \in \partial\omega\}$. In [**BN**], H. Berestycki and L. Nirenberg proved the existence of a solution in $C^0(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \backslash \Sigma)$ of the semilinear equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = f(x, u) \quad \text{in } S_a$$

with the boundary conditions (1.2)-(1.3). To prove this, they used an iteration method based on the existence of a pair of ordered sub and super solutions.

They first solved the linear problem

$$Lu = g \quad \text{in } S_a \qquad (1.6)$$

with the boundary conditions (1.2)-(1.3). For $g \in L^p(S_a)$, $p > n$, they proved the existence of a solution $u \in C^0(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \backslash \Sigma)$. In order to overcome the technical problem of corners, they considered an approximate problem in a subdomain of S_a in which the corners have been rounded off. Then they passed to the limit using a symmetric positive barrier function.

Our situation is more intricate since we need a priori estimate on the gradient to prove the existence of solutions of (1.1)-(1.3) as explained above. We can not apply the classical elliptic estimates, since the domain is no smooth (with corners). Another difficulty comes from the fact that different boundary conditions are imposed on parts of the boundary which touch each other. Indeed we can prove the existence of solutions in $C^0(\overline{\Omega}) \cap C^2(\Omega)$ of some Dirichlet problem

$$\sum_{i,j=1}^{n} a_{i,j}(x, u, \nabla u)\partial_{i,j}u + f(x, u, \nabla u) = 0 \quad \text{in } \Omega, \quad u = \phi \quad \text{on } \partial\Omega$$

when Ω is a bounded domain in $\mathbb{R}^n$ satisfying an exterior sphere condition on $\partial\Omega$ and $\phi \in C^0(\partial\Omega)$. This is proved by approximation of the domain Ω by $C^{2,\gamma}$ domains Ω_m and the function ϕ by functions $\phi_m \in C^{2,\gamma}(\Omega_m)$. Then we use the procedure of Serrin [**S**] described above to solve the problem in Ω_m (See [**GT**], Theorem 15.16).

We will first seek for solutions in $W^{2,p}(S_a)$, for some $p > n$, of the linear problem. This gives us some conditions on the coefficients $a_{i,j}$. We assume that the nonlinear term $f(x,u,p)$ is continuous on $\overline{S}_a \times \mathbb{R} \times \mathbb{R}^n$, locally lipschitz continuous in (u,p) and grows at most quadratically in p i.e.

$$|f(x,u,p)| \leq K(1+|p|^2) \quad \text{for } (x,u,p) \in \overline{S}_a \times \mathbb{R} \times \mathbb{R}^n$$

Then, we prove that the existence of sub and super solutions of (1.1)-(1.3) implies the existence of a solution in some Sobolev spaces $W^{2,p}(S_a)$, $p > n$.

Our main results are,

Theorem 1.1. *Let $\underline{u} \leq \overline{u}$ be sub and super solutions of the problem* (1.1)-(1.3) *belonging to $C^0(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \backslash \Sigma)$. That is they satisfy,*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}\underline{u} + f(x,\underline{u},\nabla\underline{u}) \geq 0 \geq$$

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}\overline{u} + f(x,\overline{u},\nabla\overline{u}) \quad \text{in } S_a$$

$$\partial_\nu\underline{u}(x_1,y) \leq 0 \leq \partial_\nu\overline{u}(x_1,y) \quad \text{for } -a < x_1 < a, y \in \partial\omega$$

$$\underline{u}(-a,y) \leq \psi_1(y) \leq \overline{u}(-a,y), \quad \underline{u}(a,y) \leq \psi_2(y) \leq \overline{u}(a,y) \text{ for } y \in \omega$$

If the coefficients $a_{i,j}$ satisfy,

$$a_{1,i}(\pm a,y) = 0 \quad \text{for } i \neq 1 \quad \text{and } y \in \omega \tag{1.7}$$

then there is a solution $u \in W^{2,p}(S_a)$ of (1.1)-(1.3) *for all $p > n$, with $\underline{u} \leq u \leq \overline{u}$.*

We do not know if the condition $a_{1,i}(\pm a,y) = 0$ for $i \neq 0$ is really needed for this result, this is still an open problem. However in the case of dimension 2, we prove a more general result,

Theorem 1.2. *Consider the case of dimension $n = 2$, and let $\underline{u} \leq \overline{u}$ be sub and super solutions of the problem* (1.1)-(1.3) *belonging to $C^0(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \backslash \Sigma)$. If the coefficients $a_{i,j}$ satisfy,*

$$a_{1,2}(S_i) = a_{2,1}(S_i) \leq 0 \quad \text{for } i \in \{1,\dots,4\} \tag{1.8}$$

where $\{S_1,\dots,S_4\}$ denote the corner points of S_a, then there is a solution u of (1.1)-(1.3) *and $\underline{u} \leq u \leq \overline{u}$. Furthermore u is in $W^{2,p}(S_a)$ for all $2 < p < p^*$ where p^* is defined by the following conditions:*

(i) *$p^* = +\infty$, if $\forall i \in \{1,\dots,4\}$ we have,*

$$-\tan\frac{\pi}{3} \leq \frac{\Lambda(S_i)}{a_{1,2}(S_i)} := \frac{[a_{1,1}(S_i)a_{2,2}(S_i) - a_{1,2}^2(S_i)]^{\frac{1}{2}}}{a_{1,2}(S_i)} < 0 \text{ or } a_{1,2}(S_i) = 0$$

(ii) $$p^* = \inf_{i\in\{1,\dots,4\}} \left[\frac{2\arctan\frac{\Lambda(S_i)}{a_{1,2}(S_i)}}{3\arctan\frac{\Lambda(S_i)}{a_{1,2}(S_i)} + \pi}\right] \text{ if } \exists i \in \{1,\dots,4\} \text{ such that,}$$

$$\frac{\Lambda(S_i)}{a_{1,2}(S_i)} < -\tan\frac{\pi}{3}$$

To prove these theorems, we use a monotone iteration which is a slight modification of the arguments of [**AC**]. In fact the main difficulty here is to prove a *shift* theorem in S_a for the linear problem (that is a theorem which ensure the existence of solutions in $W^{2,p}(S_a)$ of the linear problem (1.6)-(1.2)-(1.3) when $g \in L^p(S_a)$). This is the aim of section 2 of this paper. The proof of the two theorems is then given in section 3. Some technical results particular to the dimension 2 are proved in section 4.

2. Resolution of the linear problem

We first begin with some propositions, they will be used several times to solve the linear problem in S_a. Consider the linear problem,

$$Lu \equiv \sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u + b_i(x)\partial_i u + c(x)u = g \tag{2.1}$$

$$\partial_\nu u(x_1,y) = 0 \quad \text{for } -a < x_1 < a, y \in \partial\omega \tag{2.2}$$

$$u(-a,y) = \psi_1(y), \quad u(a,y) = \psi_2(y) \tag{2.3}$$

Here, the coefficients $a_{i,j}$, ψ_1, ψ_2 satisfy the conditions imposed in the introduction, $g \in L^p(S_a)$ and the coefficients b_i, c satisfy,

$$|b_i|, |c| < C \quad \text{and } c \leq 0 \quad \text{a.e. in } S_a \tag{2.4}$$

Proposition 2.1. *If the coefficients $a_{i,j}$ satisfy the condition* (1.7), *then the linear problem* (2.1)-(2.3) *has a unique solution $u \in W^{2,p}(S_a)$ for all $p > n$ and*

$$\|u\|_{W^{2,p}(S_a)} \leq C_1 \left(\|g\|_{L^p(S_a)} + \sum_j \|\psi_j\|_{W^{2,p}(\omega)} + \|u\|_{L^p(S_a)} \right) \tag{2.5}$$

In the particular case where S_a is a rectangle of $\mathbb{R}^2$, we have,

Proposition 2.2. *If the dimension is $n = 2$ and the coefficient $a_{1,2} = a_{2,1}$ satisfies the condition* (1.8), *then the linear problem* (2.1)-(2.3) *has a unique solution u in $W^{2,p}(S_a)$ for the values of p satisfying one of the conditions* (i), (ii) *of Theorem* 1.2 *and*

$$\|u\|_{W^{2,p}(S_a)} \leq C_1 \left(\|g\|_{L^p(S_a)} + \sum_j \|\psi_j\|_{W^{2,p}(\omega)} + \|u\|_{L^p(S_a)} \right) \tag{2.6}$$

Proof of Proposition 2.1. The uniqueness of a solution of (2.1)-(2.3) follows from the maximum principle. To prove the existence, we write

$$u = v + \frac{a - x_1}{2a}\psi_1(y) + \frac{a + x_1}{2a}\psi_2(y) \tag{2.7}$$

then v satisfies the linear problem,

$$Lv = \tilde{g} \in L^p(S_a) \tag{2.8}$$

$$\partial_\nu v(x_1, y) = 0 \quad \text{for } -a < x_1 < a, \quad y \in \partial\omega \tag{2.9}$$

$$v(x_1, y) = 0 \quad \text{for } x_1 \pm a, \quad y \in \omega \tag{2.10}$$

By reflecting across $x_1 = \pm a$, we consider an extended problem. We extend the coefficients $a_{i,j}$, b_i, c and $\tilde{g}$ to be periodic in x_1 of period $4a$ and such that $a_{1,1}$, $a_{i,j}$ for i and $j \neq 1$, $b_2, \ldots, b_n$, c are symmetric with respect to the hyperplane $\{x_1 = -a\}$, that is

$$a_{i,j}(-2a - x_1) = a_{i,j}(x_1) \quad \text{for } x_1 \in (-a, a), i \quad \text{and } j \neq 1$$

and $a_{1,i}$ for $i \neq 1$, b_1 and $\tilde{g}$ are antisymmetric with respect to the hyperplane $\{x_1 = -a\}$, that is

$$a_{1,i}(-2a - x_1) = -a_{1,i}(x_1) \quad \text{for } x_1 \in (-a, a), \quad \text{and } i \neq 1$$

We will use the same letters as a notation.

By standard elliptic theory there is a unique $4a$-periodic function $v \in W^{2,p}(]-3a, a[\times\omega)$ satisfying (2.8) in the infinite cylinder (by multiple reflections) and $\partial_\nu v = 0$ on its boundary i.e. for $y \in \partial\omega$.

We can see that $-v(-2a - x_1, y)$ is also a solution of the same periodic problem. By uniqueness, it equals v. Hence we get that $-v(-a, y) = v(-a, y)$ and then, $v(-a, y) = 0$. Similarly $v(a, y) = 0$ for $y \in \omega$. Thus v is a solution of problem (2.8)-(2.10) and we have the estimate

$$\|v\|_{W^{2,p}(S_a)} \leq C \left(\|\tilde{g}\|_{L^p(S_a)} + \|v\|_{L^p(S_a)} \right) \tag{2.11}$$

The function u given by (2.7) is then a solution in $W^{2,p}(S_a)$ of the problem (2.1)-(2.3) and satisfies the estimate (2.5). □

The proof of Proposition 2.2 is more technical, it is given in section 4.

3. Proofs of the main results

We show here how Theorems 1.1 and 1.2 follow respectively from Proposition 2.1 and 2.2.

Proof of Theorems 1.1 and 1.2. First, we wish to obtain a priori estimates for any solution u of our problem (1.1)-(1.3) satisfying $\underline{u} \leq u \leq \overline{u}$.

Let u be such a solution, then it satisfies,

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u = g = -f(x, u, \nabla u)$$

using the representation of u (2.7) and the elliptic estimates, we see that,

$$\|v\|_{W^{2,p}(S_a)} \leq C \left(\|\tilde{g}\|_{L^p(S_a)} + \|v\|_{L^p(S_a)} \right)$$

In the following C denotes various constants not depending on u but possibly depending on the ψ_j and their $W^{2,p}$ norms and on $\max|\underline{u}|$ and $\max|\overline{u}|$.

Using the argument in [**AC**] based on the Gagliardo-Nirenberg interpolation inequality [**N**],

$$\left\| |\nabla w|^2 \right\|_{L^p} \leq \gamma \|w\|_{L^\infty} \|w\|_{W^{2,p}}$$

we get the a priori estimate for u solution of (1.1)-(1.3) and $\underline{u} \leq u \leq \overline{u}$

$$\|u\|_{W^{2,p}(S_a)} \leq C(a_{i,j}, S_a, \psi_1, \psi_2, \max|\underline{u}|, \max|\overline{u}|, p, n, K) \tag{3.1}$$

In place of equation (1.1), we will solve a modified equation

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u + f(x, u, h(\nabla u)) = 0 \tag{3.2}$$

where h is a smooth mapping: $\mathbb{R}^n \to \mathbb{R}^n$ satisfying $|h(\xi)| \leq 2|\xi|$.

For this class of maps h, the preceding argument gives an a priori bound of the form (3.1) with a different constant C, but independant of the mapping h. Thus in particular we have the a priori estimate,

$$\max|\nabla u| \leq C_1$$

with C_1 independant of h. Now fix such a map h whose range is bounded and which satisfies in addition,

$$h(\xi) = \xi \text{ for } |\xi| \leq \max(C_1, |\nabla\underline{u}|_{L^\infty}, |\nabla\overline{u}|_{L^\infty})$$

With h so fixed, any solution of (3.2) satisfying (1.2)-(1.3) and $\underline{u} \leq u \leq \overline{u}$ is a solution of (1.1).

Let k be the Lipschitz constant of f in B, the subset of $\mathbb{R}^{n+1}$ defined by,

$$B = \{(u,p) \in \mathbb{R} \times \mathbb{R}^n; |u| \leq \max(|\underline{u}|_{L^\infty}, |\overline{u}|_{L^\infty}); \quad |p| \leq \max(|\nabla\underline{u}|_{L^\infty}, |\nabla\overline{u}|_{L^\infty}, \max_{\mathbb{R}^n}|h|\}$$

Rewrite the equation (3.2) as

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u - ku = -f(x, u, h(\nabla u)) - ku \tag{3.3}$$

We will solve this by monotone iteration using the following lemmas,

Lemma 3.1. *Assume that the coefficients $a_{i,j}$ satisfy the condition* (1.7). *Let $v \in C(\overline{S}_a)$ be a function satisfying $\underline{u} \leq v \leq \overline{u}$. Then the problem*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u - ku = -f(x, v, h(\nabla u)) - kv \quad \text{in } S_a \tag{3.4}$$

under the boundary conditions (1.2)-(1.3), *has a unique solution $u = T(v)$ in $W^{2,p}(S_a)$ for all $p > n$. It satisfies*

$$\underline{u} \leq u \leq \overline{u} \tag{3.5}$$

and furthermore

$$\underline{u} \leq v_1 \leq v_2 \leq \overline{u} \quad \Rightarrow \quad T(v_1) \leq T(v_2) \tag{3.6}$$

For the case $n = 2$, we have,

Lemma 3.2. *Consider the case of dimension $n = 2$ and assume that the coefficient $a_{1,2} = a_{2,1}$ satisfies the condition* (1.8). *Let $v \in C(\overline{S}_a)$ be a function satisfying $\underline{u} \leq v \leq \overline{u}$. Then the problem* (3.4) *under the boundary conditions* (1.2)-(1.3), *has a unique solution $u = T(v)$ in $W^{2,p}(S_a)$ for the values of p given by* (i) *or* (ii) *of Theorem* 1.2. *Furthermore the solution u satisfies* (3.5) *and* (3.6).

Proof of the lemmas. The uniqueness follows from the maximum principle. With the help of respectively lemmas 2.1, 2.2, the existence is easily proved using the Schauder fixed point theorem, (since the right hand side in (3.4) is bounded), the a priori estimate and the compactness of the injection $W^{2,p}(S_a) \hookrightarrow C_1(\overline{S}_a)$, since S_a is a Lipschitz domain.

To prove (3.5), we first observe that $h(\nabla\underline{u}) = \nabla\underline{u}$ and then

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}(u - \underline{u}) - k(u - \underline{u}) &\leq \\
&\leq -f(x, v, h(\nabla u)) + +f(x, \underline{u}, h(\nabla\underline{u})) + k(\underline{u} - v) \\
&= -f(x, v, h(\nabla u)) + f(x, \underline{u}, h(\nabla u)) \\
&\quad - f(x, \underline{u}, h(\nabla u)) + f(x, \underline{u}, h(\nabla\underline{u})) + k(\underline{u} - v) \\
&\leq -f(x, \underline{u}, h(\nabla u)) + f(x, \underline{u}, h(\nabla\underline{u})) \\
&= \sum_{i,j=1}^{n} b_j(x)\partial j(u - \underline{u})
\end{aligned}$$

by the mean-value theorem.

By the maximum principle and the Hopf Lemma it follows that $u - \underline{u} \geq 0$. Similarly $u - \overline{u} \leq 0$ and we prove (3.6) in the same way. □

To complete the proof of the theorem, we solve (3.3) by iteration starting with $\underline{u}$. Let the sequence (u_m) be given by $u_0 = \underline{u}$ and the recurrence relation $u_{m+1} = T(u_m)$.

Inductively, we find using Lemmas 3.1, 3.2,

$$\underline{u} \leq u_1 \leq u_2 \leq \cdots \leq \overline{u}$$

We have

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u_{m+1} - ku_{m+1} = -f(x, u_m, h(\nabla u_{m+1})) - ku_m$$

The right hand sides are uniformly bounded. Hence by Propositions 2.1, 2.2 we find that

$$\|u_{m+1}\|_{W^{2,p}(S_a)} \leq C \quad \text{for all } m$$

Thus a subsequence of u_m converges uniformly in $C^1(\overline{S}_a)$ to a function $u \in W^{2,p}(S_a)$ satisfying (1.2)-(1.3). Since the u_m form a monotone sequence, the whole sequence converges to u and the function u is a solution with all the desired properties. □

Remark. Using Lemma 7.1 in [**BN**], one can prove the following result:

Theorem 3.3. *Let f be a continuous function which is assumed to be bounded and locally Lipschitz in (u,p). Let $\underline{u} \leq \overline{u}$ be sub and super solutions of* (1.1)-(1.3) *belonging to $C^0(\overline{S}_a) \cap W^{2,p}_{loc}(\overline{S}_a \backslash \Sigma)$. Then there is a solution u in $W^{2,p}_{loc}(\overline{S}_a \backslash \Sigma) \cap C^0(\overline{S}_a)$ of* (1.1)-(1.3) *satisfying $\underline{u} \leq u \leq \overline{u}$.*

4. Proof of proposition 2.2

In this section we give the proof of Proposition 2.2.

We use the representation of u

$$u = v + \frac{a - x_1}{2a}\psi_1(y) + \frac{a + x_1}{2a}\psi_2(y)$$

and denote by $X_p(S_a)$ the functional space,

$$X_p(S_a) = \{v \in W^{2,p}(S_a); \partial_\nu v = 0 \quad \text{on } (-a,a) \times \partial\omega;$$
$$v = 0 \quad \text{on } (\pm a, y), y \in \omega\}$$

Our purpose is to find the values of $p > 2$, for which the index of the operator A defined by,

$$A = \sum_{i,j=1}^{2} a_{i,j}(x)\partial_{i,j} + \sum_{i=1}^{2} b_i(x)\partial_i + c(x)$$

from $X_p(S_a)$ into $L^p(S_a)$ is zero.

For this, we use two lemmas of technical character where we are going to deal with the particular case when the domain has only one corner. We follow in this the representation of P. Grisvard [**G**].

We consider a plane bounded domain Ω whose boundary $\partial\Omega$ is composed of two rectilinear sides Υ_1 and Υ_2, a curvilinear side Υ_3 of class C^2 and having one corner $S = \Upsilon_1 \cap \Upsilon_2$ whose measure of the angle is $\pi/2$. We denote by $Y_p(\Omega)$ the functional space:

$$\mathcal{U}: \Omega \to \mathcal{U}\Omega$$

$$(x_1, x_2) \mapsto \left(y_1 = \frac{1}{\sqrt{a_{1,1}}}x_1, y_2 = \frac{1}{\Lambda\sqrt{a_{1,1}}}(-a_{1,2}x_1 + a_{1,1}x_2) \right)$$

where $a_{i,j} = a_{i,j}(S)$, for $i,j \in \{1,2\}$ and $\Lambda = (a_{1,1}a_{2,2} - a_{1,2}^2)^{\frac{1}{2}}$.

Let $w(y) = v(x) = v(\mathcal{U}^{-1})(y)$, for $y \in \mathcal{U}\Omega$ and $\tilde{A}w(y) = Av(x)$, then we have

$$\tilde{A}_{i,j}(\mathcal{U}S) = \delta_{i,j}$$

We denote by $\alpha_S(A)$ or α the measure of the angle at $\mathcal{U}S$ of $\mathcal{U}\Omega$, then we have

$$\tan\alpha = -\frac{\Lambda}{a_{1,2}} \quad \text{and } \alpha \in (0,\pi)$$

For every $v \in Y_p(\Omega)$ the function $w(y) = v \circ \mathcal{U}^{-1}(y)$ satisfies

$$w = 0 \quad \text{on } \mathcal{U}\Upsilon_1 \cap \mathcal{U}\Upsilon_3$$
$$\partial_{y_1}w - \frac{a_{1,2}}{\Lambda}\partial_{y_2}w = 0 \quad \text{on } \mathcal{U}\Upsilon 2$$

We will prove these two lemmas:

Lemma 4.1. *There exists a constant $C > 0$ such that for all $v \in Y_p(\Omega)$,*

$$\|v\|_{W^{2,p}(\Omega)} \leq C \left(\|Av\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \right) \tag{4.1}$$

for all $p > 2$, if $0 < \alpha \leq \frac{\pi}{3}$ or $\alpha = \frac{\pi}{2}$ and all $2 < p < \frac{2}{3-\frac{\pi}{\alpha}}$, if $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$.

Lemma 4.2. *The image of* $Y_p(\Omega)$ *through the operator* A *is* $L^p(\Omega)$ *for all* $p > 2$, *if* $0 < \alpha \leq \frac{\pi}{3}$ *or* $\alpha = \frac{\pi}{2}$ *and all* $2 < p < \frac{2}{3-\frac{\pi}{\alpha}}$, *if* $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$.

Let us end the proof of Proposition 2.2. We denote by $\{\Gamma_i\}_{i\in\{1,\dots,4\}}$ the open linear segments forming the boundary of S_a.

Let $f \in L^p(S_a)$, then there exists a unique $v \in W^{2,p}(S_a\backslash V) \cap C^0(\overline{S}_a)$ solution of:

$$\begin{aligned} &Av = f \quad \text{in } S_a \\ &\partial_\nu v = 0 \quad \text{on} \quad \Gamma_2 \cup \Gamma_4 = (-a, a) \times \partial\omega \\ &v = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_3 = \{(\pm a, y), y \in \omega\} \end{aligned}$$

V denotes a closed neighbourhood of the corners. The existence of this solution is proved in Lemma 7.1 of [**BN**] by considering an approximate problem in a subdomain of S_a in which the corners have been rounded off and by passing to the limit using a symmetric positive barrier function.

Then we fix a partition of unity $\{\eta_j\}_{j=0,\dots,4}$ on $\overline{S}_a$ such that $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ for each j and

(a) the support of η_0 does not contain any corner of S_a,

(b) the support of η_j contains S_j and not contain any other corner; in addition the support of η_j does not intersect Γ_k for $k \neq j$ and $k \neq j+1$,

(c) $\partial_{\nu_k}\eta_j = 0$ on Γ_k for $k = j$ if $j \in \{2,4\}$ and for $k = j+1$ if $j+1 \in \{2,4\}$, where ν_k is the exterior unit normal on Γ_k.

It follows from the classical results of regularity [**GT**], that $\eta_0 v \in W^{2,p}(S_a)$ and we have

$$\|\eta_0 v\|_{W^{2,p}(S_a)} \leq C\left(\|Av\|_{L^p(S_a)} + \|v\|_{L^p(S_a)}\right) \tag{4.2}$$

since the support of $\eta_0 v$ is at a strictly positive distance from the corners.

Then we select an open neighbourhood W_j of a corner S_j containing the support of η_j, having only one corner S_j and such that the boundary of W_j coincides with the ∂S_a near S_j. Thus assuming that $j \in \{1,3\}$,

we have

$$\begin{aligned} &A\eta_j v = \eta_j f + [A;\eta_j]v \in L^p(W_j) \\ &\eta_j v = 0 \quad \text{on } \Gamma_j \cap \partial W_j \\ &\partial_\nu \eta_j v = 0 \quad \text{on } \Gamma_{j+1} \cap \partial W_j \\ &\eta_j v = 0 \quad \text{on } \partial W_j \backslash (\Gamma_j \cup \Gamma_{j+1}) \end{aligned}$$

We deduce using Lemma 4.2 that, $\eta_j v \in W^{2,p}(W_j)$ for the values of p given by the lemma. Furthermore by lemma 4.1, we have the estimate

$$\|\eta_j v\|_{W^{2,p}(S_a)} \leq C\left(\|A\eta_j v\|_{L^p(S_a)} + \|\eta_j v\|_{L^p(S_a)}\right) \tag{4.3}$$

Adding the inequalities (4.2) and (4.3), we obtain that $v \in W^{2,p}(S_a)$ and v satisfies the inequality (2.11) for the values of p given by (i) or (ii). This achieves the proof of Proposition 2.2. □

Now, it remains to prove the two Lemmas mentioned above.

Proof of Lemma 4.1. We choose a function $\eta \in \mathcal{D}(\mathbb{R}^2)$ with support in Ω with η identically equal to one near S and $\partial_\nu \eta = 0$ on Υ_2. We shall look separately at ηv and $1 - \eta v$. Set

$$w(y) = (\eta v)(x) = (\eta v)(\mathcal{U}^{-1}(y)), \ y \in \mathcal{U}\Omega$$

and select any plane open domain W with polygonal boundary such that

(a) $\overline{W} \subset \mathcal{U}\Omega$,

(b) $\overline{W}$ contains the support of w,

(c) ∂W coincides with $\mathcal{U}\partial\Omega$ near $\mathcal{U}S$.

It is clear that $w \in W^{2,p}(W)$ and that w satisfies

$$\partial_{\nu_2} w - \frac{a_{1,2}}{\Lambda}\partial_{\tau_2} w = 0 \quad \text{on } \mathcal{U}\Upsilon_2 \cap \partial W$$

where ν_2 is the unit normal on $\mathcal{U}\Upsilon_2$ and τ_2 denotes the unit tangent vector on $\mathcal{U}\Upsilon_2$.

$$w = 0 \quad \text{on the rest of } \partial W$$

$$\tilde{A}w = \Delta w + \sum_{i,j=1}^{2} \tilde{b}_{i,j}(y)\partial_{i,j} w + \sum_{i=1}^{2} \tilde{b}_i(y)\partial_i w + \tilde{b}_0 w = (A\eta v)\circ\mathcal{U}^{-1} = \phi \quad \text{in } W$$

where $\tilde{b}_{i,j} \in C^0(\overline{W})$, $\tilde{b}_i \in L^\infty(W)$, $0 \leq i \leq 2$ and $\tilde{b}_{i,j}(\mathcal{U}S) = 0$.

With the notations of [**G**], we have at the corner $\mathcal{U}S$, $\phi_1 = 0$ corresponding to the Dirichlet condition on $\mathcal{U}\Upsilon_1$,

$$\tan\phi_2 = -\frac{a_{1,2}}{\Lambda};$$

where $\phi_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the angle between the vectors

$$\nu_2 \quad \text{and } \mu_2 = \nu_2 - \frac{a_{1,2}}{\Lambda}\tau_2.$$

Thus we have, $\phi_2 = \frac{\pi}{2} - \alpha$ and if we denote by q the conjugate exponent of p,

$$\left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

then

$$\frac{\phi_1 - \phi_2 + 2\frac{\alpha}{q}}{\pi} = \left(1 + \frac{2}{q}\right)\frac{\alpha}{\pi}$$

is not an integer for all $p > 2$, if $0 < \alpha \leq \frac{\pi}{3}$ or $\alpha = \frac{\pi}{2}$ and all $2 < p < \frac{2}{3-\frac{\pi}{\alpha}}$, if $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$.

It is always possible to choose the other angles of W, to avoid the exceptional cases for the inequality (4.1.2) given in [**G**]. Accordingly, we have

$$\|w\|_{W^{2,p}(W)} \leq$$

$$\leq C\left(\left\|\phi - \sum_{i,j=1}^{2}\tilde{b}_{i,j}(y)\partial_{i,j}w - \sum_{i=1}^{2}\tilde{b}_i(y)\partial_i w - \tilde{b}_0 w\right\|_{L^p(W)} + \|w\|_{L^p(W)}\right) \leq$$

$$\leq C\left(\|\phi\|_{L^p(W)} + 4\max_{i,j=1,2}\left|\tilde{b}_{i,j}(y)\right|\|w\|_{W^p(W)} + \|w\|_{W^{1,p}(W)}\right)$$

Since $\tilde{b}_{i,j}(\mathcal{U}S) = 0$, we can choose the support of η small enough such that

$$\max_{\substack{i,j=1,2 \\ y\in \text{supp } \eta\circ\mathcal{U}^{-1}}} \left|\tilde{b}_{i,j}(y)\right| \leq \frac{1}{8}C$$

then we obtain

$$\|w\|_{W^{2,p}(W)} \leq 2C\left(\|\phi\|_{L^p(W)} + \|w\|_{W^{1,p}(W)}\right)$$

Going back to v, this implies that

$$\|\eta v\|_{W^{2,p}(\Omega)} \leq C\left(\|A\eta v\|_{L^p(\Omega)} + \|\eta v\|_{W^{1,p}(\Omega)}\right) \tag{4.4}$$

We choose another plane domain W' with a $C^{1,1}$ boundary. Then we have:

$$(1-\eta)v \in W^{2,p}(W') \quad \text{and } (1-\eta)v$$

satisfies either the condition of Dirichlet or Neumann on the boundary of W'. By the classical estimates, we get

$$\|(1-\eta)v\|_{W^{2,p}(\Omega)} \leq C\left(\|A(1-\eta)v\|_{L^p(\Omega)} + \|(1-\eta)v\|_{W^{1,p}(\Omega)}\right) \tag{4.5}$$

Adding inequalities (4.4) and (4.5), we obtain the desired estimate (4.1), with the help of the interpolation inequality

$$\|v\|_{W^{1,p}(\Omega)} \leq \epsilon\|v\|_{W^{2,p}(\Omega)} + K\epsilon^{-1}\|v\|_{L^p(\Omega)}, \quad \forall\epsilon > 0 \qquad \square$$

Proof of Lemma 4.2. We denote by $\tilde{\Omega}$ the domain $\mathcal{U}\Omega$ and by $\tilde{Y}_p(\tilde{\Omega})$ the functional space

$$\tilde{Y}_p(\tilde{\Omega}) = \{\tilde{v} \in W^{2,p}(\tilde{\Omega}); \tilde{v} = 0 \quad \text{on} \quad \mathcal{U}\Upsilon_1 \cup \mathcal{U}\Upsilon_3,$$
$$\partial_\nu\tilde{v} - \frac{a_{1,2}}{\Lambda}\partial_\tau\tilde{v} = 0 \quad \text{on} \quad \mathcal{U}\Upsilon_2\}$$

As $\mathcal{U}$ is invertible, we shall calculate the image of $\tilde{Y}_p(\tilde{\Omega})$ through the operator $\tilde{A}$.

Let us set

$$\tilde{A}(t) = t\tilde{A} + (1-t)\Delta, \quad t \in [0,1]$$

Then since $(\tilde{A}(t))_{i,j}(\mathcal{U}S) = \delta_{i,j}$, applying Lemma 4.1, we know that for each $t \in [0,1]$, there exists a constant C_t such that

$$\|\tilde{v}\|_{W^{2,p}(\tilde{\Omega})} \leq C_t\left(\left\|\tilde{A}(t)\tilde{v}\right\|_{L^p(\tilde{\Omega})} + \|\tilde{v}\|_{L^p(\tilde{\Omega})}\right)$$

for all $\tilde{v} \in \tilde{Y}_p(\tilde{\Omega})$. Accordingly $\tilde{A}(t)$ is a semi-Fredholm operator from $\tilde{Y}_p(\tilde{\Omega})$ into $L^p(\tilde{\Omega})$ for every $t \in [0,1]$.

By a theorem of Kato [**K**], the index of $\tilde{A}(t)$ does not depend on t since $\tilde{A}(t)$ depends continuously on t. Then index $\tilde{A}$ = index Δ.

To calculate the index of Δ, consider a function $\tilde{f} \in L^p(\tilde{\Omega})$. By the results in [**G**] using an abstract lemma of J.L. Lions [**L**], there exists

$\tilde{v} \in H^1(\tilde{\Omega})$ solution of

$$\Delta\tilde{v} = \tilde{f} \quad \text{in } \tilde{\Omega}$$
$$\tilde{v} = 0 \quad \text{on } \mathcal{U}\Upsilon_1 \cup \mathcal{U}\Upsilon_3$$
$$\partial_\nu \tilde{v} - \frac{a_{1,2}}{\Lambda}\partial_\tau \tilde{v} = 0 \quad \text{on } \mathcal{U}\Upsilon_2$$

By the classical results of regularity [**GT**], $\tilde{v} \in W^{2,p}(\tilde{\Omega}\backslash V)$, where V is any closed neighbourhood of $\mathcal{U}S$. Then choose a function $\eta \in \mathcal{D}(\mathbb{R}^2)$, η is identically equals to 1 near $\mathcal{U}S$ and

$$\partial_\nu \eta - \frac{a_{1,2}}{\Lambda}\partial_\tau \eta = 0 \quad \text{on } \mathcal{U}\Upsilon_2$$

It is clear that $(1-\eta)\tilde{v} \in W^{2,p}(\tilde{\Omega})$ and we have

$$\Delta\eta\tilde{v} = \eta\tilde{f} + [\Delta;\eta]\tilde{v} = f_1 \in L^p(W)$$

Where W is a polygonal domain which contains the support of η and such that ∂W coincides with $\partial\tilde{\Omega}$ near $\mathcal{U}S$.

Thus $\eta\tilde{v}$ satisfies also

$$\partial_\nu(\eta\tilde{v}) - \frac{a_{1,2}}{\Lambda}\partial_\tau(\eta\tilde{v}) = 0 \quad \text{on } \mathcal{U}\Upsilon_2$$
$$\tilde{v} = 0 \quad \text{on } \partial W\backslash\mathcal{U}\Upsilon_2$$

Applying Theorem 4.4.4.13 in [**G**], we know that there exist numbers c_m and functions S_m such that

$$\eta\tilde{v} - \sum_{\substack{-\frac{2}{q}<\lambda_m<0 \\ \lambda_m \neq -1}} c_m S_m \in W^{2,p}(W)$$

where λ_m is the eigenvalue defined by

$$\lambda_m = \frac{\phi_1 - \phi_2 + m\pi}{\alpha},$$

q denotes the conjugate exponent of p, $\phi_1 = 0$ and $\tan\phi_2 = \frac{-\Lambda}{a_{1,2}}$.

Furthermore, we have

$$\text{card}\{m \in \mathbb{Z}; -\frac{2}{q} < \lambda_m < 0, \lambda_m \neq -1\} = 0$$

for the values of p and α fixed in the lemma. Thus $\tilde{v} \in W^{2,p}(\tilde{\Omega})$ and index $\Delta = 0$. □

□

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