

Stable Complete Surfaces with Constant Mean Curvature

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Abstract. Let $x : M^2 \rightarrow N^3$ be a stable immersion with constant mean curvature H of a complete orientable surface M^2 into a complete oriented three dimensional Riemannian manifold N^3 . In this paper we prove that, if M^2 is compact and $H^2 > -\frac{1}{2} \inf_M \text{Ric}_N$, then M^2 has genus $g \leq 3$, here Ric_N is the Ricci curvature of N^3 . We also prove that, if M^2 is complete non compact and N^2 has bounded geometry, the area of M^2 is infinite in the metric induced by x . In this case, if $H^2 \geq -\frac{1}{2} \inf_M \text{Ric}_N$ then x is umbilic and the equality holds.

1. Introduction

The goal of this paper is to present some results on the stability of immersions $x : M^2 \rightarrow N^3$ with constant mean curvature H .

We first consider the case where M^2 is compact, orientable, and prove the following result.

"Let M^2 have genus g . If $x : M^2 \rightarrow N^3$ is stable and

$$H^2 > -\frac{1}{2} \inf_M \text{Ric}_N,$$

then $g \leq 3$. Here Ric_N is the Ricci curvature of N ".

This is related to Fischer-Colbrie and Schoen [9] where it was proved that if x is minimal and stable, and N^3 has nonnegative Ricci curvature, then $g \leq 1$. A similar result was also obtained by El Soufi and S. Ilias [6].

It was proved in [2] that if M^2 is a compact, stable surface with constant mean curvature H in the three dimensional simply-connected, complete Riemannian manifold Q_c^3 with constant sectional curvature c ,

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then $M^2 \subset Q_c^3$ is a geodesic sphere and $(H^2 + c)A = 4\pi$, where A is the area of M^2 . It is a surprising fact that this equality becomes a sharp inequality in the case that the ambient space N^3 is arbitrary.

"Let M^2 be compact, orientable and assume that $x : M^2 \rightarrow N^3$ is a stable surface with constant mean curvature H . Let A be the area of M^2 and $c = \frac{1}{2} \inf_M \text{Ric}_N$. Then

$$(H^2 + c)A \leq 4\pi$$

and equality holds if and only if M^2 has genus $g = 0$, x is umbilic, and the sectional curvature $K_N \equiv c$ in M ".

We next consider the case where M^2 is complete noncompact. For that case, we will need that the area of M^2 is infinite in the metric induced by $x : M^2 \rightarrow N^3$. This will be seen to be the case if N^3 has bounded geometry (i.e., it has sectional curvature bounded above and injectivity radius bounded below). More generally, we will prove in an Appendix to this paper the following result.

"Let M^m be a complete, noncompact manifold and let $x : M^m \rightarrow N^n$ be an immersion with mean curvature vector field bounded in norm. Assume that N^n has bounded geometry. Then the volume of M^m in the induced metric is infinite".

Now we can present our results on stability.

"Let M^2 be a complete, noncompact, orientable surface and let $x : M^2 \rightarrow N^3$ be an immersion with constant mean curvature H . Assume that N^3 has bounded geometry and that $H^2 \geq -\frac{1}{3} \inf_M S$ where S is the scalar curvature of N^3 . If M^2 has finite index, then $H^2 = -\frac{1}{3} \inf_M S$ ".

From this theorem we obtain that, if H is nonzero and N^3 has nonnegative scalar curvature, then M^2 has finite index if and only if M^2 is compact.

Our final theorem generalizes results of Fischer-Colbrie and R. Schoen [9] for stable minimal surfaces and of A.M. Silveira [12] for stable surfaces with constant mean curvature in Q_c^3 .

"Let M^2 be a complete, noncompact, orientable surface and let

$x : M^2 \rightarrow N^3$ be an immersion with constant mean curvature H . Assume that N^3 has bounded geometry and that $H^2 \geq -\frac{1}{2} \inf_M \text{Ric}_N$. If x is stable, then

$$H^2 = -\frac{1}{2} \inf_M \text{Ric}_N$$

and x is umbilic. Furthermore, M^2 is conformally equivalent to the plane or to a cylinder. If M^2 is a cylinder, then M^2 is flat.

The results of this paper are part of my Doctoral Dissertation at IMPA, as were announced in [10]. Personal problems prevented its publication at the proper time. I want to thank to M. do Carmo for his orientation.

2. Stability of Compact Surfaces with Constant Mean Curvature.

Let M^2 be a complete orientable surface and let N^3 be a three dimensional complete oriented Riemannian manifold. Let $x : M^2 \rightarrow N^3$ be an isometric immersion with constant mean curvature H . We know that such surfaces are critical points of the area function for compactly supported variation that preserve volume. We say that the immersion is *stable* if the second variation of the area function is nonnegative, i.e.,

$$I(f) = \int_M [-f\Delta f - (\text{Ric}_N(e_3) + \|B\|^2)f^2]dA \geq 0 \quad (1)$$

for any piecewise smooth function $f : M \rightarrow \mathbb{R}$ with compact support and with $\int_M f dA = 0$ (see [2]). Here Δ is the Laplacian in M , e_3 is a unit normal vector field, $\text{Ric}_N(e_3)$ is the Ricci curvature of N in the direction of e_3 and $\|B\|$ is the norm of the second fundamental form B .

Associated to the quadratic form I we have the operator

$$L = \Delta + \text{Ric}_N(e_3) + \|B\|^2,$$

called the *Jacobi operator*.

Let (M^2, ds^2) be a compact orientable surface of genus g . Then M^2 has a Riemann surface structure compatible with the metric ds^2 .

In the study of stability of a compact surface with constant mean curvature, it is fundamental to have some way of constructing smooth functions $f : M \rightarrow \mathbb{R}$ with mean value zero, i.e. $\int_M f dA = 0$. For this,

we will use in this section the following result.

Lemma 1. (see [5], Lemma (2.1).) *Let $S^2 \subset \mathbb{R}^3$ be the round sphere of radius one and let $\psi : M^2 \rightarrow S^2$ be a nonconstant meromorphic function. Then there exists a conformal transformation $\phi : S^2 \rightarrow S^2$ such that*

$$\int_M \bar{\psi}_i dA = 0$$

for $i = 1, 2, 3$, where $\bar{\psi} = \phi \circ \psi = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3) \in \mathbb{R}^3$.

Let e_1, e_2, e_3 be a positively oriented orthonormal frame defined locally on M with e_1 and e_2 tangent to M , and e_3 the unit normal vector field. Let $b_{ij} = \langle \bar{\nabla}_{e_i} e_3, e_j \rangle$, $1 \leq i, j \leq 2$, be the coefficients of the second fundamental form B , where $\bar{\nabla}$ is the Riemannian connection of N .

By the Gauss formula, we have

$$K = K_{12} + b_{11}b_{22} - b_{12}^2$$

where K is the Gaussian curvature of M and K_{ij} is the sectional curvature of N for the plane determined by e_i and e_j .

Therefore,

$$2H^2 - K + K_{12} = \frac{(b_{11} + b_{22})^2}{2} - b_{11}b_{22} + b_{12}^2 = \frac{\|B\|^2}{2} \quad (2)$$

and

$$H^2 - K + K_{12} = \frac{(b_{11} - b_{22})^2}{4} + b_{12}^2 \geq 0. \quad (3)$$

We now prove the results about stability of compact surfaces with constant mean curvature mentioned in the introduction.

Theorem 2. *Let (M^2, ds^2) be a compact orientable surface of genus g , and let $x : M^2 \rightarrow N^3$ be an isometric immersion with constant mean curvature H . If x is stable and $H^2 > -\frac{1}{2} \inf_M \text{Ric}_N$, where Ric_N is the Ricci curvature of N , then $g \leq 3$.*

Proof. Suppose that $g \geq 4$. Then there exists a nonconstant meromorphic function $\psi : M^2 \rightarrow S^2$ of degree less or equal than $(3g + 1)/4$ (see [7], pg. 118). So, by Lemma 1, there exists a nonconstant meromorphic

function $\bar{\psi} : M^2 \rightarrow S^2$ of degree less or equal to $(3g+1)/4$ such that

$$\int_M \bar{\psi}_i dA = 0,$$

for $i = 1, 2, 3$, where $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)$. Since x is stable, we have

$$\int_M |\text{grad } \bar{\psi}_i|^2 dA \geq \int_M (-2K + 4H^2 + 2K_{12} + \text{Ric}_N(e_3)) \bar{\psi}_i^2 dA,$$

$i = 1, 2, 3$, where grad denotes the gradient in M .

By summing up in i , $i = 1, 2, 3$ and by using that $|\bar{\psi}|^2 = \sum_{i=1}^3 \bar{\psi}_i^2 = 1$ we obtain that

$$\int_M |\nabla \bar{\psi}|^2 dA \geq \int_M (-2K + 4H^2 + 2K_{12} + \text{Ric}_N(e_3)) dA, \quad (4)$$

where $|\nabla \bar{\psi}|^2 = \sum_{i=1}^3 |\text{grad } \bar{\psi}_i|^2$. By using that $\int_M K dA = 4\pi(1-g)$, that

$$\int_M |\nabla \bar{\psi}|^2 dA = 8\pi(\text{degree } (\bar{\psi})),$$

and that

$$2K_{12} + \text{Ric}_N(e_3) = \text{Ric}_N(e_1) + \text{Ric}_N(e_2) \geq -4c,$$

we obtain

$$8\pi(\text{degree } (\bar{\psi})) + 8\pi(1-g) \geq 4(H^2 - c)A, \quad (5)$$

where A is the area of M^2 and $c = -\frac{1}{2} \inf_M \text{Ric}_N$.

If $g = 4$, $\text{degree } (\bar{\psi}) \leq 13/4$. So $\text{degree } (\bar{\psi}) \leq 3$, and by (5), $4(H^2 - c)A \leq 0$, which is a contradiction. Now suppose that $g \geq 5$. By (5),

$$4(H^2 - c)A \leq 8\pi \frac{3g+1}{4} + 8\pi(1-g) \leq 2\pi(-g+5) \leq 0,$$

which is again a contradiction. It follows that $g \leq 3$ and the theorem is proved. \square

Corollary 3. *There is no compact orientable stable surface of genus $g \geq 4$ with constant mean curvature in a three-dimensional manifold with positive Ricci curvature.*

Remark 4. Corollary 3 also holds if the ambient space has nonnegative Ricci curvature and the compact surface has *nonzero* constant mean curvature.

Remark 5. M. Ross proved in [14] that the classical Schwarz \mathcal{P} -minimal surface of genus three in the flat three torus is a stable constant mean curvature surface. He also mentions ([14], p. 193) that the constant mean curvature companions of the Schwarz \mathcal{P} -minimal surface that are close enough to it are stable. Thus, the result of Theorem 2 is sharp.

Remark 6. It follows from the above Remark that it is unlikely that an explicit description for stable surface can be found even in the flat 3-torus. However, M. Ritoré and A. Ros proved in [13] that if M^2 is a stable compact orientable surface with nonzero constant mean curvature in the real projective space \mathbf{RP}^3 , then either M^2 is a geodesic sphere ($g = 0$) or is an embedded flat torus ($g = 1$) of radius r , with $\pi/6 \leq r \leq \pi/3$, about a geodesic. If $H = 0$, they also proved that M^2 is a two fold covering of a real projective plane ($g = 0$).

Theorem 7. *Let M^2 be a compact, orientable surface, and let $x : M^2 \rightarrow N^3$ be an isometric immersion with constant mean curvature H . If x is stable, then*

$$(H^2 + c)A \leq 4\pi,$$

where $c = \frac{1}{2} \inf_M \text{Ric}_N$ and $A = \text{area}(M)$. Furthermore, the equality holds if and only if M^2 has genus zero, x is umbilic and $K_N \equiv c$ in M^2 .

Proof. Let g be the genus of M . From the Riemann-Roch Theorem, there exists a finite number of points p_1, \dots, p_k , called Weierstrass points of M , such that, if $p \in M - \{p_1, \dots, p_k\}$, then there exists a meromorphic function $\psi : M \rightarrow S^2$ such that ψ is holomorphic in $M - \{p\}$, and p is a pole of order $g + 1$ of ψ . Then $\deg \psi = g + 1$. By Lemma 1, we obtain a conformal transformation $\phi : S^2 \rightarrow S^2$ such that

$$\int_M \bar{\psi} dA = 0$$

where $\bar{\psi} = \phi \circ \psi$. Then, by (4),

$$\int_M |\nabla \bar{\psi}|^2 dA \geq \int_M (-2K + 4H^2 + 2K_{12} + \text{Ric}_N(e_3)) dA,$$

since x is stable. Using the facts that

$$\int_M |\nabla \bar{\psi}|^2 dA = 8\pi(g+1), \quad \int_M K dA = 4\pi(1-g),$$

and $\text{Ric}_N \geq 2c$ in M , we obtain

$$8\pi(g+1) \geq -8\pi(1-g) + 4(H^2 + c)A.$$

Thus $(H^2 + c)A \leq 4\pi$. Now we suppose that $(H^2 + c)A = 4\pi$. Then $K_N \equiv c$ in M , and equality holds in (4). Therefore,

$$\Delta \bar{\psi} + (-2K + 4H^2 + 4c)\bar{\psi} = 0 \text{ in } M. \quad (6)$$

In the other hand, since $\bar{\psi}$ is a meromorphic function we have

$$\Delta \bar{\psi} + |\nabla \bar{\psi}|^2 \bar{\psi} = 0 \text{ in } M. \quad (7)$$

Equalities (6) and (7) imply that

$$|\nabla \bar{\psi}|^2(q) = (-2K + 4H^2 + 4c)(q),$$

for any point $q \in M$.

If $g > 0$, $|\nabla \bar{\psi}|^2(p) = 0$, since p is a pole of order $(g+1) \geq 2$. Then $K(p) = 2(H^2 + c)$. Because this equality holds for every point $p \in M - \{p_1, \dots, p_k\}$ we obtain that $K \equiv 2(H^2 + c)$ in M . Since $H^2 - K + K_{12} \geq 0$, we have that $H^2 + c \leq 0$, which is a contradiction. Then $g = 0$.

Moreover, since $\|B\|^2 + \text{Ric}_N(e_3) = -K + 3H^2 + 3c + (H^2 - K + K_{12})$, and the equality holds in (4), we have

$$\int_M [|\nabla \bar{\psi}|^2 + K - 3H^2 - 3c - (H^2 - K + K_{12})] dA = 0.$$

By using that $\int_M \|\nabla \bar{\psi}\|^2 dA = 8\pi$, that $\int_M K dA = 4\pi$, and that $(H^2 + c)A = 4\pi$, we conclude that $\int_M (H^2 - K + K_{12}) dA = 0$. Then, by (3), $H^2 - K + K_{12} \equiv 0$ in M , i.e., x is umbilic.

Now suppose that M^2 is an umbilic sphere in N^3 and $K_N \equiv c$ in M . Then

$$4\pi = \int_M K dA = \int_M (H^2 + c) dA = (H^2 + c)A,$$

and this completes the proof of the theorem. \square

3. Stability of Complete Noncompact Surfaces with Constant Mean Curvature.

Let M^2 be a complete orientable surface and let $T = \Delta + q$ be an operator in M , where $q : M \rightarrow \mathbb{R}$ is a smooth function. Let D be a relatively compact domain in M , with smooth boundary. The *index* of T in D , denoted by $\text{Ind}_T(D)$, is the number of negative eigenvalues of T with Dirichlet boundary condition. We define the *index* of T in M by

$$\text{Ind}_T(M) = \sup_{D \subset M} \text{Ind}_T(D),$$

where D is any relatively compact domain in M .

When T is the Jacobi operator L , the index of T in M is called the *index* of M , and is denoted by $\text{Ind}(M)$. One can prove (see [13]) that if the immersion $x : M^2 \rightarrow N^3$ is stable with constant mean curvature then $\text{Ind}(M)$ is at most one. For minimal surfaces, we know that the condition $I(f) \geq 0$ for all compactly supported function f is a necessary and sufficient condition for stability. Thus, a minimal immersion is stable if and only if $\text{Ind}(M) = 0$.

To prove our results we will need the following theorems:

Theorem 1. (A.M. da Silveira [15], pg.630.) *Let (M^2, ds^2) be a complete surface and let $T = \Delta - K + q$ be an operator as above, where K is the Gaussian curvature of M . Assume that the operator has finite index and that the function q is nonnegative. Then M^2 is conformally equivalent to a compact Riemannian surface minus a finite number of points. Furthermore,*

$$\int_M q dA < \infty,$$

where dA is the area element in the metric ds^2 .

Theorem 2. (A.M. da Silveira [15], pg. 630.) *Let (M^2, ds^2) be a complete surface conformally equivalent to a compact Riemann minus a finite number of points, and let $T = \Delta + q$. Assume that q is nonnegative, $q \neq 0$, and that the area of M is infinite. Then there exists a piecewise*

smooth function $f : M \rightarrow \mathbb{R}$ with compact support such that

$$\int_M -fT(f)dA < 0, \text{ and } \int_M fdA = 0.$$

Now we will give the proof of the theorems mentioned in the introduction.

Theorem 3. *Let M^2 be a complete, noncompact, orientable surface, and let $x : M^2 \rightarrow N^3$ be an isometric immersion with constant mean curvature H . Assume that N^3 has bounded geometry and $H^2 \geq -\frac{1}{3} \inf_M S$, where S is the scalar curvature of N^3 . If M^2 has finite index, then $H^2 = -\frac{1}{3} \inf_M S$.*

Proof. By (2) Jacobi's operator may be rewritten in the form

$$L = \Delta - K + q,$$

where $q = 4H^2 - K + K_{12} + S$, and S is the scalar curvature of N given by $S = K_{12} + K_{13} + K_{23} = K_{12} + \text{Ric}_N(e_3)$. Since $3H^2 + \inf_M S \geq 0$ and $H^2 - K + K_{12} \geq 0$, we obtain that $q \geq 3H^2 + \inf_M S \geq 0$. Because the operator L has finite index, we obtain by Theorem 1 that

$$\int_M (3H^2 + \inf_M S) dA \leq \int_M q dA < \infty.$$

On the other hand, since N has bounded geometry, we have that M has infinite area (see Theorem 1 in the Appendix). So $H^2 = -\frac{1}{3} \inf_M S$, and this completes the proof. \square

Corollary 4. *Let M^2 be a complete orientable surface, and let $x : M^2 \rightarrow N^3$ be an isometric immersion with constant nonzero mean curvature H . Assume that N^3 has bounded geometry and nonnegative scalar curvature. Then M^2 has finite index if and only if M^2 is compact.*

Proof. Suppose that M is compact. Since L is an elliptic operator, the index of L in M is finite.

If M has finite index and it is noncompact, we obtain, by Theorem 3, that $H = 0$. So M is compact. \square

It should be remarked that when the immersion is minimal the above situation changes. In fact, it has been proved in [8] and [11] that if M^2 is a minimal surface in \mathbb{R}^3 , then $\text{Ind}(M) < \infty$ if and only if $\int_M (-K) dA < \infty$.

∞ . Thus, there exist several examples of noncompact minimal surfaces in \mathbb{R}^3 , with finite index.

Corollary 5. *If M^2 is a complete surface in the hyperbolic three-space of sectional curvature -1 , with constant mean curvature $H > 1$, then $\text{Ind}(M) < \infty$ if and only if M is compact.*

If $H = 1$, M. do Carmo and A.M. Silveira proved in [4] that $\text{Ind}(M) < \infty$ if and only if $\int_M (-K) dA < \infty$. Thus there are examples of noncompact surfaces with $H = 1$ in the hyperbolic space with finite index.

Corollary 6. *There is no complete, noncompact surface with constant mean curvature and finite index in a three-dimensional compact Riemannian manifold with positive scalar curvature.*

Theorem 7. *Let M^2 be a complete, noncompact surface with constant mean curvature and let $x : M^2 \rightarrow N^3$ be an isometric immersion with constant mean curvature H . Assume that N^3 has bounded geometry. If $H^2 \geq -\frac{1}{2} \text{Ric}_N$, where Ric_N is the Ricci curvature of N , and x is stable, then $H^2 = -\frac{1}{2} \inf_M \text{Ric}_N = -\frac{1}{2} \text{Ric}_N(e_3)$ and x is umbilic. Furthermore, M^2 is conformally equivalent to the complex plane or the cylinder. If M^2 is a cylinder, then M^2 is flat.*

Proof. Since x is stable and $H^2 \geq -\frac{1}{2} \text{Ric}_N \geq -\frac{1}{3} \inf_M S$, we have, by Theorem 3 that $H^2 = -\frac{1}{3} \inf_M S = -\frac{1}{2} \inf_M \text{Ric}_N$. From Theorem 1 we also have that M^2 is conformally equivalent to a compact Riemann surface minus a finite number of points.

The Jacobi operator L can be rewritten in the form $L = \Delta + q$, where $q = 2(H^2 - K + K_{12}) + 2H^2 + \text{Ric}_N(e_3)$. Since x is stable, $H^2 - K + K_{12} \geq 0$ and $2H^2 + \text{Ric}_N(e_3) \geq 0$, we obtain by Theorem 2 that $H^2 - K + K_{12} \equiv 0$ and $H^2 \equiv -\frac{1}{2} \text{Ric}_N(e_3)$ on M^2 . So, the immersion x is umbilic.

On the other hand, since

$$-3H^2 \leq S = K_{12} + \text{Ric}_N(e_3) = K_{12} - 2H^2,$$

we have that $K = K_{12} + H^2 \geq 0$. It follows that the operator $\Delta - K$ is positive semidefinite. So, there is a smooth positive function $f : M \rightarrow \mathbb{R}$ that satisfies $\Delta f - Kf = 0$ on M^2 (see Theorem 1 in [9]). Therefore, by Theorem 2 in [9], we know that the universal covering of M^2 is

conformally equivalent to the complex plane. So, M^2 is conformally equivalent to the complex plane or the cylinder. If M^2 is conformally equivalent to the cylinder we have by the Cohn-Vossen inequality, that $\int_M K dA \leq 0$. Since $K \geq 0$, it follows that $K = 0$. This completes the proof. \square

Remark 8 . If N^3 has nonnegative Ricci curvature, the theorem above extends to surfaces with constant mean curvature the result obtained by F. Colbrie and R. Schoen in [9].

Remark 9 . If $N^3 = \mathbb{Q}^3(c)$, $c = -1, 0, 1$, the theorem above yields the results proved by A.M da Silveira in [15].

Appendix

Let N^n be a complete Riemannian manifold of dimension n . We denote by $i_N(p)$ the injectivity radius of N in p and by K_N the sectional curvature of N . As mentioned in the introduction, a manifold N^n has bounded geometry if there exist positive real numbers δ and λ such that $K_N \leq \delta^2$ and $i_N \geq \lambda$ on N^n .

The purpose of this section is to prove the following result.

Theorem 1. *Let M^m be a complete, noncompact manifold, and let $x : M^m \rightarrow N^n$ be an isometric immersion with mean curvature vector field bounded in norm. If N^n has bounded geometry, the volume of M^m is infinite.*

Remark 2 . In the case that N has nonpositive sectional curvature, the result above has been proved by D. Hoffman and R. Schoen (personal communication).

In the proof of Theorem 1 we will use the following result which gives us a bound from below to the volume of a geodesic ball in M . We denote by $B_\mu(p)$ the geodesic ball in M^m with radius μ and center p , and by w_m the volume of the unit ball in \mathbb{R}^m .

Theorem 3. *Let $x : M^m \rightarrow N^n$ be an isometric immersion with mean curvature vector field H bounded in norm. Assume that N^n has sectional*

curvature $K_N \leq \delta^2$, where δ is a positive real number. Then

$$\text{vol}_m(B_\mu(p)) \geq \delta^{-m} w_m (\sin \mu \delta)^m e^{-H_0 \mu}, \quad (1)$$

where $\mu \leq \min\{\frac{\pi}{2\delta}, i_N(p)\}$ and $|H| \leq H_0$.

Remark 4. When N^n has nonpositive sectional curvature we obtain, by letting δ tend to zero,

$$\text{vol}_m(B_\mu(p)) \geq w_m \mu^m e^{-H_0 \mu},$$

where $\mu \leq i_N(p)$.

For the proof of theorem 3 we will use the two lemmas below. First we give the following definitions.

Let $x : M^m \rightarrow N^n$ be an isometric immersion. Given a vector field $V : M \rightarrow TN$, its gradient $\bar{\nabla}V : TM \rightarrow TN$ is the map $w \mapsto \bar{\nabla}_w V$, where $\bar{\nabla}$ is the covariant differentiation on N . The divergence of V on M , denoted by $\text{div}_M V$, is the trace of $\bar{\nabla}V$ on TM . If e_1, \dots, e_m is an orthonormal frame of $T_p M$, then

$$\text{div}_M V(p) = \sum_{i=1}^m \langle \bar{\nabla}_{e_i} V, e_i \rangle(p).$$

Lemma 5. (see [12], pg. 719.) If $V : M \rightarrow TN$ is a vector field on M , then

$$\text{div}_M V^T = \text{div}_M V + \langle V, H \rangle, \quad (2)$$

where V^T denote the projection of V onto TM .

Lemma 6. (see [12], pg. 721.) Let $x : M^m \rightarrow N^n$ be an isometric immersion, $p_0 \in M^m$ and $r(\cdot) = d_N(\cdot, p_0)$, where d_N is the geodesic distance in N^n . Let $V = r \text{grad} r$ be the radial vector field centered at p_0 , where grad is the gradient in N^n . If N^n has sectional curvature $K_N \leq \delta^2$, $\delta > 0$, then

$$\text{div}_M V(p) \geq m \delta r(p) \cot(\delta r(p)),$$

for any point $p \in M$ such that $r(p) < \min\{\pi/\delta, i_N(p_0)\}$.

Remark. In the case that N^n has nonpositive sectional curvature, we have

$$\text{div}_M V(p) \geq m,$$

for any point $p \in M$ such that $r(p) < i_N(p_0)$. For this, we let δ tend to zero and observe that $s \cdot \cot(s) \rightarrow 1$ as s tend to zero.

Proof of Theorem 3. Let p_0 be a point in M , $r(p) = d_N(p, p_0)$ and $V(p) = r(p) \operatorname{grad} r(p)$. Let ρ be the function r restricted to M . Since $\operatorname{grad}_M \rho = (\operatorname{grad} r)^T$, we obtain by (2) that

$$\begin{aligned} \Delta_M \rho^2(p) &= 2 \operatorname{div}_M(\rho(\operatorname{grad} r)^T)(p) \\ &= 2 \operatorname{div}_M V^T(p) = 2(\operatorname{div}_M V + \langle V, H \rangle)(p) \\ &\geq 2(m\delta \rho \cot(\delta\rho) - \rho H_0)(p), \end{aligned}$$

for any point $p \in B_\mu(p_0)$, where Δ_M is the Laplacian on M and $\mu \leq \min\{\pi/\delta, i_N(p_0)\}$. By integrating the above expression,

$$\int_{B_\mu(p_0)} \Delta_M \rho^2 dA \geq 2m\delta C(\mu) - 2H_0 \int_{B_\mu(p_0)} \rho dA, \quad (3)$$

where $C(\mu) = \int_{B_\mu(p_0)} \rho \cot(\delta\rho) dA$. On the other hand, by using Stokes theorem, we obtain

$$\int_{B_\mu(p_0)} \Delta_M \rho^2 dA = \int_{\partial B_\mu(p_0)} \operatorname{grad}_M \rho^2 \vec{\eta} dS,$$

where $\vec{\eta}$ is the exterior normal vector field to $B_\mu(p_0)$ on $\partial B_\mu(p_0)$. So, since $|\operatorname{grad}_M \rho| \leq 1$ and $\rho(p) \leq d_M(p, p_0)$, we obtain

$$\int_{B_\mu(p_0)} \Delta_M \rho^2 dA \leq 2\mu A(\mu), \quad (4)$$

where $A(\mu) = \operatorname{vol}_{m-1}(\partial B_\mu(p_0))$.

Moreover, since $\rho(p) < \mu < \pi/2\delta$ and $p \in B_\mu(p_0)$,

$$\rho(p) = \rho(p) \cot(\delta\rho)(p) \tan(\delta\rho)(p) \leq \rho(p) \cot(\delta\rho)(p) \tan(\delta\mu). \quad (5)$$

By (3), (4) and (5),

$$\mu A(\mu) \geq (m\delta - H_0 \tan \mu\delta) C(\mu). \quad (6)$$

From the formula co-area formula (see [3] pag. 80), we obtain

$$\frac{dC}{d\mu} \geq \mu \cot(\delta\mu) A(\mu), \quad (7)$$

since the function $s \mapsto s \cdot \cot(s)$ is decreasing in $[0, \pi/2]$ and $\rho(p) \leq d_M(p, p_0)$. So by (6) and (7),

$$\frac{d}{d\mu}(C(\mu)(\sin \mu\delta)^{-m+1}) \geq (\sin \mu\delta)^{-m}(\delta \cos \mu\delta - H_0 \sin \mu\delta)C(\mu). \quad (8)$$

Therefore,

$$\frac{\frac{d}{d\mu}(C(\mu)(\sin \mu\delta)^{-m+1})}{C(\mu)(\sin \mu\delta)^{-m+1}} \geq \delta \cot \mu\delta - H_0.$$

By integrating the above expression from ϵ to μ , we obtain

$$\log \left(\frac{C(\mu)(\sin \mu\delta)^{-m}}{C(\epsilon)(\sin \epsilon\delta)^{-m}} \right) \geq -H_0(\mu - \epsilon).$$

It follows that

$$C(\mu)(\sin \mu\delta)^{-m} \geq C(\epsilon)(\sin \epsilon\delta)^{-m}e^{-H_0(\mu-\epsilon)}.$$

We now consider the function $\bar{\rho}(p) = d_M(p, p_0)$ and the function

$$\bar{C}(\epsilon) = \int_{B_\epsilon(p_0)} \bar{\rho} \cot(\bar{\rho}\delta) dA.$$

Since $C(\epsilon) \geq \bar{C}(\epsilon)$ and

$$\lim_{\epsilon \rightarrow 0^+} \bar{C}(\epsilon)(\sin(\epsilon\delta))^{-m} = w_m \delta^{-m-1},$$

we obtain

$$C(\mu) \geq \delta^{-m-1} w_m \sin(\mu\delta)^m e^{-H_0\mu}.$$

On the other hand, since the function $s \mapsto s \cdot \cot(s)$ is decreasing and $s \cdot \cot(s) \rightarrow 1$, as s tend to 0, we have

$$\text{vol}_m(B_\mu(p_0)) \geq \delta C(\mu).$$

Then

$$\text{vol}_m(B_\mu(p_0)) \geq \delta^{-m} w_m (\sin \mu\delta)^m e^{-H_0\mu},$$

and this completes the proof of Theorem 3. \square

Proof of Theorem 1. Let $\lambda > 0$ be a real number such that $i_N(q) \geq \lambda$ for any point $q \in N^n$, and let $\mu_0 = \min\{\pi/2\delta, \lambda\}$. Then, by (1),

$$\text{vol}_m(B_{\mu_0}(p)) \geq L > 0, \quad (9)$$

for any point $p \in M^m$, where

$$L = \delta^{-m} w_m (\sin \mu_0 \delta)^m e^{-H_0 \mu_0}.$$

Since M^m is complete and noncompact, it has a geodesic ray $\gamma : [0, \infty) \rightarrow M^m$. Consider the sequence of points $p_j = \gamma(3j\mu_0)$, $j \geq 0$. Then, if $j \neq k$,

$$B_{\mu_0}(p_j) \cap B_{\mu_0}(p_k) = \emptyset.$$

So, by (9), for any positive integer k ,

$$\text{vol}_m(M) \geq \text{vol}_m \left(\bigcup_{i=0}^k B_{\mu_0}(p_i) \right) \geq (k+1)L.$$

Since k is arbitrary, the volume of M^m is infinite, and this completes the proof of Theorem 1. \square

Corollary 8. *Let M^m be a complete, noncompact Riemannian manifold, and let N^n be a compact Riemannian manifold. Let $x : M^m \rightarrow N^n$ be an isometric immersion with mean curvature vector field bounded in norm. Then the volume of M^m is infinite.*

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