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Flatness of Families Induced By Hypersurfaces on Flag Varieties

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Abstract. We show the family of tangent flags to smooth quadric hypersurfaces extends to a flat family parametrized by the variety of complete quadrics. This answers a question posed by S. Kleiman.

Introduction

Let $\mathbf{S}$ be the variety of complete quadrics, $\mathbf{S}^{nd}$ the open subset of nondegenerate quadrics and $\mathbf{F}_n$ the variety of complete flags in $\mathbf{P}^n$. Let $f_0 : \mathbf{S}^{nd} \to \mathbf{Hilb}(\mathbf{F}_n)$ be the morphism that assigns to each nondegenerate quadric the locus of its tangent flags. We prove the following.

Theorem. *f_0 extends to a morphism $f : \mathbf{S} \to \mathbf{Hilb}(\mathbf{F})$.*

This answers affirmatively a question S. Kleiman asked in ([**K**], p.362).

Let $\mathbf{F}_{0,n-1} \subset \mathbf{P}^n \times \check{\mathbf{P}}^n$ be the partial flag variety "point $\in$ hyperplane". We first show that $\mathbf{S}$ parametrizes a flat family

$$\begin{array}{ccc} \mathbf{K} & \subset & \mathbf{S} \times \mathbf{F}_{0,n-1} \\ & \searrow \quad \swarrow & \\ & \mathbf{S} & \end{array}$$

that restricts, over $\mathbf{S}^{nd}$, to the family of the graphs of the Gauss map (point $\mapsto$ tangent hyperplane) of nondegenerate quadric hypersurfaces. The family $\widetilde{\mathbf{K}} \to \mathbf{S}$ pertinent to Kleiman's question is obtained by pull-

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back in the fiber square,

$$\begin{array}{ccc} \widetilde{\mathbf{K}} & \hookrightarrow & \mathbf{F}_n \times \mathbf{S} \\ \downarrow & & \downarrow \\ \mathbf{K} & \hookrightarrow & \mathbf{F}_{0,n-1} \times \mathbf{S}, \end{array}$$

where the vertical maps are flag bundles.

Our proof of flatness for the completed family of graphs relies on Laksov's description [**L**] of Semple–Tyrrell's "standard" affine open cover of $\mathbf{S}$.

The space of complete conics has recently reappeared as a simple instance of Kontsevich's spaces of stable maps (cf. Pandharipande [**P**]). It is also instrumental for the counting of rational curves on a K3 surface double cover of the plane (cf. [**V1**]). Complete quadric surfaces play a role in Narasimhan–Trautmann [**NT**] study of a compactification of a space of instanton bundles.

We also show that any flat family of hypersurfaces on Grassmann varieties induces a flat family of subschemes of the corresponding flag variety. Precisely, we have the following.

Proposition. *Let $\mathbf{G}_{r,n}$ denote the grassmannian of projective subspaces of dimension r of $\mathbf{P}^n$. For each $r = 0 \dots n-1$, let $\mathbf{W}_r \subset \mathbf{T}_r \times \mathbf{G}_{r,n}$ be the total space of a flat family of hypersurfaces in $\mathbf{G}_{r,n}$ parametrized by a variety $\mathbf{T}_r$. Then*

$$\mathbf{W} := \left(\mathbf{W}_0 \times \cdots \times \mathbf{W}_{n-1}\right) \times \left(\mathbf{T} \times \mathbf{F}_n\right) \longrightarrow \mathbf{T} := \mathbf{T}_0 \times \cdots \times \mathbf{T}_{n-1}$$

where $\times$ stands for fiber product over $\mathbf{G}_{0,n} \times \cdots \times \mathbf{G}_{n-1,n} \times \mathbf{T}$, is flat.

This statement was first obtained as an earlier attempt to answer Kleiman's question. The reason we include it here is that, in one hand, the proof rests on a nice, sharp count of constants, akin to dimension estimates of Fano varieties of linear subspaces of a hypersurface (cp. Harris [**JH**], thm. 12.8, p.154).

On the other hand, for the specific case envisaged here, take $\mathbf{W}_r \to \mathbf{T}_r$ to be the family defined by intersections of $\mathbf{G}_{r,n}$ with the complete system of quadric hypersurfaces for the Plücker embedding. Recall that we have $\mathbf{S} \subset \mathbf{T}$ (cf. Kleiman-Thorup [**KT**], (7.9) p.314, Laksov [**L**] p.375, [**V**], 6.3 p. 214). Now it is fun and instructive to realize that the family $\mathbf{W} \subset \mathbf{T} \times \mathbf{F}_n \to \mathbf{T}$ described in the proposition, does *not* restrict to the family of tangent flags. In fact, for conics ($n = 2$) its fibers are of arithemtic genus 1. It yields a double structure on the graph of the Gauss map. For $n = 3$ (and conceivably for higher n) the fiber of $\mathbf{W}$ over a point of $\mathbf{S}$ representing a smooth quadric contains the tangent flag as one of its two components. (cf. §7.4 for details).

In section 1 we compute the Hilbert polynomial of the graph of the Gauss map of a general quadric. In section 2 we do the same for the subscheme defined by the initial ideal of the ideal of 2×2 minors that cut out the diagonal subvariety of $\mathbf{P}^n$. In section 3 we recall Laksov's description of the standard open cover of $\mathbf{S}$ introduced by Semple and Tyrrel. This is used in section 5 to study a torus action compatible with the family of graphs defined in section 4. The proof of the theorem is accomplished in section 6 by comparing Hilbert polynomials at the generic and special points. The final section contains the proof of the proposition and some observations for the cases $n = 2, 3$. Thanks are due to the referee for his help in clarifying and correcting several points.

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1. The tangent flag to a smooth quadric

Write $x = (x_1, \dots, x_{n+1})$ (resp. $y = (y_1, \dots, y_{n+1})$) for the vector of homogeneous coordinates in $\mathbf{P}^n$ (resp. $\check{\mathbf{P}}^n$). Let $\mathbf{F}_{0,n-1} \subset \mathbf{P}^n \times \check{\mathbf{P}}^n$ be the incidence correspondence "point $\in$ hyperplane". It is the zeros of the incidence section $x \cdot y$ of $\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathcal{O}_{\check{\mathbf{P}}^n}(1)$.

Let $\mathcal{K} \subset \mathbf{P}^n$ denote a smooth quadric represented by a symmetric matrix a. The Gauss map $\gamma : \mathcal{K} \to \check{\mathbf{P}}^n$ is given by $x \mapsto y = x \cdot a$. Hence we have

$$\gamma^*(\mathcal{O}_{\check{\mathbf{P}}^n}(1)) = \mathcal{O}_{\mathbf{P}^n}(1)_{|\mathcal{K}}.$$

The tangent flag $\widetilde{\mathcal{K}} \subset \mathbf{F}_n$ of $\mathcal{K}$ is equal to the restriction of the flag bundle

$$\mathbf{F}_n \to \mathbf{F}_{0,n-1} \subset \mathbf{P}^n \times \check{\mathbf{P}}^n$$

over the graph Γ_κ of γ. Consequently, flatness of the family $\{\widetilde{\mathcal{K}}\}$ of tangent flags is equivalent to flatness of the family of graphs $\{\Gamma_\kappa\}$ as long as we stay over the open set $\mathbf{S}^{nd}$. The family $\{\Gamma_\kappa\}_{\kappa\in\mathbf{S}^{nd}}$ will be handled in §4: we will show it extends flatly over $\mathbf{S}$; therefore so does $\{\widetilde{\mathcal{K}}\}_{\kappa\in\mathbf{S}^{nd}}$.

We proceed to compute the Hilbert polynomial of the graph Γ_κ of the Gauss map of a general quadric hypersurface $\mathcal{K} \subset \mathbf{P}^n$.

1.1 Lemma. *Notation as above, the Hilbert polynomial* $\chi(\mathcal{O}_{\Gamma_\kappa}(\mathcal{L}^{\otimes t}))$ *with respect to*

$$\mathcal{L} = \left(\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathcal{O}_{\check{\mathbf{P}}^n}(1)\right)_{|\Gamma}$$

is equal to

$$\binom{2t+n}{n} - \binom{2(t-1)+n}{n}.$$

Proof. We have $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}^n}(2)_{|\mathcal{K}}$ under the identification $\Gamma \cong \mathcal{K}$. Thus we may compute

$$\begin{aligned}\chi(\mathcal{L}^{\otimes t}) &= \chi\big(\mathcal{O}_{\mathbf{P}^n}(2t)\big)_{|\mathcal{K}} \\ &= \chi\big(\mathcal{O}_{\mathbf{P}^n}(2t)\big) - \chi\big(\mathcal{O}_{\mathbf{P}^n}(2t-2)\big) \\ &= \binom{2t+n}{n} - \binom{2(t-1)+n}{n}. \qquad \square\end{aligned}$$

2. Hilbert polynomial of loci of rank 1 matrices

The image of the Segre embedding $\mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^N$ is the variety of matrices of rank one. The image Δ of the diagonal $\mathbf{P}^n \to \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^N$ is the subvariety of *symmetric* matrices of rank one. Its Hilbert polynomial is easily found to be given by

$$\dim\left(H^0(\Delta, \mathcal{O}_{\mathbf{P}^N}(t)\right) = \binom{2t+n}{n} \tag{1}$$

for $t >> 0$. The bi-homogeneous ideal I_Δ of the diagonal is generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & \dots & x_{n+1} \\ y_1 & y_2 & \dots & y_{n+1} \end{bmatrix}. \qquad (2)$$

Write

$$S = k[x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}]$$

for the polynomial ring in $2n+2$ variables, and let $S_{i,j}$ denote the space of bi-homogeneous polynomials of bi-degree (i, j). We have for $t >> 0$

$$\dim_k S_{t,t}/(I_\Delta)_{t,t} = \binom{2t+n}{n}. \qquad (3)$$

Indeed, quite generally, for a closed subscheme $X \subseteq \mathbf{P}^m \times \mathbf{P}^n$ defined by a bi-homogeneous ideal $I \subseteq S$ we have, by Serre's theorem (cf. Kleiman-Thorup [**KTB**], (4.2) p. 189),

$$H^0\big(X, \mathcal{O}_{\mathbf{P}^m}(i) \otimes \mathcal{O}_{\mathbf{P}^n}(j)_{|X}\big) = S_{i,j}/(I)_{i,j} \quad \text{for all } \ i, j >> 0.$$

Thus (3) follows from

$$H^0\big(X, \mathcal{O}_{\mathbf{P}^N}(t)_{|X}\big) = H^0\big(X, \mathcal{O}_{\mathbf{P}^m}(t) \otimes \mathcal{O}_{\mathbf{P}^n}(t)_{|X}\big).$$

2.1 Lemma. *Let Γ_0 be the subscheme of $\mathbf{P}^n \times \check{\mathbf{P}}^n$ defined by the ideal*

$$\langle x_i y_j \,|\, 1 \le i < j \le n+1 \rangle + \langle \textstyle\sum x_i y_i \rangle.$$

Then we have

$$\varphi_{\Gamma_0}(t) = \binom{2t+n}{n} - \binom{2(t-1)+n}{n}.$$

Proof. The whole point is to notice[1] that the $x_i y_j$ span the ideal of initial terms of I_Δ with respect to a suitable order. In fact, the set of 2×2 minors of (2) is known to be a (universal) Gröbner basis for I_Δ (see Sturmfels [**BS**], thm.1, p.137 or [**BS1**]). By (1), we may write (cf. Eisenbud [**E**], thm. 15.26, p.356),

$$\varphi_{in(I_\Delta)}(t) = \varphi_{I_\Delta}(t) = \binom{2t+n}{n}.$$

[1] I'm indebted to P. Gimenez for his precious help on this matter.

One checks at once that $\sum x_i y_i$ is a nonzero divisor mod the initial ideal $in(I_\Delta)$ (see 7(i)). Therefore

$$\varphi_{\Gamma_0}(t) = \varphi_{in(I_\Delta)}(t) - \varphi_{in(I_\Delta)}(t-1). \quad \square$$

We will deduce flatness for the "completed" family of Gauss maps from the fact that the above Hilbert polynomial at the special point Γ_0 coincides with the generic one (1.1).

3. Semple-Tyrrell-Laksov cover of S

Let $\mathbf{U}_n$ denote the group of lower triangular unipotent $(n+1)$-matrices. Thus, $\mathbf{U}_n$ is isomorphic to the affine space $\mathbb{A}^{n(n+1)/2}$ with coordinate functions $u_{i,j}$, $1 \le j \le i-1$, $i = 2 \dots n+1$. These are thought of as entries of the matrix,

$$u = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{2,1} & 1 & 0 & \cdots & 0 \\ u_{3,1} & u_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n+1,1} & u_{n+1,2} & u_{n+1,3} & u_{n+1,n} & 1 \end{bmatrix}.$$

Let $d_1, \dots, d_n$ be coordinate functions in $\mathbb{A}^n$. Put

$$d^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & d_1 & 0 & \cdots & 0 \\ 0 & 0 & d_1 d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_1 d_2 \cdots d_n \end{bmatrix}. \tag{4}$$

For a matrix A let its ith adjugate be the matrix $\overset{i}{\wedge} A$ of all $i \times i$ minors. We denote by $d^{(i)}$ the matrix obtained from $\overset{i}{\wedge} d^{(1)}$ by removing the common factor $d_1^{i-1}d_2^{i-2}\cdots d_{i-1}$. *E.g.*, for $n=3$ we have

$$\begin{aligned} d^{(1)} &= diag(1,\, d_1,\, d_1d_2,\, d_1d_2d_3) \\ d^{(2)} &= diag(d_1,\, d_1d_2,\, d_1d_2d_3,\, d_1^2d_2,\, d_1^2d_2d_3,\, d_1^2d_2^2d_3)/(d_1) \\ &= diag(1,\, d_2,\, d_2d_3,\, d_1d_2,\, d_1d_2d_3,\, d_1d_2^2d_3) \\ d^{(3)} &= diag(1,\, d_3,\, d_2d_3,\, d_1d_2d_3). \end{aligned}$$

The map $\mathbf{U}_n \times \mathbb{A}^n \to \mathbf{S} \subset \prod_{i=1}^{i=n} \mathbf{P}(S_2(\bigwedge^i k^{n+1*}))$ defined by sending $(u,\, d)$ to

$$(u\, d^{(1)}\, u^t,\, (\overset{2}{\wedge}\, u)\, d^{(2)}\, \overset{2}{\wedge} u^t, \ldots, (\overset{n}{\wedge} u)\, d^{(n)}\, \overset{n}{\wedge} u^t)$$

is an isomorphism onto an affine open subset $\mathbf{S}^0$ of $\mathbf{S}$. The variety of complete quadrics may be covered by translates of $\mathbf{S}^0$ (cf. Laksov [**L**], p. 376-377).

Let $\mathbf{S}^0_d \cong \mathbf{U}_n \times \mathbb{A}^n{}_d$ be the principal open piece defined by $d_1d_2\cdots$ $\cdots d_n \neq 0$. It maps isomorphically onto an open subvariety of $\mathbf{S}^{nd}$.

4. Graph of the Gauss map

The variety $\mathbf{S}^{nd}$ of nondegenerate quadrics parametrizes a flat family of graphs of Gauss maps. For a nondegenerate quadric represented by a symmetric matrix $a \in \mathbf{S}^{nd}$ the Gauss map is given by $x \mapsto y = x \cdot a$. We define $\mathbf{K}^{nd} \subset \mathbf{S}^{nd} \times \mathbf{P}^n \times \check{\mathbf{P}}^n$ by the bi-homogeneous ideal generated by the incidence relation $x \cdot y$ together with the 2×2 minors of the $2\times(n+1)$ matrix with rows y, $x \cdot z$, where z denotes the generic symmetric matrix. Clearly $\mathbf{K}^{nd} \to \mathbf{S}^{nd}$ is a map of $\mathbf{GL}_{n+1}-$homogeneous spaces.

Now write $a = vc^{(1)}v^t$ with $v \in \mathbf{U}_n$, $c \in \mathbb{A}^n{}_d$ ($c^{(1)}$ as in (4)), and put $x' = xv$, $y' = y(v^{-1})^t$. We have $y = xa$ iff $y' = x'c^{(1)}$. Let

$$\mathbf{K}^0_d \subset \mathbf{S}^0_d \times \mathbf{P}^n \times \check{\mathbf{P}}^n. \tag{5}$$

be defined by $x \cdot y$ together with the 2×2 minors of the $2\times(n+1)$ matrix

$$\begin{bmatrix} x'_1 & d_1x'_2 & d_1d_2x'_3 & \ldots & d_1\cdots d_nx'_{n+1} \\ y'_1 & y'_2 & y'_3 & \ldots & y'_{n+1} \end{bmatrix} \tag{6}$$

where we put $x'_j = \sum_i u_{ij}x_i$ and likewise y'_j denotes the jth entry of $y(u^{-1})^t$. Thus $\mathbf{K}^0_d$ is the total space of the family of Gauss maps parametrized by $\mathbf{S}_d^0$. Note that $\mathbf{K}_d^0 \to \mathbf{S}_d^0$ is a smooth quadric bundle. Its fiber over $(I,(1,\dots,1)) \in \mathbf{U}_n \times \mathbb{A}_d^n$ is equal to the quadric given by $\sum x_i^2$ inside the "diagonal" $y_1 = x_1, \dots, y_{n+1} = x_{n+1}$ of $\mathbf{P}^n \times \check{\mathbf{P}}^n$.

Let

$$\mathbf{K}^0 \subset \mathbf{S}^0 \times \mathbf{P}^n \times \check{\mathbf{P}}^n \tag{7}$$

be defined by $x \cdot y$ together with the ideal

$$\begin{aligned} J = \langle & x_1'y_2' - d_1y_1'x_2', \dots, x_1'y_{n+1}' - d_1 \cdots d_n y_1' x_{n+1}', \\ & x_2'y_3' - d_2y_2'x_3', \dots, x_n'y_{n+1}' - d_n y_n' x_{n+1}' \rangle \end{aligned} \tag{8}$$

obtained by cancelling all d_i factors occurring in the above 2×2 minors. We obviously have $\mathbf{K}^0_{|\mathbf{S}_d^0} = \mathbf{K}_d^0$.

We will show that $\mathbf{K}^0$ is the scheme theoretic closure of $\mathbf{K}_d^0$ in $\mathbf{S}^0 \times \mathbf{P}^n \times \check{\mathbf{P}}^n$ (cf. 6.2).

5. A torus action

Notation as in (4), embed $\mathbb{G}_m^{\times n}$ in $\mathbf{GL}_{n+1}$ by sending $c = (c_1, \dots, c_n) \in \mathbb{G}_m^{\times n}$ to $c^{(1)} = diag(1, c_1, c_1c_2, \dots)$. We let $\mathbb{G}_m^{\times n}$ act on $\mathbf{S}^0$ by

$$c \cdot (v,b) = (c^{(1)}\, v\, (c^{(1)})^{-1},\ (c_1^2 b_1, \dots c_n^2 b_n)).$$

This action is compatible with the natural action of $\mathbf{GL}_{n+1}$ on the space $\mathbf{P}(S_2(k^{n+1*}))$ of quadrics, *i.e.*, for a symmetric matrix $a(v,b) := v\, b^{(1)}\, v^t$ as above, we have

$$\begin{aligned} c^{(1)} \cdot a(v,b) &= c^{(1)}\, a(v,b)\, (c^{(1)})^t = c^{(1)}\, v\, b^{(1)}\, v^t\, (c^{(1)})^t \\ &= c^{(1)}\, v\, (c^{(1)})^{-1}\, c^{(1)}\, b^{(1)}\, c^{(1)}\, ((c^{(1)})^t)^{-1}\, v^t\, (c^{(1)})^t \\ &= c^{(1)}\, v\, (c^{(1)})^{-1}\, (c^{(1)})^2\, b^{(1)}\, ((c^{(1)})^t)^{-1}\, v^t\, (c^{(1)})^t \\ &= a(c \cdot (v,b)). \end{aligned}$$

It can be also easily checked that $\mathbb{G}_m^{\times n}$ acts compatibly on $\mathbf{S}^0 \times \mathbf{P}^n \times \check{\mathbf{P}}^n$ and $\mathbf{K}^0$ is invariant. Indeed, let $((v,b),x,y) \in \mathbf{K}^0$. Pick $c \in \mathbb{G}_m^{\times n}$. We have

$$c \cdot ((v,b),x,y) = ((c^{(1)}\, v\, (c^{(1)})^{-1},\ (c_1^2 b_1, \dots c_n^2 b_n)),\ x\, (c^{(1)})^{-1},\ y\, (c^{(1)})^t).$$

Now $x' = xv$ changes to

$$x'' = (x\, (c^{(1)})^{-1})\, (c^{(1)}\, v\, (c^{(1)})^{-1}) = x\, v\, (c^{(1)})^{-1} = x'\, (c^{(1)})^{-1}$$

so that the first row $x'\, b^{(1)}$ in (6) (evaluated at $((v,b),x,y)$) changes to

$$x''\,(b^{(1)}\,(c^{(1)})^2) \;=\; x'\,(c^{(1)})^{-1}\,(b^{(1)}\,(c^{(1)})^2) \;=\; x'\,(b^{(1)}\,c^{(1)}).$$

Similarly, $y' \;=\; y\,(v^{-1})^t$ changes to

$$y'' \;=\; (y\,(c^{(1)})^t)\,((c^{(1)}\,v\,(c^{(1)})^{-1})^{-1})^t \;=\; y\,(v^{-1})^t\,(c^{(1)})^t) \;=\; y'\,c^{(1)}.$$

Therefore (6) changes to the matrix with rows $x'\,(b^{(1)}\,c^{(1)})$ and $y'\,c^{(1)}$. Thus evaluation of (8) at $c\cdot((v,b),x,y)$ and at $((v,b),x,y)$ differ only by nonzero multiples.

5.1 Lemma. *The orbit of* $(I,0)\in \mathbf{S}^0$ *is the unique closed orbit where* I *is the identity matrix.*

Proof. Conjugation of $v\in \mathbf{U}_n$ by the diagonal matrix $c^{(1)}$ replaces each entry $v_{ij},\ j<i$ by

$$\begin{aligned}(c^{(1)}\,v\,(c^{(1)})^{-1})_{ij} &= c^{(1)}_{ii}\,(v\,(c^{(1)})^{-1})_{ij} = c^{(1)}_{ii}\,v_{ij}\,((c^{(1)})^{-1})_{jj}\\ &= v_{ij}\,c^{(1)}_{ii}/c^{(1)}_{jj} = v_{ij}\,c_{i-1}\cdots c_j.\end{aligned}$$

Thus, letting $c\to 0$, we see that $(I,0)$ is in the orbit closure $\overline{\mathbb{G}_m^{\times n}\cdot(v,b)}$. □

6. Proof of the theorem

6.1 Lemma. *Notation as in (7), the family* $\mathbf{K}^0\to\mathbf{S}^0$ *is flat.*

Proof. Since $\mathbf{K}^0\to\mathbf{S}^0$ is equivariant for the $\mathbb{G}_m^{\times n}$–action, it suffices to check that the Hilbert polynomial of the fiber over the representative $(I,0)$ of the unique closed orbit is right, *i.e.*, coincides with the generic one (cf. Hartshorne [**H**], thm. 9.9, p.261). Evaluating (8) at $(I,0)$ yields the monomial ideal in 2.1. We are done by virtue of 1.1. □

6.2 Lemma. *Notation as in (7) and (5), we have that* $\mathbf{K}^0$ *is equal to the scheme theoretic closure of* $\mathbf{K}^0_d$.

Proof. In view of 6.1, we may apply to $\mathbf{K}^0\to\mathbf{S}^0\supset\mathbf{S}^0_d$ the general observation that the formation of scheme theoretic closure commutes with flat base change (cf. [**EGA**], (11.10.5), p. 171, [**EGA-I**], p. 325). □

6.3 Lemma. *Let G be an algebraic group and let*

$$\begin{array}{ccc} X^0 & \subset & X \\ \downarrow & & \downarrow \\ Y^0 & \subset & Y \end{array}$$

be a commutative diagram of maps of $G-$varieties. Let $\overline{X}$, $\overline{Y}$ denote the closures of X^0, Y^0. If $\overline{X} \to \overline{Y}$ is flat over a neighborhood of a point in each closed orbit then $\overline{X} \to \overline{Y}$ is flat.

Proof. Immediate. □

We may now finish the proof of the theorem. Let $\mathbf{K} \subset \mathbf{S} \times \mathbf{P}^n \times \check{\mathbf{P}}^n$ be the scheme theoretic closure of $\mathbf{K}^0$. We have $\mathbf{K} \cap (\mathbf{S}^0 \times \mathbf{P}^n \times \check{\mathbf{P}}^n) = \mathbf{K}^0$ flat over $\mathbf{S}^0$ by 6.1. The latter is a neighborhood of a point in the unique closed orbit of $\mathbf{S}$. Now apply the previous lemma to $G = \mathbf{GL}_{n+1}$, $X = \mathbf{S} \times \mathbf{P}^n \times \check{\mathbf{P}}^n$, $Y = \mathbf{S}$, $Y^0 = \mathbf{S}^{nd}$, $X^0 = \mathbf{K}^{nd}$. Finally, since the family of tangent flags is defined by the fiber square,

$$\begin{array}{ccc} \widetilde{\mathbf{K}} & \longrightarrow & \mathbf{F}_n \times \mathbf{S} \\ \downarrow & & \downarrow \\ \mathbf{K} & \longrightarrow & \mathbf{F}_{0,n-1} \times \mathbf{S} \end{array}$$

the composition $\widetilde{\mathbf{K}} \to \mathbf{K} \to \mathbf{S}$ is flat.

7. Final remarks and proof of the proposition

7.1. (i) The primary decomposition of the monomial ideal in 2.1 can be checked to be given by

$$\langle x_1, x_2, \ldots, x_n \rangle \cap \cdots \cap \langle x_1, \ldots, x_i, y_{i+2}, \ldots, y_{n+1} \rangle \cap \cdots$$
$$\cdots \cap \langle y_2, y_3, \ldots, y_{n+1} \rangle.$$

Thus enlarging it to include the nonzero divisor $x \cdot y$ we see that the special fiber Γ_0 presents no embedded component.

(ii) In the situation of 6.3, let $X \to Y = \mathbf{P}^n$ be the blowup of a point, acted on by the stabilizer of that point. Of course flatness fails over any neighborhood of the unique closed orbit. This might clarify why we had to show first that $\mathbf{K}^0 \to \mathbf{S}^0$ is flat, instead of trying to show directly that the closure of $\mathbf{K}^{nd}$ is flat over $\mathbf{S}^{nd}$.

(iii) For $n = 2$ we may write the following global equations for $\mathbf{K}$. Let z, w be a pair of symmetric 3×3 matrices of independent indeterminates. Then $\mathbf{K} \subset \mathbf{P}^5\times\check{\mathbf{P}}^5\times\mathbf{P}^2\times\check{\mathbf{P}}^2$ is given by the 2×2 minors of the 2×3 matrices with rows $x \cdot z$, y and x, $y \cdot w$, in addition to the incidence relation $x \cdot y$ together with the equation $3z \cdot w = \mathrm{trace}(z \cdot w)I$ for $\mathbf{S} \subset \mathbf{P}^5\times\check{\mathbf{P}}^5$. Indeed, the equations for $\mathbf{S}$ are right because they are invariant, they are satisfied for $z = I, w = I$ hence on the open orbit of $\mathbf{S}$, therefore on all of $\mathbf{S}$. Moreover, the solutions with $z = \mathrm{diag}(1,1,0)$ and $z = \mathrm{diag}(1,0,0)$ also lie in $\mathbf{S}$. Letting $\mathbf{K}'$ be the subscheme defined by those equations, one checks at once that the fiber $\mathbf{K}'_{(I,I)}$ over the representative of the open orbit of $\mathbf{S}$ is equal to the graph of the Gauss map. The fiber over the representative of the closed orbit, given by $z = \mathrm{diag}(1,0,0)$, $w = \mathrm{diag}(0,0,1)$, is cut out in $\mathbf{P}^2\times\check{\mathbf{P}}^2$ by $x \cdot y$ in addition to the 2×2 minors of the matrices

$$\begin{pmatrix} x_1 & 0 & 0 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad , \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & y_3 \end{pmatrix}.$$

The ideal is precisely the one described in 2.1. It would be nice to give a similar description for higher dimension.

(iv) Still assuming $n = 2$, put

$$\Gamma = \{(P, \ell, \kappa, \kappa') \in \mathbf{P}^2\times\check{\mathbf{P}}^2\times\mathbf{P}^5\times\check{\mathbf{P}}^5 \mid P \in \kappa\cap\ell, \ \ell \in \kappa'\}.$$

It is easy to check that $\Gamma_{|\mathbf{S}} = \mathbf{K}$ as sets. Furthermore, Γ may be endowed with a natural scheme structure such that $\Gamma \to \mathbf{P}^5\times\check{\mathbf{P}}^5$ is flat and with Hilbert polynomial of its fibers equal to $4t$ (cf. (9) below). Thus, $\Gamma_{|\mathbf{S}} \to \mathbf{S}$ is a family of double structures of genus one on the fibers of $\mathbf{K}$. See in (7.4) below a similar discussion for $n = 3$.

We proceed to prove the proposition stated at the introduction.

Before considering the general case, we describe the situation in the projective plane. Thus, let

$$\mathbf{F}_2 \subset \mathbf{P}^2 \times \check{\mathbf{P}}^2$$

be the incidence correspondence "point $\in$ line". Let f_0 (resp. f_1) denote a curve in $\mathbf{P}^2$ (resp. $\check{\mathbf{P}}^2$). Set

$$\Gamma_{\underline{f}} := (f_0 \times f_1) \cap \mathbf{F}_2.$$

Then $\Gamma_{\underline{f}}$ is easily seen to be regularly embedded of codimension 2 in $\mathbf{F}_2$ (cf. 7.2). Moreover, its Hilbert polynomial with respect to the ample sheaf $\mathcal{O}_{\mathbf{P}^2}(1) \otimes \mathcal{O}_{\check{\mathbf{P}}^2}(1)$ restricted to $\mathbf{F}_2$ depends only on the degrees, say d_0, d_1 of f_0, f_1. In fact, the Koszul complex that resolves the ideal of $f_0 \times f_1$ in $\mathbf{P}^2 \times \check{\mathbf{P}}^2$ restricts to a resolution of $\Gamma_{\underline{f}}$ in $\mathbf{F}_2$. One finds the Hilbert polynomial,

$$\chi_{\underline{f}}(t) = (d_0 + d_1)t - d_0 d_1 (d_0 + d_1 - 4)/2. \qquad (9)$$

Therefore, as in the final argument for the proof of 6.1, the parameter space of pairs (f_0, f_1), call it $\mathbf{T}$ $(=\mathbf{P}^{n_0} \times \mathbf{P}^{n_1}$ for suitable $n_0, n_1)$, carries a flat family of curves on $\mathbf{F}_2$. Precisely, let

$$\mathbf{W}_0 \subset \mathbf{P}^2 \times \mathbf{P}^{n_0} \text{ and } \mathbf{W}_1 \subset \check{\mathbf{P}}^2 \times \mathbf{P}^{n_1}$$

denote the total spaces of the universal plane curve parametrized by $\mathbf{P}^{n_i}$. Then

$$\Gamma := (\mathbf{W}_0 \times \mathbf{W}_1) \underset{\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{T}}{\times} (\mathbf{F}_2 \times \mathbf{T}) \longrightarrow \mathbf{T}$$

is a flat family of curves in $\mathbf{F}_2$, with fiber $\Gamma_{\underline{f}}$.

Recall that the dimension of the variety of complete flags $\mathbf{F}_n \subset \prod \mathbf{G}_{i,n}$ is

$$\dim \mathbf{F}_n = 1 + \cdots + n.$$

The proposition is an easy consequence of the following.

7.2 Lemma. *Let $f_0, f_1, \ldots, f_n$ be arbitrary hypersurfaces of points, lines, ..., hyperplanes in the appropriate grassmannians of subspaces of $\mathbf{P}^{n+1}$. Then the intersection*

$$\Gamma_{\underline{f}} := (f_0 \times \cdots \times f_n) \cap \mathbf{F}_{n+1}$$

is of codimension $n+1$ in $\mathbf{F}_{n+1}$.

Proof. We shall argue by induction on n.

We may assume all f_i irreducible. For, if $f_0 = f_{0,1} \cup f_{0,2}$, say, we have $\Gamma_{\underline{f}} := (f_{0,1} \times \cdots \times f_n) \cap \mathbf{F}_{n+1} \cup (f_{0,2} \times \cdots \times f_n) \cap \mathbf{F}_{n+1}$.

Let $n = 1$. Pick a line $h \in f_1$. Set

$$h^{(0)} = \{P \in \mathbf{P}^2 \mid P \in h\}.$$

The fiber $(\Gamma_{\underline{f}})_h \simeq h^{(0)} \cap f_0$ is zero dimensional unless $h^{(0)} = f_0$. This occurs for at most one $h \in f_1$, hence $\Gamma_{\underline{f}}$ is 1–dimensional (otherwise most of its fibers over f_1 would be at least 1–dimensional).

For the inductive step, we set for $h \in \check{\mathbf{P}}^{n+1}$,

$$h^{(r)} = \{g \in \mathbf{G}_{r,n+1} | g \subseteq h\} \simeq \mathbf{G}_{r,n}. \tag{10}$$

If the intersection

$$f'_r = h^{(r)} \cap f_r$$

were proper for all r and $h \in f_n$ then we would be done by induction. Indeed, we have

$$(\Gamma_{\underline{f}})_h \simeq (f'_0 \times \cdots \times f'_{n-1}) \cap \mathbf{F}_n.$$

By the induction hypothesis, this is of the right dimension

$$1 + \cdots + n - n = 1 + \cdots + (n-1).$$

Since h varies in the n–dimensional hypersurface f_n of $\mathbf{G}_{n,n+1} = \check{\mathbf{P}}^{n+1}$, we would have

$$\dim \Gamma_{\underline{f}} = \big(1 + \cdots + (n-1)\big) + n = \big(1 + \cdots + (n+1)\big) - (n+1)$$

as desired.

However, just as in the case $n = 1$, it may well happen that the intersection $h^{(r)} \cap f_r$ be *not* proper for some h, r. Thus it remains to be shown that, whenever $\dim (\Gamma_{\underline{f}})_h$ exceeds the right dimension, say by δ, the hyperplane h is restricted to vary in a locus of codimension at least δ in f_n. This is taken care of in (7.3) below.

Consider the stratification of f_n by the condition of improper intersection of f_r with $h^{(r)}$, namely,

$$\begin{aligned} f_{n,0} &= \{h \in f_n \mid h^{(0)} \subseteq f_0\}, \\ f_{n,1} &= \{h \in f_n \mid h^{(1)} \subseteq f_1\} \setminus f_{n,0}, \\ &\vdots \\ f_{n,n} &= \{h \in f_n \mid h^{(n)} \subseteq f_n\} \setminus \bigcup_{j<n} f_{n,j}. \end{aligned}$$

We will be done if we show

$$\dim\,(\Gamma_{\underline{f}})_h \ \leq 1 + \cdots + n - r \quad \forall\ h \in f_{n,r}.$$

We have already seen that $\dim\,(\Gamma_{\underline{f}})_h = 1+\cdots+n-1$ for h in $f_{n,n}$. Also, for $r = 0$, the desired estimate holds because we have $(\Gamma_{\underline{f}})_h \subseteq (\mathbf{F}_{n+1})_h \simeq \mathbf{F}_n$ and $\dim \mathbf{F}_n = 1 + \cdots + n$. Let $r > 0$ and pick a hyperplane $h \in f_{n,r}$. Then the intersections,

$$f_i' = h^{(i)} \cap f_i,$$

are proper for $i = 0, \ldots, r-1$, whereas for the subsequent index, we have

$$h^{(r)} \cap f_r = h^{(r)} \simeq \mathbf{G}_{r,n}.$$

Thus, we may write,

$$(\Gamma_{\underline{f}})_h \hookrightarrow \left(f_0' \times \cdots \times f_{r-1}' \times \mathbf{G}_{r,n} \times \cdots \times \mathbf{G}_{n-1,n}\right) \bigcap \mathbf{F}_n.$$

By the induction hypothesis the intersection above is of dimension $\dim \mathbf{F}_n - r$ in view of the following easy

Remark. *The validity of 7.2 for a given n implies properness of the "partial" intersection*

$$(f_0 \times \cdots \times \mathbf{G}_{r,n+1} \times \cdots \times f_n) \cap \mathbf{F}_{n+1},$$

where one (or more) of the hypersurfaces $f_r \subset \mathbf{G}_{r,n+1}$ is replaced by the corresponding full grassmannian. □

7.3 Lemma. *Notation as in (10), for $r = 0, \ldots, n$ we have*

$$dim\,\{h \in \check{\mathbf{P}}^{n+1} \mid h^{(r)} \subseteq\ f_r\} \leq r.$$

Proof. Let $\mathbf{F}_{r,n} \subset \check{\mathbf{P}}^{n+1} \times \mathbf{G}_{r,n+1}$ be the partial flag variety. Form the diagram with natural projections,

$$\begin{array}{ccc} & \mathbf{F}_{r,n} & \\ {}^{\pi_n}\swarrow & & \searrow^{\pi_r} \\ \check{\mathbf{P}}^{n+1} & & \mathbf{G}_{r,n+1} \end{array}$$

For $g_r \in \mathbf{G}_{r,n+1}$, set

$$g_r^{(n)} = \{h \in \check{\mathbf{P}}^{n+1} \mid g_r \subseteq h\}.$$

We have $g_r^{(n)} \simeq \mathbf{P}^{n-r}$ whence it hits any subvariety of $\check{\mathbf{P}}^{n+1}$ of dimension $\geq r+1$. In other words, for any subvariety $\mathbf{Z} \subseteq \check{\mathbf{P}}^{n+1}$ such that dim $\mathbf{Z} \geq r+1$, we have

$$\begin{aligned} \pi_r \pi_n^{-1} \mathbf{Z} &= \{g_r \mid \exists\, h \in \mathbf{Z} \text{ s.t. } h \supseteq g_r\} \\ &= \{g_r \mid g_r^{(n)} \cap \mathbf{Z} \neq \varnothing\} = \mathbf{G}_{r,n+1}. \end{aligned}$$

The lemma follows by taking $\mathbf{Z} = \{h \in \check{\mathbf{P}}^{n+1} \mid h^{(r)} \subseteq f_r\}$. Indeed, if $\dim \mathbf{Z} \geq r+1$, then for all $g_r \in \mathbf{G}_{r,n+1}$ there exists $h \in \mathbf{Z}$ s.t. $h \supseteq g_r$, so $g_r \in h^{(r)} \subseteq f_r$, contradicting that f_r is a hypersurface of $\mathbf{G}_{r,n+1}$. □

7.4 Remark. Let (f_0, f_1, f_2) represent a nondegenerate, complete quadric surface κ. Thus, $f_0 \subset \mathbf{P}^3$ is a smooth quadric surface, $f_1 \subset \mathbf{G}_{1,3}$ is the hypersurface parametrizing the family of lines tangent to f_0 and $f_2 \subset \check{\mathbf{P}}^3$ is the dual quadric. We have that

$$(f_0 \times f_1 \times f_2) \bigcap \mathbf{F}_3 \subset \mathbf{P}^3 \times \mathbf{G}_{1,3} \times \check{\mathbf{P}}^3$$

contains an extra component in addition to the fiber of $\mathbf{K}$ over κ. In fact, it contains

$$\{(P, \ell, \pi) \in (f_0 \times f_1 \times f_2) \mid P \in \ell \subset f_0 \cap \pi\}$$

which is of dimension 3 (= 2 for the choice of $P \in \ell \subset f_0$ plus 1 for the pencil of planes containing ℓ). The point is that a plane π containing a ruling ℓ through a point P need not be tangent to f_0 at P, so that (P, ℓ, π) need not to belong to the tangent flag.

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References

[**E**] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, GTM # 150 Springer–Verlag, (1995).

[**EGA**] A. Grothendieck (avec J. Dieudonné), *Élements de Géométrie Algébrique IV (Troisième Partie)*, Publ. Math. IHES **28**, (1966).

[**EGA-I**] A. Grothendieck & J. Dieudonné , *Élements de Géométrie Algébrique I*, Grundlehren **166**, Springer-Verlag, (1971).

[**JH**] J. Harris, *Algebraic Geometry: a first course*, GTM # 133 Springer–Verlag, (1992).

[**H**] R. Hartshorne, *Algebraic Geometry*, GTM # 52 Springer–Verlag, (1977).

[**K**] S.L. Kleiman with A. Thorup, *"Intersection theory and enumerative geometry: A decade in review"*, *in* Algebraic geometry: Bowdoin 1985, S. Bloch, ed., AMS Proc. of Symp. Pure Math, **46-2**, 321-370, (1987).

[**KT**] S. L. Kleiman & A. Thorup, *"Complete bilinear forms"*, in Algebraic Geometry, Sundance, 1986, eds. A. Holme and R. Speiser, Lect. Notes in Math. 1311, 253-320, Springer–Verlag, (1988).

[**KTB**] ___________ , *"A Geometric Theory of the Buchsbaum–Rim Multiplicity"*, J. Algebra **167**, 168-231, (1994).

[**L**] D. Laksov *"Completed quadrics and linear maps"*, in Algebraic geometry: Bowdoin 1985, S. Bloch, ed., AMS Proc. of Symp. Pure Math., **46-2**, 371-387, (1987).

[**NT**] M. S. Narasimhan & G. Trautmann, *"Compactification of $M_{\mathbf{P}^3}(0,2)$ and Poncelet pairs of conics"*, Pacific J. Math. **145-2**, 255-365, (1990).

[**P**] R. Pandharipande, *"Notes on Kontsevich's Compactification Of The Space Of Maps"*, Course notes, Univ. Chicago, (1994).

[**BS**] B. Sturmfels, *"Gröbner basis and Stanley decompositions of determinantal rings"*, Math. Z., **205**(1), 137-144, (1995).

[**BS1**] ___________ , *"Gröbner basis and convex polytopes"*, Lectures notes at the Holiday Symp. at N. Mexico State Univ., Las Cruces (1994), AMS University Lecture Series, **8**, Providence, RI, (1995).

[**V**] I. Vainsencher, *Schubert calculus for complete quadrics*, in Enumerative Geometry and Classical Algebraic Geometry, ed. P. Le Barz & Y. Hervier, Progress in Math. **24**, Birkhäuser, 199–235, (1982)

[**V1**] ___________ , *"Conics multitangent to a plane curve"*, in preparation.

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